

# Solution for Navier-Stokes Equations – Lagrangian and Eulerian Descriptions

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**Abstract** – We find an exact solution for the system of Navier-Stokes equations, following the description of the Lagrangian movement of an element of fluid, for spatial dimension  $n = 3$ . As we had seen in other previous articles, there are infinite solutions for pressure and velocity, given only the condition of initial velocity.

**Keywords** – Navier-Stokes equations, velocity, pressure, Eulerian description, Lagrangian description, formulation, classical mechanics, Newtonian mechanics, Newton's law, second law of Newton, equivalent systems, exact solutions, Millennium Problem, existence, smoothness.

## §

Essentially the Navier-Stokes equations relate to the velocity  $u$  and pressure  $p$  suffered for a volume element  $dV$  at position  $(x, y, z)$  and time  $t$ . In the formulation or description Eulerian the position  $(x, y, z)$  is fixed in time, running different volume elements of fluid in this same position, while the time varies. In the Lagrangian formulation the position  $(x, y, z)$  refers to the instantaneous position of a specific volume element  $dV = dx dy dz$  at time  $t$ , and this position varies with the movement of this same element  $dV$ .

Basically, the deduction of the Navier-Stokes equations is a classical mechanics problem, a problem of Newtonian mechanics, which use the 2nd law of Newton  $F = ma$ , force is equal to mass multiplied by acceleration. We all know that the force described in Newton's law may have different expressions, varying only in time or also with the position, or with the distance to the source, varying with the body's velocity, etc. Each specific problem must to define how the forces involved in the system are applied and what they mean. I suggest consulting the classic Landau & Lifshitz<sup>[1]</sup> or Prandtl book<sup>[2]</sup> for a more detailed description of the deduction of these equations.

In spatial dimension  $n = 3$ , the Navier-Stokes equations can be put in the form of a system of three nonlinear partial differential equations, as follows:

$$(1) \quad \begin{cases} \frac{\partial p}{\partial x} + \frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_1}{\partial y} + u_3 \frac{\partial u_1}{\partial z} = \nu \nabla^2 u_1 + \nu (\nabla(\nabla \cdot u))_1 + f_1 \\ \frac{\partial p}{\partial y} + \frac{\partial u_2}{\partial t} + u_1 \frac{\partial u_2}{\partial x} + u_2 \frac{\partial u_2}{\partial y} + u_3 \frac{\partial u_2}{\partial z} = \nu \nabla^2 u_2 + \nu (\nabla(\nabla \cdot u))_2 + f_2 \\ \frac{\partial p}{\partial z} + \frac{\partial u_3}{\partial t} + u_1 \frac{\partial u_3}{\partial x} + u_2 \frac{\partial u_3}{\partial y} + u_3 \frac{\partial u_3}{\partial z} = \nu \nabla^2 u_3 + \nu (\nabla(\nabla \cdot u))_3 + f_3 \end{cases}$$

where  $u(x, y, z, t) = (u_1(x, y, z, t), u_2(x, y, z, t), u_3(x, y, z, t))$ ,  $u: \mathbb{R}^3 \times [0, \infty) \rightarrow \mathbb{R}^3$ , is the velocity of the fluid, of components  $u_1, u_2, u_3$ ,  $p$  is the pressure,  $p: \mathbb{R}^3 \times [0, \infty) \rightarrow \mathbb{R}$ , and  $f(x, y, z, t) = (f_1(x, y, z, t), f_2(x, y, z, t), f_3(x, y, z, t))$ ,  $f: \mathbb{R}^3 \times [0, \infty) \rightarrow \mathbb{R}^3$ , is the density of external force applied in the fluid in point  $(x, y, z)$  and at the instant of time  $t$ , for example, gravity force per mass unity, with  $x, y, z, t \in \mathbb{R}$ ,  $t \geq 0$ . The coefficient  $\nu \geq 0$  is the viscosity coefficient, and in the special case that  $\nu = 0$  we have the Euler equations.  $\nabla \equiv \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$  is the nabla operator and  $\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \equiv \Delta$  is the Laplacian operator.

The non-linear terms  $u_1 \frac{\partial u_i}{\partial x} + u_2 \frac{\partial u_i}{\partial y} + u_3 \frac{\partial u_i}{\partial z}$ ,  $1 \leq i \leq 3$ , are a natural consequence of the Eulerian formulation of motion, and corresponds to part of the total derivative of velocity with respect to time of a volume element  $dV$  in the fluid, i.e., its acceleration. If  $u = (u_1(x, y, z, t), u_2(x, y, z, t), u_3(x, y, z, t))$  and these  $x, y, z$  also vary in time,  $x = x(t)$ ,  $y = y(t)$ ,  $z = z(t)$ , then, by the chain rule,

$$(2) \quad \frac{Du}{Dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt} + \frac{\partial u}{\partial z} \frac{dz}{dt}.$$

Defining  $\frac{dx}{dt} = u_1$ ,  $\frac{dy}{dt} = u_2$ ,  $\frac{dz}{dt} = u_3$ , comes

$$(3) \quad \frac{Du}{Dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} u_1 + \frac{\partial u}{\partial y} u_2 + \frac{\partial u}{\partial z} u_3,$$

and therefore

$$(4) \quad \frac{Du_i}{Dt} = \frac{\partial u_i}{\partial t} + u_1 \frac{\partial u_i}{\partial x} + u_2 \frac{\partial u_i}{\partial y} + u_3 \frac{\partial u_i}{\partial z}, \quad 1 \leq i \leq 3,$$

which contain the non-linear terms that appear in (1).

Numerically, searching a computational result, i.e., in practical terms, there can be no difference between the Eulerian and Lagrangian formulations for the evaluate of  $\frac{Du}{Dt}$  (or  $\frac{du}{dt}$ , it is the same physical and mathematical entity). Only conceptually and formally there is difference in the two approaches. I agree, however, that you first consider  $(x, y, z)$  variable in time (Lagrangian formulation) and then consider  $(x, y, z)$  fixed (Eulerian formulation), seems to be subject to criticism. As in the Newton equations of motion we can consider that there are

forces traveling with a body, and also there are forces that may be fixed in the each space position, in a same configuration or system model, in our present formulation the pressure, and its corresponding gradient, they do not travel with the volume element  $dV = dx dy dz$ , i.e., obeys to the Eulerian description of motion, and the external force  $f$  can do it or not, depending on the definition of the problem. The velocity  $u$ , however, will obey to the Lagrangian description, and it is representing the velocity of a generic volume element  $dV$  over time, initially at position  $(x_0, y_0, z_0)$  and with initial velocity  $u^0 = u(0) = cte., u = u(t)$ .

Following this definition, the system (1) above is transformed into

$$(5) \quad \begin{cases} \frac{\partial p}{\partial x} + \frac{Du_1}{Dt} = \nu (\nabla^2 u_1)|_t + \nu ((\nabla(\nabla \cdot u))_1)|_t + f_1 \\ \frac{\partial p}{\partial y} + \frac{Du_2}{Dt} = \nu (\nabla^2 u_2)|_t + \nu ((\nabla(\nabla \cdot u))_2)|_t + f_2 \\ \frac{\partial p}{\partial z} + \frac{Du_3}{Dt} = \nu (\nabla^2 u_3)|_t + \nu ((\nabla(\nabla \cdot u))_3)|_t + f_3 \end{cases}$$

thus (1) and (5) are equivalent systems, according (4) validity. The nabla and Laplacian operators are considered calculated in Lagrangian formulation, i.e., in the variable time. We first choose  $u \in C^\infty(\mathbb{R}^3 \times [0, \infty))$  (say, for example, or belonging to Schwartz space, etc.), calculate the operators in variables  $x, y, z, t$  in the Eulerian formulation and then we convert  $x = x(t), y = y(t), z = z(t)$  to Lagrangian formulation. Likewise for the calculation of  $\frac{Du}{Dt}$ , following (4).

The system (5) always has a solution if the external force  $f$  is a gradient function, for example, dependent only on the time variable, and the components velocity are  $C^2([0, \infty))$  class.

Given  $u = u(t) \in C^2([0, \infty))$  obeying the initial conditions and a gradient vector function  $f$ , the system solution (5) is

$$(6) \quad \begin{aligned} p &= \int_L S \cdot dl + \theta(t), \\ S &= \nu(\nabla^2 u)|_t + \nu(\nabla(\nabla \cdot u))|_t + f - \frac{Du}{Dt}, \end{aligned}$$

where  $L$  is any continuous path linking a point  $(x_0, y_0, z_0)$  to  $(x, y, z)$  and  $\theta(t)$  is a generic time function, physically and mathematically reasonable, for example with  $\theta(0) = 0$ .

In special case when  $f$  is a constant vector or a function dependent only on the time variable, we come to

$$(7) \quad \begin{aligned} p &= p^0 + S_1(t) (x - x_0) + S_2(t) (y - y_0) + S_3(t) (z - z_0), \\ S_i(t) &= \nu(\nabla^2 u_i)|_t + \nu((\nabla(\nabla \cdot u))_i)|_t + f_i - \frac{Du_i}{Dt}, \end{aligned}$$

where  $p^0 = p^0(t)$  is the pressure in the point  $(x_0, y_0, z_0)$  at time  $t$ . Note that the variables  $x, y, z$  in (7) are in Eulerian description, i.e. are fixed coordinates in space while time  $t$  varies.

Again we have seen that the system of Navier-Stokes equations has no unique solution. We can choose different velocities that have the same initial velocity and also result, in general, in different pressures.

How to return to the Eulerian formulation obtained a solution in the Lagrangian formulation? Previously we already have the solution for velocity in Eulerian formulation, except the pressure value. We can choose appropriate  $u(x, y, z, t)$  and  $x(t), y(t), z(t)$  to the velocities and positions of the system, next we calculate (4) as a time function as well as the differential operations  $(\nabla^2 u)|_t$  and  $(\nabla(\nabla \cdot u))|_t$  and then we carry these results in (7) for the pressure calculation. This is not completely free because will be necessary to calculate a system of ordinary differential equations to obtain the correct set of  $x(t), y(t), z(t)$ , such that

$$(8) \quad \begin{cases} \frac{dx}{dt} = u_1(x, y, z, t) \\ \frac{dy}{dt} = u_2(x, y, z, t) \\ \frac{dz}{dt} = u_3(x, y, z, t) \end{cases}$$

Nevertheless, this yet can save lots calculation time.

Apply this method to the famous 6th Millenium Problem<sup>[3]</sup> on existence and smoothness of the Navier-Stokes equations is not so difficult at the same time also it is not absolutely trivial. It takes some time. I hope to do it soon, with dignity. This includes the cases (C) and (D): if the external force  $f$  is a non-gradient vector function then, in general, there is no solution for the Navier-Stokes equations, valid for all  $t \geq 0$ , a conclusion that comes to help my first article on the subject<sup>[4]</sup>. On the other hand, apply this method to the case  $n = 2$  or  $\nu = 0$  (Euler equation) is almost immediate.

*To Leonard Euler, in memorian,  
the greatest mathematician of all time.  
309th anniversary of his birth,  
April-15-1707-2016.*

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