Solution for Navier-Stokes Equations – Lagrangian and Eulerian Descriptions

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Abstract – We find an exact solution for the system of Navier-Stokes equations, following the description of the Lagrangian movement of an element of fluid, for spatial dimension n = 3. As we had seen in other previous articles, there are infinite solutions for pressure and velocity, given only the condition of initial velocity.

Keywords – Navier-Stokes equations, velocity, pressure, Eulerian description, Lagrangian description, formulation, classical mechanics, Newtonian mechanics, Newton's law, second law of Newton, equivalent systems, exact solutions, Millennium Problem, existence, smoothness, Bernoulli's law, Laplace's equation, harmonic functions.

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Essentially the Navier-Stokes equations relate to the velocity u and pressure p suffered for a volume element dV at position (x, y, z) and time t. In the formulation or description Eulerian the position (x, y, z) is fixed in time, running different volume elements of fluid in this same position, while the time varies. In the Lagrangian formulation the position (x, y, z) refers to the instantaneous position of a specific volume element $dV = dx\,dy\,dz$ at time t, and this position varies with the movement of this same element dV.

Basically, the deduction of the Navier-Stokes equations is a classical mechanics problem, a problem of Newtonian mechanics, which use the 2nd law of Newton $F = ma$, force is equal to mass multiplied by acceleration. We all know that the force described in Newton's law may have different expressions, varying only in time or also with the position, or with the distance to the source, varying with the body's velocity, etc. Each specific problem must to define how the forces involved in the system are applied and what they mean. I suggest consulting the classic Landau & Lifshitz[1] or Prandtl book[2] for a more detailed description of the deduction of these equations. Note that the deduction by Landau & Lifshitz [1] contain more parameters than the shown in the references [2] and [3].

In spatial dimension $n = 3$, the Navier-Stokes equations can be put in the form of a system of three nonlinear partial differential equations, as follows:

$$(1)\quad \begin{cases} \frac{\partial p}{\partial x}+\frac{\partial u_1}{\partial t}+u_1\frac{\partial u_1}{\partial x}+u_2\frac{\partial u_1}{\partial y}+u_3\frac{\partial u_1}{\partial z}=\nu\nabla^2 u_1+\frac{1}{3}\nu\big(\nabla(\nabla\cdot u)\big)_1+f_1 \\ \frac{\partial p}{\partial y}+\frac{\partial u_2}{\partial t}+u_1\frac{\partial u_2}{\partial x}+u_2\frac{\partial u_2}{\partial y}+u_3\frac{\partial u_2}{\partial z}=\nu\nabla^2 u_2+\frac{1}{3}\nu\big(\nabla(\nabla\cdot u)\big)_2+f_2 \\ \frac{\partial p}{\partial z}+\frac{\partial u_3}{\partial t}+u_1\frac{\partial u_3}{\partial x}+u_2\frac{\partial u_3}{\partial y}+u_3\frac{\partial u_3}{\partial z}=\nu\nabla^2 u_3+\frac{1}{3}\nu\big(\nabla(\nabla\cdot u)\big)_3+f_3 \end{cases}$$

where $u(x,y,z,t)=\big(u_1(x,y,z,t),u_2(x,y,z,t),u_3(x,y,z,t)\big)$, $u:\mathbb{R}^3\times[0,\infty)\to\mathbb{R}^3$, is the velocity of the fluid, of components u_1,u_2,u_3, p is the pressure, $p:\mathbb{R}^3\times[0,\infty)\to\mathbb{R}$, and $f(x,y,z,t)=\big(f_1(x,y,z,t),f_2(x,y,z,t),f_3(x,y,z,t)\big)$, $f:\mathbb{R}^3\times[0,\infty)\to\mathbb{R}^3$, is the density of external force applied in the fluid in point (x,y,z) and at the instant of time t, for example, gravity force per mass unity, with $x,y,z,t\in\mathbb{R}$, $t\geq 0$. The coefficient $\nu\geq 0$ is the viscosity coefficient, and in the special case that $\nu=0$ we have the Euler equations. $\nabla\equiv\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)$ is the nabla operator and $\nabla^2=\nabla\cdot\nabla=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\equiv\Delta$ is the Laplacian operator.

The non-linear terms $u_1\frac{\partial u_i}{\partial x}+u_2\frac{\partial u_i}{\partial y}+u_3\frac{\partial u_i}{\partial z}$, $1\leq i\leq 3$, are a natural consequence of the Eulerian formulation of motion, and corresponds to part of the total derivative of velocity with respect to time of a volume element dV in the fluid, i.e., its acceleration. If $u=(u_1(x,y,z,t),u_2(x,y,z,t),u_3(x,y,z,t))$ and these x,y,z also vary in time, $x=x(t)$, $y=y(t)$, $z=z(t)$, then, by the chain rule,

$$(2)\qquad \frac{Du}{Dt}=\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}\frac{dx}{dt}+\frac{\partial u}{\partial y}\frac{dy}{dt}+\frac{\partial u}{\partial z}\frac{dz}{dt}.$$

Defining $\frac{dx}{dt}=u_1$, $\frac{dy}{dt}=u_2$, $\frac{dz}{dt}=u_3$, comes

$$(3)\qquad \frac{Du}{Dt}=\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}u_1+\frac{\partial u}{\partial y}u_2+\frac{\partial u}{\partial z}u_3,$$

and therefore

$$(4)\qquad \frac{Du_i}{Dt}=\frac{\partial u_i}{\partial t}+u_1\frac{\partial u_i}{\partial x}+u_2\frac{\partial u_i}{\partial y}+u_3\frac{\partial u_i}{\partial z},\ 1\leq i\leq 3,$$

which contain the non-linear terms that appear in (1).

Numerically, searching a computational result, i.e., in practical terms, there can be no difference between the Eulerian and Lagrangian formulations for the evaluate of $\frac{Du}{Dt}$ (or $\frac{du}{dt}$, it is the same physical and mathematical entity). Only conceptually and formally there is difference in the two approaches. I agree, however, that you first consider (x,y,z) variable in time (Lagrangian formulation) and then consider (x,y,z) fixed (Eulerian formulation), seems to be subject to criticism. As in the Newton equations of motion we can consider that there are forces traveling with a body, and also there are forces that may be fixed in the each space position, in a same configuration or system model, in our present formulation the pressure, and its corresponding gradient, they travel with the volume element $dV = dx\,dy\,dz$, i.e., obeys to the Lagrangian description of motion, as well as the external force f, in order to avoid contradictions. The velocity u also will obey to the Lagrangian description, and it is representing the velocity of a generic volume element dV over time, initially at position (x_0, y_0, z_0) and with initial velocity $u^0 = u(0) = cte.$, $u = u(t)$.

Following this definition, the system (1) above is transformed into

$$(5) \qquad \begin{cases} \frac{\partial p}{\partial x} + \frac{Du_1}{Dt} = \nu\,(\nabla^2 u_1)|_t + \frac{1}{3}\nu\,((\nabla(\nabla\cdot u))_1)|_t + f_1 \\ \frac{\partial p}{\partial y} + \frac{Du_2}{Dt} = \nu\,(\nabla^2 u_2)|_t\ + \frac{1}{3}\nu\,((\nabla(\nabla\cdot u))_2)|_t + f_2 \\ \frac{\partial p}{\partial z} + \frac{Du_3}{Dt} = \nu\,(\nabla^2 u_3)|_t + \frac{1}{3}\nu\,((\nabla(\nabla\cdot u))_3)|_t + f_3 \end{cases}$$

thus (1) and (5) are equivalent systems, according (4) validity. The nabla and Laplacian operators are considered calculated in Lagrangian formulation, i.e., in the variable time. We first choose $u \in C^\infty(\mathbb{R}^3 \times [0,\infty))$ (say, for example, or belonging to Schwartz space, etc.), calculate the operators in variables $x,\ y,\ z,\ t$ in the Eulerian formulation, which is a standard procedure, and then we convert these results to Lagrangian formulation using $x = x(t),\ y = y(t),\ z = z(t)$, the our special method. Likewise for the calculation of $\frac{Du}{Dt}$, following (4).

The system (5) always has a solution if the external force f is a gradient function[4], for example, dependent only on the time variable, and the components velocity are $C^2([0,\infty))$ class.

Given $u = u(t) \in C^2([0,\infty)^3)$ obeying the initial conditions and a gradient vector function f, the system solution (5) is

$$(6) \qquad \begin{aligned} &p = \int_L\ S\cdot dl + \theta(t), \\ &S = \nu(\nabla^2 u)|_t + \frac{1}{3}\nu(\nabla(\nabla\cdot u))|_t + f - \frac{Du}{Dt}, \end{aligned}$$

where L is any continuous path linking a point (x_0, y_0, z_0) to (x, y, z) and $\theta(t)$ is a generic time function, physically and mathematically reasonable, for example with $\theta(0) = 0$.

In special case when f is a constant vector or a dependent function only on the time variable, we come to

$$(7) \qquad \begin{aligned} &p = p^0 + S_1(t)\,(x - x_0) + S_2(t)\,(y - y_0) + S_3(t)\,(z - z_0), \\ &S_i(t) = \nu(\nabla^2 u_i)|_t + \frac{1}{3}\nu((\nabla(\nabla\cdot u))_i)|_t + f_i - \frac{Du_i}{Dt}, \end{aligned}$$

where $p^0 = p^0(t)$ is the pressure in the point (x_0, y_0, z_0) at time t. Note that the variables x, y, z in (7) are in Lagrangian description, i.e. are the coordinates of a volume element dV in movement, the position of a chosen particle of fluid, and we can eliminate the dependence of the position substituting in (6)

(8) $$dl = (dx,\ dy,\ dz) = (u_1 dt,\ u_2 dt,\ u_3 dt)$$

and integrating over time. The result is

(9) $$p(t) = p^0 + \int_0^t \sum_{i=1}^3 S_i(t)\, u_i(t)\, dt,$$

$$p^0 = p(0) = cte.$$

This calculation can be more facilitated making $u_i \frac{Du_i}{Dt} dt = u_i du_i$ and $\int_0^t u_i \frac{Du_i}{Dt} dt = \int_{u_i^0}^{u_i} u_i du_i = \frac{1}{2}\left(u_i^2 - u_i^{0\,2}\right)$, so (9) is equal to

(10) $$p(t) = p^0 - \frac{1}{2}\sum_{i=1}^3 \left(u_i^2 - u_i^{0\,2}\right) + \int_0^t \sum_{i=1}^3 R_i(t)\, u_i(t)\, dt,$$
$$R_i(t) = \nu(\nabla^2 u_i)|_t + \frac{1}{3}\nu((\nabla(\nabla \cdot u))_i)|_t + f_i,$$

i.e.,

(11) $$p(t) = p^0 - \frac{1}{2}(u^2 - u^{0\,2}) + \int_0^t R \cdot u\, dt,$$
$$R = \nu(\nabla^2 u)|_t + \frac{1}{3}\nu(\nabla(\nabla \cdot u))|_t + f,$$

$p, p^0 \in \mathbb{R}, u, u^0, f, R \in \mathbb{R}^3, u = (u_1, u_2, u_3),\ u^0 = (u_1^0, u_2^0, u_3^0) = u(0)$, $f = (f_1, f_2, f_3)$, in Lagrangian description, as well as

(12) $$p(x, y, z, t) = p^0(x, y, z) - \frac{1}{2}(u^2 - u^{0\,2}) + \int_L \ R \cdot dl,$$
$$R = \nu\nabla^2 u + \frac{1}{3}\nu\nabla(\nabla \cdot u) + f,$$

in Eulerian description, $p^0(x, y, z) = p(x, y, z, 0)$, $u^0 = u^0(x, y, z) = u(x, y, z, 0)$, both formulations supposing R a gradient vector function ($\nabla \times R = 0,\ R = \nabla\phi,\ \phi$ potential function of R) when in Eulerian description (because we do not expect there is a contradiction between the Lagrangian and Eulerian descriptions). $u^2 = u \cdot u$ and $u^{0\,2} = u^0 \cdot u^0$ are the square modules of the respective vectors u and u^0.

When $f = 0$ and $\nu = 0$ (or most in general $R = 0$) it is simply

(13) $$p = p^0 - \frac{1}{2}(u^2 - u^{0\,2}),$$

which then can be considered an exact solution for Euler equations in a general format, in Lagrangian and Eulerian descriptions, and according Bernoulli's law without external force (for example, gravity).

Again we have seen that the system of Navier-Stokes equations has no unique solution, only given initial conditions. We can choose different velocities that have the same initial velocity and also result, in general, in different pressures.

How to return to the Eulerian formulation if only was obtained a complete solution in the Lagrangian formulation? Previously we already have the solution for velocity in Eulerian formulation, except the pressure value. We can choose appropriate $u(x, y, z, t)$ and $x(t), y(t), z(t)$ to the velocities and positions of the system, next we calculate (4) as a time function as well as the differential operations $(\nabla^2 u)|_t$ and $(\nabla(\nabla \cdot u))|_t$ and then we carry these results in (7) or (11) for the pressure calculation. This choose is not completely free because will be necessary to calculate a system of ordinary differential equations to obtain the correct set of $x(t), y(t), z(t)$, such that

(14)
$$\begin{cases} \frac{dx}{dt} = u_1(x, y, z, t) \\ \frac{dy}{dt} = u_2(x, y, z, t) \\ \frac{dz}{dt} = u_3(x, y, z, t) \end{cases}$$

Nevertheless, this yet can save lots calculation time.

It will be necessary find solutions of (14) such that it is always possible to make any point (x, y, z) of the velocity domain can be achieved for each time t, introducing for this initial positions (x_0, y_0, z_0) conveniently calculated according to (14). Thus we will have velocities and pressures that, in principle, can be calculated for any position and time, independently of one another, not only for a single position for each time. For different values of (x, y, z) and t we will, in the general case, obtain the velocity and pressure of different volume elements dV, starting from different initial positions (x_0, y_0, z_0).

We can escape the need to solve (14), but admitting its validity and the corresponding existence of solution, previously choosing differentiable functions $x = x(t), y = y(t), z = z(t)$ and then calculating directly the solution for velocity in the Lagrangian formulation,

(15)
$$\begin{cases} u_1(t) = \frac{dx}{dt} \\ u_2(t) = \frac{dy}{dt} \\ u_3(t) = \frac{dz}{dt} \end{cases}$$

hereafter calculating $\frac{\partial u_i}{\partial x_j}, \frac{\partial^2 u_i}{\partial x_j^2}$ and the differential operations $\nabla \cdot u, \nabla^2 u$ and $\nabla(\nabla \cdot u)$ through of the transformations

(16) $$\frac{\partial u_i}{\partial x_j} = \frac{\partial u_i/\partial t}{\partial x_j/\partial t} = \frac{1}{u_j}\frac{\partial u_i}{\partial t}$$

and

(17) $$\frac{\partial^2 u_i}{\partial x_j^2} = \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j}\right) = \frac{\partial/\partial t}{\partial x_j/\partial t}\left(\frac{1}{u_j}\frac{\partial u_i}{\partial t}\right) =$$

$$= \frac{1}{u_j^2}\left[-\frac{1}{u_j}\left(\frac{\partial u_i}{\partial t}\right)\left(\frac{\partial u_j}{\partial t}\right) + \frac{\partial^2 u_i}{\partial t^2}\right],$$

and finally calculating the pressure in (7) or (11), with $\frac{Du_i}{Dt} \equiv \frac{du_i}{dt}$, supposing finites the limits in (16) and (17) when $u_j \to 0$. Remembering, this method calculates the pressure related to the position $(x(t), y(t), z(t))$ in Lagrangian description. Also note that perhaps the denominators appearing in (16) and (17) explaining the occurrence of *blowup time* reported in the literature[3], when the limits are not finites.

Concluding, answering the question, in the result of pressure in Lagrangian formulation, conveniently transforming the initial position $(x_0,\ y_0,\ z_0)$ as function of a generic position (x, y, z) and time t, we will have a correct value of the pressure in Eulerian formulation. The same is valid for the velocity in Lagrangian formulation, if the correspondent Eulerian formulation was not previously obtained.

Another way to solve (1) seems to me to be the best of all, for its extreme ease of calculation, without we need to resort to Lagrangian formulation and its conceptual difficulties. If $u(x, y, z, 0) = u^0(x, y, z)$ is the initial velocity of the system, valid solution in $t = 0$, then $u(x, y, z, t) = u^0(x + t, y + t, z + t)$ is a solution for velocity in $t \geq 0$, a non-unique solution. Similarly, $p(x, y, z, t) = p^0(x + t, y + t, z + t)$ is the correspondent solution for pressure in $t \geq 0$, being $p^0(x, y, z)$ the initial condition for pressure. The velocities $u^0(x + t, y, z)$, $u^0(x, y + t, z)$ and $u^0(x, y, z + t)$ are also solutions, and respectively also the pressures $p^0(x + t, y, z)$, $p^0(x, y + t, z)$ and $p^0(x, y, z + t)$. Other solutions may be searched, for example in the kind $u(x, y, z, t) = u^0\big(x + T_1(t), y + T_2(t), z + T_3(t)\big)$, $T_i(0) = 0$, and therefore $p(x, y, z, t) = p^0\big(x + T_1(t), y + T_2(t), z + T_3(t)\big)$.

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A very useful and interesting special solution for equation (1) and derived from (12) is

(18) $$u(x,t) = A(t)\, u^0(x) + B(t),$$

where A, B are time dependent real functions, $A(0) = 1,\ B(0) = 0, x \in \mathbb{R}^3$ and the initial velocity is $u^0(x) = u(x,0)$, which respective solution for pressure is

(19) $$p(x,t) = p^0(x) + \tfrac{1}{2}(u^2 - u^{0\,2}) + \int_L\ R \cdot dl =$$
$$= p^0(x) + \tfrac{1}{2}[(A^2 - 1)u^{0\,2} + B^2] + ABu^0 +$$
$$+ \int_L\ \left\{\nu\, A(t) \left[\nabla^2 u^0 + \tfrac{1}{3}\nabla(\nabla \cdot u^0)\right] + f\right\} \cdot dl,$$

$p^0(x) = p(x,0)$ is the initial pressure.

The solution (12) must be valid for all $t \geq 0$ and for all solution (u,p) of (1), so in $t = 0$ must be satisfied (in Eulerian formulation)

(20) $$0 = \int_L\ R^0 \cdot dl$$

where

(21) $$R^0 = \nu\, \nabla^2 u^0 + \tfrac{1}{3}\nu\, \nabla(\nabla \cdot u^0) + f^0,$$

$f^0 = f(x,0),\ R^0 = R(x,0)$, thus the initial velocity must verify the equation

(22) $$\nu\, \nabla^2 u^0 + \tfrac{1}{3}\nu\, \nabla(\nabla \cdot u^0) + f^0 = 0.$$

The equation (22) applied in (19) gives

(23) $$p(x,t) = p^0(x) + \tfrac{1}{2}[(A^2 - 1)u^{0\,2} + B^2] + ABu^0 +$$
$$+ \int_L\ (f - Af^0) \cdot dl,$$

or in most elegant format

(24) $$p(x,t) = p^0(x) + \tfrac{1}{2}(u^2 - u^{0\,2}) + \int_L\ (f - Af^0) \cdot dl,$$

again according Bernouilli's law in a generalized form.

In special case when don't have external force and the fluid is incompressible the equation (22) becomes

(25) $$\nabla^2 u^0 = 0,$$

the Laplace's equation, which nontrivial solutions are the harmonic functions[5].

Apply these methods to the famous 6th Millenium Problem[3] on existence and smoothness of the Navier-Stokes equations is not so difficult at the same time also it is not absolutely trivial. It takes some time. I hope to do it soon. On the other hand, apply these methods to the case $n = 2$ or $\nu = 0$ (Euler equation) is almost immediate.

It is no longer true that the Navier-Stokes and Euler equations yet do not have general solutions known.

To Leonard Euler, in memorian,
the greatest mathematician of all time.
309th anniversary of his birth,
April-15-1707-2016.

Last update: June-12-2016.

Euler, forgive me for my mistakes...
This subject is very difficult!

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