Solution for Navier-Stokes Equations – Lagrangian and Eulerian Descriptions

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Abstract – We find an exact solution for the system of Navier-Stokes equations, supposing that there is some solution, following the Eulerian and Lagrangian descriptions, for spatial dimension $n = 3$. As we had seen in other previous articles, it is possible that there are infinite solutions for pressure and velocity, given only the condition of initial velocity.

Keywords – Navier-Stokes equations, Euler equations, velocity, pressure, Eulerian description, Lagrangian description, formulation, classical mechanics, Newtonian mechanics, Newton's law, second law of Newton, equivalent systems, exact solutions, Millennium Problem, existence, smoothness, Bernoulli's law, Turbulence Theory, Theory of Perturbations, Numerical Methods, Computational Fluid Dynamics.

§ 1

Essentially the Navier-Stokes equations relating the velocity u and pressure p suffered by a volume element dV at position (x, y, z) and time t. In the formulation or description Eulerian the position (x, y, z) is fixed in time, running different volume elements of fluid in this same position, while the time varies. In the Lagrangian formulation the position (x, y, z) refers to the instantaneous position of a specific volume element $dV = dx\, dy\, dz$ at time t, and this position varies with the movement of this same element dV.

Basically, the deduction of the Navier-Stokes equations is a classical mechanics problem, a problem of Newtonian mechanics, which use the 2nd law of Newton $F = ma$, force is equal to mass multiplied by acceleration. We all know that the force described in Newton's law may have different expressions, varying only in time or also with the position, or with the distance to the source, varying with the body's velocity, etc. Each specific problem must to define how the forces involved in the system are applied and what they mean. I suggest consulting the classic Landau & Lifshitz[1] or Prandtl's book[2] for a more detailed description of the deduction of these equations. Note that the deduction by Landau & Lifshitz [1] contain more parameters than the shown in the references [2] and [3].

In spatial dimension $n = 3$, the Navier-Stokes equations in rectangular coordinates can be put in the form of a system of three nonlinear partial differential equations, as follows:

(1) $$\begin{cases} \frac{\partial p}{\partial x} + \frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_1}{\partial y} + u_3 \frac{\partial u_1}{\partial z} = \nu \nabla^2 u_1 + \frac{1}{3} \nu \nabla_1 (\nabla \cdot u) + f_1 \\ \frac{\partial p}{\partial y} + \frac{\partial u_2}{\partial t} + u_1 \frac{\partial u_2}{\partial x} + u_2 \frac{\partial u_2}{\partial y} + u_3 \frac{\partial u_2}{\partial z} = \nu \nabla^2 u_2 + \frac{1}{3} \nu \nabla_2 (\nabla \cdot u) + f_2 \\ \frac{\partial p}{\partial z} + \frac{\partial u_3}{\partial t} + u_1 \frac{\partial u_3}{\partial x} + u_2 \frac{\partial u_3}{\partial y} + u_3 \frac{\partial u_3}{\partial z} = \nu \nabla^2 u_3 + \frac{1}{3} \nu \nabla_3 (\nabla \cdot u) + f_3 \end{cases}$$

where $u(x,y,z,t) = \big(u_1(x,y,z,t), u_2(x,y,z,t), u_3(x,y,z,t)\big)$, $u: \mathbb{R}^3 \times [0,\infty) \to \mathbb{R}^3$, is the velocity of the fluid, of components u_1, u_2, u_3, p is the pressure, $p: \mathbb{R}^3 \times [0,\infty) \to \mathbb{R}$, and $f(x,y,z,t) = \big(f_1(x,y,z,t), f_2(x,y,z,t), f_3(x,y,z,t)\big)$, $f: \mathbb{R}^3 \times [0,\infty) \to \mathbb{R}^3$, is the density of external force applied in the fluid in point (x,y,z) and at the instant of time t, for example, gravity force per mass unity, with $x,y,z,t \in \mathbb{R}$, $t \geq 0$. The coefficient $\nu \geq 0$ is the viscosity coefficient, and in the special case that $\nu = 0$ we have the Euler equations. $\nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ is the nabla operator and $\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \equiv \Delta$ is the Laplacian operator. We are using fluid mass density $\rho = 1$ (otherwise substitute p by p/ρ and ν by ν/ρ, supposing ρ is a constant).

The non-linear terms $u_1 \frac{\partial u_i}{\partial x} + u_2 \frac{\partial u_i}{\partial y} + u_3 \frac{\partial u_i}{\partial z}$, $1 \leq i \leq 3$, are a natural consequence of the Eulerian formulation of motion, and corresponds to part of the total derivative of velocity with respect to time of a volume element dV in the fluid, i.e., its acceleration. If $u = (u_1(x,y,z,t), u_2(x,y,z,t), u_3(x,y,z,t))$ and these x, y, z also vary in time, $x = x(t)$, $y = y(t)$, $z = z(t)$, then, by the chain rule,

(2) $$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}.$$

Defining

(3) $$\begin{cases} \frac{dx}{dt} = u_1 \\ \frac{dy}{dt} = u_2 \\ \frac{dz}{dt} = u_3 \end{cases}$$

or synthetically $\frac{dx_i}{dt} = u_i$, using $x_1 \equiv x$, $x_2 \equiv y$, $x_3 \equiv z$, comes

(4) $$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} u_1 + \frac{\partial u}{\partial y} u_2 + \frac{\partial u}{\partial z} u_3,$$

and therefore

(5) $$\frac{Du_i}{Dt} = \frac{\partial u_i}{\partial t} + u_1 \frac{\partial u_i}{\partial x} + u_2 \frac{\partial u_i}{\partial y} + u_3 \frac{\partial u_i}{\partial z}, 1 \leq i \leq 3,$$

which contain the non-linear terms that appear in (1).

Numerically, searching a computational result, i.e., in practical terms, there can be no difference between the Eulerian and Lagrangian formulations for the evaluate of $\frac{Du}{Dt}$ (or $\frac{du}{dt}$, it is the same physical and mathematical entity). Only conceptually and formally there is difference in the two approaches. I agree, however, that you first consider (x, y, z) variable in time (Lagrangian formulation) and then consider (x, y, z) fixed (Eulerian formulation), seems to be subject to criticism. In our present deduction, starting from Navier-Stokes equations in Eulerian description, implicitly with a solution (u, p), next the pressure, and its corresponding gradient, they travel with the volume element $dV = dx\, dy\, dz$, i.e., obeys to the Lagrangian description of motion, as well as the external force f, in order to avoid contradictions. The velocity u also will obey to the Lagrangian description, and it is representing the velocity of a generic volume element dV over time, initially at position (x_0, y_0, z_0) and with initial velocity $u^0 = u(0) = const.$, $u = u(t)$. Done the solution in Lagrangian description, the solution for pressure in Eulerian description will be given explicitly (§ 4) and next a solution in function of the initial data (§ 5).

The equation (3), $\frac{dx_i}{dt} = u_i$, show us that the velocity's component u_i is dependent only of coordinate x_i, regardless of the values of others $x_j,\ j \neq i$. Based on this, we should have

$$(6) \qquad \begin{cases} \dfrac{\partial u_i}{\partial x_j} = 0,\ i \neq j, \\ \partial x_i = u_i \partial t \end{cases}$$

which will greatly simplify our problem, enabling find its exact solution in a fast way.

Following this idea, the system (1) above can be transformed in

$$(7) \qquad \begin{cases} \dfrac{1}{u_1}\dfrac{\partial p}{\partial t} + \dfrac{Du_1}{Dt} = \nu\ (\nabla^2 u_1)|_t + \frac{1}{3}\nu(\nabla_1(\nabla \cdot u))|_t + f_1|_t \\ \dfrac{1}{u_2}\dfrac{\partial p}{\partial t} + \dfrac{Du_2}{Dt} = \nu\ (\nabla^2 u_2)|_t\ + \frac{1}{3}\nu(\nabla_2(\nabla \cdot u))|_t + f_2|_t \\ \dfrac{1}{u_3}\dfrac{\partial p}{\partial t} + \dfrac{Du_3}{Dt} = \nu\ (\nabla^2 u_3)|_t + \frac{1}{3}\nu(\nabla_3(\nabla \cdot u))|_t + f_3|_t \end{cases}$$

thus (1) and (7) are equivalent systems, according validity of (5), since that the partial derivatives of the pressure and velocities were correctly transformed to the variable time, using $\partial x = u_1 \partial t,\ \partial y = u_2 \partial t,\ \partial z = u_3 \partial t$. The nabla and Laplacian operators are considered calculated in Lagrangian formulation, i.e., in the variable time. Likewise for the calculation of $\frac{Du}{Dt}$, following (5), and external force f, using $x = x(t), y = y(t), z = z(t)$. The integration of the system (7) shows that anyone of its equations can be used for solve it, and the results must be equals each other.

Then this is a condition to the occurrence of solutions. In the sequence the procedure in more details for obtaining the pressure in Lagrangian formulation, a time dependent function.

§ 2

Given $u = u(x, y, z, t) \in C^\infty(\mathbb{R}^3 \times [0, \infty))$ obeying the initial conditions and an integrable vector function f, the system's solution (1) for p, using the condensed notation given by (5), is

(8)
$$p = \int_L S \cdot dl + \theta(t),$$
$$S = \nu\, \nabla^2 u + \frac{1}{3}\nu\, \nabla(\nabla \cdot u) + f - \frac{Du}{Dt},$$

where L is any continuous path linking a point (x_0, y_0, z_0) to (x, y, z) and $\theta(t)$ is a generic time function, physically and mathematically reasonable, for example with $\theta(0) = 0$. We are supposing that the vector S is a gradient vector function $(\nabla \times S = 0,\ S = \nabla p,\ p$ potential function of $S)$.

In Eulerian description and in special case when the integrand S in (8) is a constant vector or a dependent function only on the time variable, we come to

(9)
$$p = p^0 + S_1(t)\,(x - x_0) + S_2(t)\,(y - y_0) + S_3(t)\,(z - z_0),$$
$$S_i(t) = \nu\, \nabla^2 u_i + \frac{1}{3}\nu\, \nabla_i(\nabla \cdot u) + f_i - \frac{Du_i}{Dt},$$

where $p^0 = p^0(t)$ is the pressure in the point (x_0, y_0, z_0) at time t.

When the variables x, y, z in (8) as well as f and u are in Lagrangian description, representing a motion over time of a hypothetical volume element dV or particle of fluid, we need eliminate the dependence of the position substituting in (8)

(10)
$$dl = (dx,\ dy,\ dz) = (u_1 dt,\ u_2 dt,\ u_3 dt)$$

and integrating over time. The result is

(11)
$$p(t) = p^0 + \int_0^t \sum_{i=1}^3 S_i(t)\, u_i(t)\, dt,$$

$p^0 = p(0) = const.$

This calculation can be more facilitated making $u_i \frac{Du_i}{Dt} dt = u_i du_i$ and $\int_0^t u_i \frac{Du_i}{Dt} dt = \int_{u_i^0}^{u_i} u_i du_i = \frac{1}{2}\left(u_i^2 - u_i^{0\,2}\right)$, so (11) is equal to

(12) $$p(t) = p^0 - \frac{1}{2}\sum_{i=1}^{3}\left(u_i^2 - u_i^{0\,2}\right) + \int_0^t \sum_{i=1}^{3} R_i(t)\, u_i(t)\, dt,$$
$$R_i(t) = \nu(\nabla^2 u_i)|_t + \frac{1}{3}\nu(\nabla_{\rm i}(\nabla \cdot u))|_t + f_i|_t,$$

i.e.,

(13) $$p(t) = p^0 - \frac{1}{2}(u^2 - u^{0\,2}) + \int_0^t R \cdot u\, dt,$$
$$R = \nu(\nabla^2 u)|_t + \frac{1}{3}\nu(\nabla(\nabla \cdot u))|_t + f|_t,$$

$p, p^0 \in \mathbb{R}, u, u^0, f, R \in \mathbb{R}^3, u = (u_1, u_2, u_3)(t)$, $u^0 = (u_1^0, u_2^0, u_3^0) = u(0)$, $f = (f_1, f_2, f_3)(t)$, in Lagrangian description. $u^2 = u \cdot u$ and $u^{0\,2} = u^0 \cdot u^0$ are the square modules of the respective vectors u and u^0.

When $f = 0$ and $\nu = 0$ (or most in general $R = 0$) it is simply

(14) $$p = p^0 - \frac{1}{2}(u^2 - u^{0\,2}),$$

which then can be considered an exact solution for Euler equations in Lagrangian description, and similarly to Bernoulli's law without external force (gravity, in special) and independent of a velocity's potential ϕ.

Unfortunately, in Eulerian description, neither

(15) $$p(x, y, z, t) = p^0(x, y, z) - \frac{1}{2}(u^2 - u^{0\,2}) + \int_L \; R \cdot dl,$$

$p^0(x, y, z) = p(x, y, z, 0)$, $u^0 = u^0(x, y, z) = u(x, y, z, 0)$, nor

(16) $$p(x, y, z, t) = p^0(t) - \frac{1}{2}(u^2 - u^{0\,2}) + \int_L \; R \cdot dl,$$

$p^0(t) = p(x_0, y_0, z_0, t)$, $u^0 = u^0(t) = u(x_0, y_0, z_0, t)$, solve (1) for all cases of velocities, both formulations supposing $R = \nu\nabla^2 u + \frac{1}{3}\nu\nabla(\nabla \cdot u) + f$ a gradient vector function ($\nabla \times R = 0$, $R = \nabla\phi$, ϕ potential function of R).

For example, for $R = 0$ the solution (16) is valid only when

(17) $$\frac{\partial p}{\partial x_i} = -\sum_{j=1}^{3} u_j \frac{\partial u_j}{\partial x_i} = -\left(\frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j}\right),$$

i.e.,

(18) $$\frac{\partial u_i}{\partial t} = \sum_{j=1}^{3} u_j \left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j}\right).$$

How to return to the Eulerian formulation if only was obtained a complete solution in the Lagrangian formulation? As well as we can choose any convenient velocity $u(t) = (u_1(t),\ u_2(t),\ u_3(t))$ to calculate the pressure (13) that complies with the initial conditions (Lagrangian formulation), we also can choose appropriates $u(x,y,z,t)$ (Eulerian formulation) and $x(t), y(t), z(t)$ to the velocities and positions of the system and taking the corresponding inverse functions in the obtained solution. This choose is not completely free because will be necessary to calculate a system of ordinary differential equations to obtain the correct set of $x(t), y(t), z(t)$, such that

$$(19) \qquad \begin{cases} \dfrac{dx}{dt} = u_1(x,y,z,t) \\ \dfrac{dy}{dt} = u_2(x,y,z,t) \\ \dfrac{dz}{dt} = u_3(x,y,z,t) \end{cases}$$

Nevertheless, this yet can save lots calculation time.

It will be necessary find solutions of (19) such that it is always possible to make any point (x,y,z) of the velocity domain can be achieved for each time t, introducing for this initial positions (x_0,y_0,z_0) conveniently calculated according to (19). Thus we will have velocities and pressures that, in principle, can be calculated for any position and time, independently of one another, not only for a single position for each time. For different values of (x,y,z) and t we will, in the general case, obtain the velocity and pressure of different volume elements dV, starting from different initial positions (x_0,y_0,z_0).

We can escape the need to solve (19), but admitting its validity and the corresponding existence of solution, previously choosing differentiable functions $x = x(t), y = y(t), z = z(t)$ and then calculating directly the solution for velocity in the Lagrangian formulation,

$$(20) \qquad \begin{cases} u_1(t) = \dfrac{dx}{dt} \\ u_2(t) = \dfrac{dy}{dt} \\ u_3(t) = \dfrac{dz}{dt} \end{cases}$$

hereafter calculating $\frac{\partial u_i}{\partial x_j}, \frac{\partial^2 u_i}{\partial x_j^2}$ and the differential operations $\nabla \cdot u, \nabla(\nabla \cdot u)$ and $\nabla^2 u$ through of the transformations

$$(21.1) \qquad \frac{\partial u_i}{\partial x_j} = \begin{cases} \dfrac{\partial u_i/\partial t}{\partial x_i/\partial t} = \dfrac{1}{u_i}\dfrac{\partial u_i}{\partial t},\ i = j \\ 0,\ i \neq j \end{cases}$$

$$(21.2) \qquad \nabla \cdot u = \sum_{j=1}^{3} \frac{\partial u_j}{\partial x_j} = \sum_{j=1}^{3} \frac{1}{u_j}\frac{\partial u_j}{\partial t}$$

(21.3) $$\nabla_i(\nabla \cdot u) = \frac{\partial}{\partial x_i}\left(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z}\right) = \frac{\partial}{\partial x_i}\frac{\partial u_i}{\partial x_i} = \frac{\partial/\partial t}{\partial x_i/\partial t}\frac{1}{u_i}\frac{\partial u_i}{\partial t} = \frac{1}{u_i^2}\left[-\frac{1}{u_i}\left(\frac{\partial u_i}{\partial t}\right)^2 + \frac{\partial^2 u_i}{\partial t^2}\right]$$

and

(22.1) $$\frac{\partial^2 u_i}{\partial x_j^2} = \begin{cases} \frac{1}{u_i^2}\left[-\frac{1}{u_i}\left(\frac{\partial u_i}{\partial t}\right)^2 + \frac{\partial^2 u_i}{\partial t^2}\right], & i = j \\ 0, & i \neq j \end{cases}$$

(22.2) $$\nabla^2 u_i = \frac{\partial^2 u_i}{\partial x_i^2} = \frac{1}{u_i^2}\left[-\frac{1}{u_i}\left(\frac{\partial u_i}{\partial t}\right)^2 + \frac{\partial^2 u_i}{\partial t^2}\right]$$

and finally calculating the pressure in (11) or (13), with $\frac{Du_i}{Dt} \equiv \frac{du_i}{dt}$, supposing finites the limits in equations (21) and (22) when $u_j \to 0$. Note that perhaps the denominators appearing in (21) and (22) explaining the occurrence of *blowup time* reported in the literature[3], when the limits are not finites.

Concluding, answering the question, in the result of pressure in Lagrangian formulation given by (11) or (13), conveniently transforming the initial position (x_0, y_0, z_0) as function of a generic position (x, y, z) and time t, we will have a correct value of the pressure in Eulerian formulation. The same is valid for the velocity in Lagrangian formulation, if the correspondent Eulerian formulation was not previously obtained.

§ 3

It is worth mentioning that the Navier-Stokes equations in the standard Lagrangian format, traditional one, are different than previously deduced.

Based on [5] the Navier-Stokes equations without external force and with mass density $\rho = 1$ are

(23.1) $$\frac{\partial^2 X_i}{\partial t^2} = -\sum_{j=1}^{3}\frac{\partial A_j}{\partial x_i}\frac{\partial p}{\partial a_j} + \\ + \nu \sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\left(\frac{\partial^2 A_l}{\partial x_k \partial x_k}\frac{\partial u_i}{\partial a_l} + \frac{\partial A_j}{\partial x_k}\frac{\partial A_l}{\partial x_k}\frac{\partial^2 u_i}{\partial a_j \partial a_l}\right),$$

(23.2) $$\frac{\partial A_j}{\partial x_i} \equiv \frac{\partial}{\partial x_i} X_j(x_n, t)|_{x_n = X_n(a_m, s|t)},$$

where a_m is the label given to the fluid particle at time s. Its position and velocity at time t are, respectively, $X_n(a_m, s|t)$ and $u_n(a_m, s|t)$.

The significant difference between (23) and (7) is that our pressure (7) is varying only with time, as the initial position is a constant for each particle, not variable. In (23) the pressure varies with the initial position (label) and there is a summation on the three coordinates. We did in (7) $\partial x_i = u_i \partial t$. The nabla operator has also a very difficult expression in the traditional Lagrangian formulation, a triple summation varying on positions (functions of time, evidently) and initial positions.

§ 4

Without passing through the Lagrangian formulation, given a velocity $u(x, y, z, t)$ at least two times differentiable with respect to spatial coordinates and one respect to time and an integrable external force $f(x, y, z, t)$, perhaps a better expression for the solution of the equation (1) is, in fact,

$$(24) \qquad p(x, y, z, t) = \sum_{i=1}^{3} \int_{P_i^0}^{P_i} S_i dx_i + \theta(t),$$

$$S_i = -\left(\frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j}\right) + \nu(\nabla^2 u_i) + \frac{1}{3}\nu\big(\nabla_i(\nabla \cdot u)\big) + f_i,$$

supposing possible the integrations and that the vector $S = -\left[\frac{\partial u}{\partial t} + (u \cdot \nabla)u\right] + \nu\nabla^2 u + \frac{1}{3}\nu\nabla(\nabla \cdot u) + f$ is a gradient function. This is the development of the solution (8) for the specific path L going parallelly (or perpendicularly) to axes X, Y and Z from $(x_1^0, x_2^0, x_3^0) \equiv (x_0, y_0, z_0)$ to $(x_1, x_2, x_3) \equiv (x, y, z)$, since that the solution (8) is valid for any piecewise smooth path L. We can choose $P_1^0 = (x_0, y_0, z_0)$, $P_2^0 = (x, y_0, z_0)$, $P_3^0 = (x, y, z_0)$ for the origin points and $P_1 = (x, y_0, z_0)$, $P_2 = (x, y, z_0)$, $P_3 = (x, y, z)$ for the destination points. $\theta(t)$ is a generic time function, physically and mathematically reasonable, for example with $\theta(0) = 0$ or adjustable for some given condition. Again we have seen that the system of Navier-Stokes equations has no unique solution, only given initial conditions, supposing that there is some solution. We can choose different velocities that have the same initial velocity and also result, in general, in different pressures.

The remark given for system (7), when used here, leads us to the following conclusion: the integration of the system (1), confronting with (7), shows that anyone of its equations can be used for solve it, and the results must be equals each other. Then again this is a condition to the occurrence of solutions, which shows to us the possibility of existence of breakdown solutions, as will become clearer in §6.

By other side, using the first condition (6), $\frac{\partial u_i}{\partial x_j}=0$ if $i\neq j$, due to Lagrangian formulation, where $u_i=\frac{dx_i}{dt}$, the original system (1) is simplified as

$$(25)\quad \begin{cases} \frac{\partial p}{\partial x}+\frac{\partial u_1}{\partial t}+u_1\frac{\partial u_1}{\partial x}=\frac{4}{3}\nu\frac{\partial^2 u_1}{\partial x^2}+f_1 \\ \frac{\partial p}{\partial y}+\frac{\partial u_2}{\partial t}+u_2\frac{\partial u_2}{\partial y}=\frac{4}{3}\nu\frac{\partial^2 u_2}{\partial y^2}+f_2 \\ \frac{\partial p}{\partial z}+\frac{\partial u_3}{\partial t}+u_3\frac{\partial u_3}{\partial z}=\frac{4}{3}\nu\frac{\partial^2 u_3}{\partial z^2}+f_3 \end{cases}$$

where u_i is a function only of the respective x_i and t, but not x_j if $j\neq i$. When is required the incompressibility condition, $\nabla\cdot u=\left(\frac{\partial u_1}{\partial x}+\frac{\partial u_2}{\partial y}+\frac{\partial u_3}{\partial z}\right)=0$, then the constant $\frac{4}{3}$ in (25) should be replaced by 1.

If the external force has potential, $f=\nabla V$, then the system (25) has solution

$$(26)\qquad p=\sum_{i=1}^{3}\int_{P_i^0}^{P_i}\left[-\left(\frac{\partial u_i}{\partial t}+u_i\frac{\partial u_i}{\partial x_i}\right)+\frac{4}{3}\nu\frac{\partial^2 u_i}{\partial x_i^2}+f_i\right]dx_i+\theta(t)$$
$$=V+\sum_{i=1}^{3}\int_{x_i^0}^{x_i}\left[-\left(\frac{\partial u_i}{\partial t}+u_i\frac{\partial u_i}{\partial x_i}\right)+\frac{4}{3}\nu\frac{\partial^2 u_i}{\partial x_i^2}\right]dx_i+\theta(t),$$

$V=\int_L f\cdot dl$, which although similar to (24) has the solubility guaranteed by the special functional dependence of the components of the vector u, i.e., $u_i=u_i(x_i,t)$, with $\frac{\partial u_i}{\partial x_j}=0$ if $i\neq j$, supposing u, its derivatives and f integrable vectors. In this case the vector S described in (24) is always a gradient function. Note that if f is not an irrotational or gradient vector, i.e., if it does not have a potential, then the system (25), with $u_i=u_i(x_i,t)$, it has no solution, the case of "breakdown" solution in [3].

When the incompressibility condition is imposed we have, using (6), a small variety of possible solutions for velocity, of the form

$$(27)\qquad u_i(x_i,t)=\alpha_i(t)x_i+\beta_i(t),$$

$\alpha_i,\beta_i\in C^\infty([0,\infty[)$. In this case is valid $\nabla^2 u=0$, i.e., the system of equations has a solution for velocity independent of viscosity coefficient, equal to Euler equations, and except when $u=0$ (for some or all $t\geq 0$) we have always $\int_{\mathbb{R}^3}|u|^2\,dxdydz\to\infty$, the occurrence of unbounded or unlimited energy, what is not difficult to see.

§ 5

Another way to solve (1) with $f = 0$ seems to me to be the best of all, for its extreme ease of calculation, also without we need to resort to Lagrangian formulation and its conceptual difficulties. If $u(x,y,z,0) = u^0(x,y,z)$ is the initial velocity of the system, valid solution in $t = 0$, then $u(x,y,z,t) = u^0(x+t, y+t, z+t)$ is a solution for velocity in $t \geq 0$, a non-unique solution, where specifically there is the additional initial condition

$$(28) \qquad \frac{\partial u_i}{\partial t}\Big|_{t=0} = \sum_{j=1}^{3} \frac{\partial u_i^0}{\partial x_j}.$$

Similarly, $p(x,y,z,t) = p^0(x+t, y+t, z+t)$ is the correspondent solution for pressure in $t \geq 0$, being $p^0(x,y,z)$ the initial condition for pressure. See reference [6] for a proof of this theorem.

The velocities $u^0(x+t,y,z)$, $u^0(x,y+t,z)$ and $u^0(x,y,z+t)$ are also solutions, and respectively also the pressures $p^0(x+t,y,z)$, $p^0(x,y+t,z)$ and $p^0(x,y,z+t)$, each one with its respective additional initial condition

$$(29) \qquad \frac{\partial u_i}{\partial t}\Big|_{t=0} = \frac{\partial u_i^0}{\partial x} \quad \text{or} \quad \frac{\partial u_i}{\partial t}\Big|_{t=0} = \frac{\partial u_i^0}{\partial y} \quad \text{or} \quad \frac{\partial u_i}{\partial t}\Big|_{t=0} = \frac{\partial u_i^0}{\partial z},$$

whose proof requires only a small adaptation of [6], for the particular index which occurs the transformation $x_j \longmapsto x_j + t$.

Other solutions may be searched, without external force, for example in the kind $u(x,y,z,t) = u^0\big(x + T_1(t),\ y + T_2(t),\ z + T_3(t)\big)$, $T_i(0) = 0$, and therefore $p(x,y,z,t) = p^0\big(x + T_1(t),\ y + T_2(t),\ z + T_3(t)\big)$, supposing $T_i(t)$ smooth.

§ 6

This article would not be complete without mentioning the potential flows. When there is a potential function ϕ such that $u = \nabla\phi$ then $\nabla \times u = 0$, i.e., the velocity is an irrotational field. When the incompressibility condition is required, i.e., $\nabla \cdot u = 0$, the velocity is solenoidal, and if the field is also irrotational then $\nabla^2 u = \nabla(\nabla \cdot u) - \nabla \times (\nabla \times u) = 0$, i.e., the Navier-Stokes equations are reduced to Euler's equations and the velocity-potential ϕ must satisfied the Laplace's equation, $\nabla^2\phi = 0$, as well as the velocity.

According Courant[7] (p.241), for $n = 2$ the "general solution" of the potential equation (or Laplace's equation) is the real part of any analytic function of the complex variable $x + iy$. For $n = 3$ one can also easily obtain solutions which depend on arbitrary functions. For example, let $f(w,t)$ be analytic in the complex variable w for fixed real t. Then, for arbitrary values of t, both the real and imaginary parts of the function

$$u = f(z + ix\cos t + iy\sin t\,, t) \tag{30}$$

of the real variables x, y, z are solutions of the equation $\nabla^2 u = 0$. Further solutions may be obtained by superposition:

$$u = \int_a^b f(z + ix\cos t + iy\sin t\,, t)dt. \tag{31}$$

For example, if we set

$$f(w, t) = w^n e^{iht}, \tag{32}$$

where n and h are integers, and integrate from $-\pi$ to $+\pi$, we get homogeneous polynomials

$$u = \int_{-\pi}^{\pi}(z + ix\cos t + iy\sin t)^n\, e^{iht}dt \tag{33}$$

in x, y, z, following example given by Courant. Introducing polar coordinates $z = r\cos\theta, x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi$, we obtain

$$\begin{aligned} u &= 2r^n e^{ih\phi}\int_0^{\pi}(\cos\theta + i\sin\theta\cos t)^n \cos ht\;\; dt \\ &= r^n e^{ih\phi} P_{n,h}(\cos\theta), \end{aligned} \tag{34}$$

where $P_{n,h}(\cos\theta)$ are the associated Legendre functions.

On the other hand, according Tokaty[8], Lagrange[9] came to the conclusion that Euler´s equations could be solved only for two specific conditions: (1) for potential (irrotational) flows, and (2) for non-potential (rotational) but steady flows. The external force in [9] is considered with potential, $f = \nabla V$, and the fluid is incompressible.

Lagrange also proved, as well as Laplace (*Mécanique Céleste*), Poisson (*Traité de Méchanique*), Cauchy (*Mémoire sur la Théorie des Ondes*) and Stokes (*On the Friction of Fluids in Motion and the Equilibrium and Motion of Elastic Solids*), that if the differential of the fluid's velocity $u_1dx + u_2dy + u_3dz$ is a differential exact in some instant of time (for example, in $t = 0$) then it is also for all time ($t \geq 0$) of this movement on the same conditions. This means that a potential flow is always potential flow, since $t = 0$. Then, from the previous paragraph, if the initial velocity have not an exact differential (i.e., if the initial velocity is not a gradient function, irrotational, with potential) and the external force have potential then the Euler equations have no solution in this case of incompressible and potential flows, for non-steady flows.

And what happens with respect to Navier-Stokes equations in incompressible case, which is the major problem?

For stationary (say, steady) flows, where $\frac{\partial u}{\partial t} \equiv 0$ and $u = u^0$ for all $t \geq 0$, the condition for existence of solution (obtaining the pressure) is that

(35) $$\frac{\partial S_i}{\partial x_j} = \frac{\partial S_j}{\partial x_i}$$

for all pair (i, j), $1 \leq i, j \leq 3$, defining

(36) $$S_i = \nu \nabla^2 u_i^0 - \sum_{j=1}^{3} u_j^0 \frac{\partial u_i^0}{\partial x_j} + f_i,$$

where $f \equiv f^0$ is the stationary external force. This is a common condition for existence of solution for a system $\nabla p = S$, representing the stationary Navier-Stokes equations, that is $\nabla \times S = 0$.

For non-stationary flows it is knows that the Lagrange's theorem, as well as the Kelvin's circulation theorem, is not valid for Navier-Stokes equations, but here it is implied that $\nu \nabla^2 u \neq 0$, the general case.

The vorticity $\omega = \nabla \times u \neq 0$ is generated at solid boundaries[10], thus without boundaries $(\Omega = \mathbb{R}^3)$ no generation of vorticity, and without vorticity there is potential flow and vanishes the Laplacian of velocity if $\nabla \cdot u = 0$, then it is possible again the validity of Lagrange's theorem in an unlimited domain without boundaries and with both smooth and irrotational initial velocities and external forces, for incompressible fluids.

Regardless, the general condition for existence of solution for pressure in $t = 0$ is (35), for all pair (i, j), $1 \leq i, j \leq 3$, substituting (36) by

(37) $$S_i = \nu \nabla^2 u_i^0 - \left(\frac{\partial u_i}{\partial t} \Big|_{t=0} + \sum_{j=1}^{3} u_j^0 \frac{\partial u_i^0}{\partial x_j} \right) + f_i^0,$$

$f_i^0 = f_i(x_1, x_2, x_3, 0)$.

It is important the application of (35), (36) and (37) in Turbulence Theory and Theory of Perturbations. According Landau & Lifshitz[1], in the chapter III (Turbulence), article § 26 (Stability of steady flow), of his famous book on Fluids Mechanics,

For any problem of viscous flow under given steady conditions there must in principle exist an exact steady solution of the equations of fluid dynamics. These solutions formally exist for all Reynolds numbers. Yet not every solution of the equations of motion, even if it is exact, can actually occur in Nature. Those which do must not only the equations of fluid dynamics, but also be stable. Any small *perturbations which arise must decrease in the course of time. If, on the contrary, the small perturbations which inevitably occur in the flow tend to increase with time, the flow is unstable and cannot actually exist.*

But it is not true that any given initial velocity, yet small in module, or large velocity, is according with relation (35), or further approximations following perturbative methods, with S_i given by (36) or (37), and for this reason we sometimes (or yet oftentimes) cannot obtain the necessary solution to the pressure or else we obtain a wrong solution. The same is said about the Numerical Methods in Computational Fluid Dynamics. For any time $t \geq 0$ need be valid the relation (35) with

(38) $$S_i = \nu \nabla^2 u_i - \left(\frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j} \right) + f_i.$$

§ 7

Apply some of these methods to the famous 6th Millenium Problem[3] on existence and smoothness of the Navier-Stokes equations apparently is not so difficult at the same time also it is not absolutely trivial. It takes some time. I hope to do it soon. On the other hand, apply these methods to the case $n = 2$ or $\nu = 0$ (Euler equation) is almost immediate.

In special, we saw that even if it were widely free choose a movement for a fluid's particle, following the Lagrangian description, it is very restrictive the correspondent velocity in Eulerian description, principally if the condition of incompressibility is required. Except when the velocity is equal to zero (for some or all $t \geq 0$), there is always the occurrence of unlimited energy involving the whole space. Thus we realize that it is possible to exist velocities in the Eulerian formulation that do not correspond to a real movement of particles of a fluid, according to the Lagrangian formulation.

I think that this is better than nothing... It is no longer true that the Navier-Stokes and Euler equations do not have a general solution (when there is some).

To Leonard Euler, in memorian,
the greatest mathematician of all time.
309th anniversary of his birth,
April-15-1707-2016.

Last update: April-03-2017.

Euler, and mathematical community,
forgive me for my mistakes...
This subject is very difficult!

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