

# Generalized Uncertainty Relations, Curved Phase-Spaces and Quantum Gravity

Carlos Castro

Center for Theoretical Studies of Physical Systems  
Clark Atlanta University, Atlanta, Georgia. 30314, perelmanc@hotmail.com

April 2016

## Abstract

Modifications of the Weyl-Heisenberg algebra  $[\mathbf{x}^i, \mathbf{p}^j] = i\hbar g^{ij}(\mathbf{p})$  are proposed where the classical limit  $g_{ij}(p)$  corresponds to a metric in (curved) momentum spaces. In the simplest scenario, the  $2D$  de Sitter metric of constant curvature in momentum space furnishes a hierarchy of modified uncertainty relations leading to a minimum value for the position uncertainty  $\Delta x$ . The first uncertainty relation of this hierarchy has the same functional form as the *stringy* modified uncertainty relation with a Planck scale minimum value for  $\Delta x = L_P$  at  $\Delta p = p_{Planck}$ . We proceed with a discussion of the most general curved phase space scenario (cotangent bundle of spacetime) and provide the noncommuting phase space coordinates algebra in terms of the symmetric  $g_{(\mu\nu)}$  and nonsymmetric  $g_{[\mu\nu]}$  metric components of a Hermitian complex metric  $g_{\mu\nu} = g_{(\mu\nu)} + ig_{[\mu\nu]}$ , such  $g_{\mu\nu} = (g_{\nu\mu})^*$ . Yang's noncommuting phase-space coordinates algebra, combined with the Schrodinger-Robertson inequalities involving angular momentum eigenstates, reveals how a quantized area operator in units of  $L_P^2$  emerges like it occurs in Loop Quantum Gravity (LQG). Some final comments are made about Fedosov deformation quantization, Noncommutative and Nonassociative gravity.

Keywords : Uncertainty Relations, Gravity, Finsler Geometry, Born Reciprocity, Phase Space.

Recently, we studied the generalized gravitational field equations in curved phase spaces (the cotangent bundle of spacetime) [1]. A nontrivial solution generalizing the Hilbert-Schwarzschild black hole metric in spacetime was found. The most relevant physical consequence is that the metric becomes momentum-dependent (observer dependent) which is what one should aim for in trying to *quantize* geometry (gravity) : the observer must play an important role in any measurement (observation) process of the spacetime he/she lives in.

Most of the work devoted to Quantum Gravity has been focused on the geometry of spacetime rather than phase space per se. The first indication that phase space should play a role in Quantum Gravity was raised by [3]. The principle of Born's reciprocal relativity [3] was proposed long ago based on the idea that coordinates and momenta should be unified on the same footing, and consequently, if there is a limiting speed (temporal derivative of the position coordinates) in Nature there should be a maximal force as well, since force is the temporal derivative of the momentum. A *maximal* speed limit (speed of light) must be accompanied with a *maximal* proper force (which is also compatible with a *maximal* and *minimal* length duality).

It is better understood now that the Planck-scale modifications of the particle dispersion relations can be encoded in the nontrivial geometrical properties of momentum space [9]. When both spacetime curvature and Planck-scale deformations of momentum space are present, it is expected that the nontrivial geometry of momentum space and spacetime get intertwined. The interplay between spacetime curvature and non-trivial momentum space effects was essential in the notion of "relative locality" and in the deepening of the relativity principle [9]. Recently the authors [10] described the Hamilton geometry of the phase space of particles whose motion is characterized by general dispersion relations. Explicit examples of two models for Planck-scale modified dispersion relations, inspired from the  $q$ -de Sitter and  $\kappa$ -Poincare quantum groups, were considered. In the first case they found the expressions for the momentum and position dependent curvature of spacetime and momentum space, while for the second case the manifold is flat and only the momentum space possesses a nonzero, momentum dependent curvature.

We shall focus in this work on two main points. Firstly, on solutions to the field equations in *momentum* space with the inclusion of the momentum analog of a cosmologically constant  $\tilde{\Lambda}$

$$R_{\mu\nu}(p) - \frac{1}{2} g_{\mu\nu}(p) R + \tilde{\Lambda} g_{\mu\nu}(p) = 0, \quad \mu, \nu = 1, 2, \dots, D \quad (1)$$

the solutions to the above field equations will be used in the modified uncertainty relations. The momentum-space analog  $\tilde{\Lambda}$  of the cosmological constant should *not* be confused with the spacetime one.

Secondly, on the rotationally invariant commutator of the form [11]

$$[\mathbf{x}_i, \mathbf{p}_j] = i\hbar (f(\mathbf{p}^2) \delta_{ij} + g(\mathbf{p}^2) \mathbf{p}_i \mathbf{p}_j), \quad \mathbf{p}^2 = \mathbf{p}_i \mathbf{p}^i, \quad i, j = 1, 2, \dots, D-1 \quad (2)$$

one can see that under rotations

$$\mathbf{x}_i \rightarrow \mathbf{x}'_i = M_i^k \mathbf{x}_k, \quad \mathbf{p}_j \rightarrow \mathbf{p}'_j = M_j^l \mathbf{p}_l, \quad \mathbf{p}_i \rightarrow \mathbf{p}'_i = M_i^k \mathbf{p}_k \quad (3)$$

the left and right hand side of eq-(2) become

$$[\mathbf{x}'_i, \mathbf{p}'_j] = i\hbar (f(\mathbf{p}'^2) \delta_{ij} + g(\mathbf{p}'^2) \mathbf{p}'_i \mathbf{p}'_j), \quad \mathbf{p}^2 = \mathbf{p}_i \mathbf{p}^i = \mathbf{p}'^2 = \mathbf{p}'_i \mathbf{p}'^i \quad (4)$$

and the commutator relations remain invariant. Consequently, if one is to set  $[\mathbf{x}_i, \mathbf{p}_j] = i\hbar g_{ij}(\mathbf{p}_k)$ , a rotationally invariant commutator can be associated to a classical momentum space metric of the form

$$(ds)_{(p)}^2 = - h(p^2) (dE)^2 + ( f(p^2) \delta_{ij} + g(p^2) p_i p_j ) dp^i dp^j \quad (5)$$

A close inspection reveals that the  $4D$  momentum-space metric analog of the de Sitter metric in a  $4D$  spacetime (written in static coordinates and using the momentum-space analog of the cosmological constant  $\tilde{\Lambda}$ )

$$(ds)_{(p)}^2 = - ( 1 - \frac{\tilde{\Lambda}}{3} p_r^2 ) (dE)^2 + \frac{(dp_r)^2}{1 - \frac{\tilde{\Lambda}}{3} p_r^2} + p_r^2 (d\Omega)_{(p)}^2 \quad (6)$$

does *not* have the required form indicated by eq-(5). To verify this one simply rewrites the de Sitter metric in Cartesian coordinates. One then finds that the rotationally invariant commutation relations, leading to the metrics (5), are *not* compatible with a spherically symmetric momentum space de Sitter metric (6).

One may insert the metric (5) into the field equations in momentum space in order to determine whether or not there exist actual functions  $f(p^2), g(p^2), h(p^2)$  which solve the field equations (1). However, for our purposes it is *not* necessary to do so, and it is much simpler just to write down the momentum space analog of the de Sitter metric in  $2D$  in natural units  $\hbar = c = 1$

$$(ds)_{(p)}^2 = - ( 1 - L^2 p^2 ) (dE)^2 + ( 1 - L^2 p^2 )^{-1} (dp)^2, \quad \hbar = c = 1 \quad (7)$$

which is trivially rotational invariant.  $\tilde{\Lambda} = 0$  in  $2D$ . There is a cosmological *horizon* in momentum space when  $L^2 p_h^2 = 1 \Rightarrow p_h = \frac{1}{L}$ . We shall choose the length scale  $L = L_p$  to coincide with the Planck length  $L_P$  so that the momentum horizon  $p_h = p_{Planck}$  is the Planck momentum.

Inspired by the  $2D$  de Sitter momentum space metric (7), and by *promoting* the classical momentum variable  $p$  to an operator  $\mathbf{p}$ , such that  $g_{11}(p^2) \rightarrow g_{11}(\mathbf{p}^2)$ , the Schrodinger-Robertson inequality yields the modified uncertainty relations after performing a series expansion (with  $\hbar = c = 1$ )

$$\begin{aligned} \Delta x \Delta p &\geq \frac{1}{2} | \langle \Psi | [ \mathbf{x}, \mathbf{p} ] | \Psi \rangle | = \\ \frac{1}{2} | \langle \Psi | i g_{11}(\mathbf{p}^2) | \Psi \rangle | &= \frac{1}{2} | \langle \Psi | \frac{1}{1 - L^2 \mathbf{p}^2} | \Psi \rangle | = \\ \frac{1}{2} | \langle \Psi | 1 + L^2 \mathbf{p}^2 + (L^2 \mathbf{p}^2)^2 + (L^2 \mathbf{p}^2)^3 + \dots | \Psi \rangle | & \quad (8) \end{aligned}$$

One may notice that since  $\langle \Psi | L^{2n} \mathbf{p}^{2n} | \Psi \rangle \geq 0$  for  $n \geq 0$ , given a self-adjoint (Hermitian) momentum operator  $\mathbf{p}$ , one may drop the absolute value symbol in last terms of eq-(8). The (geometric) series expansion in  $\mathbf{p}^2$  converges as an operator if

$$L^2 \mathbf{p}^2 < \mathbf{1} \Rightarrow \langle \Psi | \mathbf{p}^2 | \Psi \rangle < \frac{1}{L^2} = p_h^2 \quad (9)$$

consistent with the cosmological momentum-horizon  $p_h$  being an ultraviolet cutoff value for the momentum. The unit operator is  $\mathbf{1}$  and the states are normalized to unity  $\langle \Psi | \Psi \rangle = 1$  so that  $\langle \Psi | \mathbf{1} | \Psi \rangle = 1$ .

Inserting the inequality of the equation below

$$(\Delta p)^2 \equiv \langle \Psi | \mathbf{p}^2 | \Psi \rangle - \langle \Psi | \mathbf{p} | \Psi \rangle^2 \Rightarrow \langle \Psi | \mathbf{p}^2 | \Psi \rangle \geq (\Delta p)^2 \quad (10)$$

into eq-(8), yields to *leading* order in  $L^2$ , a *modified* uncertainty relation

$$\begin{aligned} \Delta x \Delta p &> \frac{1}{2} \langle \Psi | (1 + L^2 \mathbf{p}^2) | \Psi \rangle \geq \frac{1}{2} (1 + (L^2 \Delta p)^2) \Rightarrow \\ \Delta x &> \frac{1}{2} \left( \frac{1}{\Delta p} + L^2 \Delta p \right), \quad \hbar = c = 1 \end{aligned} \quad (11)$$

which has the *same* functional form as the *stringy* modified uncertainty relations [2], with the main difference being that now one has the cosmological momentum-horizon  $p_h = \frac{1}{L}$  as an ultraviolet cutoff for  $\Delta p$ , and there is an strict inequality in eq-(11).

The *minimum* value for the position uncertainty is  $(\Delta x)_{min} = L$  at  $\Delta p = \frac{1}{L} = p_h$  and which coincides with the location of the cosmological momentum horizon. If one equates the *minimum* value of the position uncertainty to the Planck scale length it gives  $(\Delta x)_{min} = L = L_P$ , and which is consistent with the fact that we chose the length scale  $L$  to coincide with the Planck length  $L_P$ .

To sum up, to leading order in  $L$ , the de Sitter momentum space metric in 2D furnishes : (i) a cosmological momentum-horizon  $p_h = \frac{1}{L_P} = p_{Planck}$  as an ultraviolet cutoff; (ii) a Planck scale minimal length uncertainty for the position coordinate  $(\Delta x)_{min} = L_P$  at  $\Delta p = p_{Planck}$ .

The next-to-leading order term can be obtained after using the inequality

$$\langle \Psi | \mathbf{p}^4 | \Psi \rangle \geq (\Delta p)^4 \quad (12a)$$

that simply follows from

$$(\Delta A)^2 \equiv \langle \Psi | (A - \bar{A})^2 | \Psi \rangle \geq 0, \quad \bar{A} \equiv \langle \Psi | A | \Psi \rangle \quad (12b)$$

after replacing  $A \rightarrow \mathbf{p}^2$  and recurring to  $\langle \Psi | \mathbf{p}^2 | \Psi \rangle \geq (\Delta p)^2$ . Upon doing so one obtains another modified uncertainty relation given by

$$\Delta x \Delta p > \frac{1}{2} (1 + L_P^2 (\Delta p)^2 + L_P^4 (\Delta p)^4) \quad (12c)$$

The minimum position uncertainty now turns out to be  $(\Delta x)_{min} > L_P$  at

$$(\Delta p)_* = \left( \frac{-1 + \sqrt{13}}{6} \right)^{\frac{1}{2}} p_{Planck} < p_{Planck} \quad (12d)$$

The value of  $(\Delta p)_*$  lies between  $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}) p_{Planck}$ . Repeating the procedure based on eqs-(12a, 12b), by a process of successive squaring, a hierarchy of modified uncertainty relations of the form are derived

$$\Delta x \Delta p > \frac{1}{2} \left( 1 + L_P^2 (\Delta p)^2 + L_P^4 (\Delta p)^4 + \dots + L_P^{2k} (\Delta p)^{2k} \right), \quad k = 1, 2, \dots \quad (12d)$$

The most salient feature of the modified uncertainty relations (11,12c,12d) is that there is a *minimum* value for the position uncertainty  $\Delta x$ . The Scale Relativity Theory [8] is based on the postulate that the Planck scale is the minimum length resolution. Generalized uncertainty relations in spacetime were derived from the Extended Relativity Theory in Clifford spaces (*C*-spaces) in [13]. Such Clifford space Extended Relativity Theory has two universal parameters : the speed of light and the Planck length.

In general one can postulate the following modification of the Weyl-Heisenberg algebra

$$[\mathbf{x}_i, \mathbf{p}_j] = i\hbar g_{ij}(\mathbf{p}) \quad (13)$$

combined with the additional commutation relations

$$[\mathbf{x}_i, \mathbf{x}_j] \neq 0, \quad [\mathbf{p}_i, \mathbf{p}_j] = 0 \quad (14)$$

with the provision that the above commutators obey the Jacobi identities [11]. A nonvanishing  $[\mathbf{x}_i, \mathbf{x}_j] \neq 0$  is compatible with a *curved momentum space*. The de Sitter momentum space metric yields a constant scalar curvature  $2/p_{Planck}^2 = 2L_P^2$  in momentum space. The vanishing  $[\mathbf{p}_i, \mathbf{p}_j] = 0$  commutator is consistent with a flat spacetime. A nontrivial problem is to find the most general solutions to the field equations in momentum space (with and without the  $\tilde{\Lambda}$ ) for the metric which has the form of eq-(5), in order to yield rotationally symmetric commutators  $[\mathbf{x}_i, \mathbf{p}_j]$ , after promoting the classical momentum variables  $p_i, p_j$  to self-adjoint operators  $\mathbf{p}_i, \mathbf{p}_j$ . When the momentum space is still commutative, one can find a Hilbert space representation in the spectral representation of the momentum operator [11]. The states  $|\Psi(p)\rangle$  were analyzed in detail by [11]. Upon performing the expectation values  $\langle \Psi(p) | g_{ij}(\mathbf{p}) | \Psi(p) \rangle = g_{ij}(p)$  one recovers the classical metric in momentum space. These momentum eigenstates have for momentum uncertainty  $\Delta p = 0$  so that  $\Delta x = \infty$ , as expected.

The more general commutator than the one in eq-(13)

$$[\mathbf{x}_i, \mathbf{p}_j] = i\hbar g_{ij}(\mathbf{x}, \mathbf{p}) \quad (15)$$

may be chosen such that the classical limit  $g_{ij}(\mathbf{x}, \mathbf{p}) \rightarrow g_{ij}(x, p)$  (involving  $c$  variables) furnishes a classical phase space metric obeying the full-fledged gravitational field equations in curved phase spaces (cotangent bundle of spacetime). This poses more difficulties due to the *ordering* ambiguities of the  $\mathbf{x}, \mathbf{p}$  operators inside  $g_{ij}(\mathbf{x}, \mathbf{p})$ . One can solve this ordering ambiguity by performing a Weyl ordering procedure, like  $\mathbf{x}\mathbf{p} \rightarrow \frac{1}{2}(\mathbf{x}\mathbf{p} + \mathbf{p}\mathbf{x})$  to ensure that the latter ordering is Hermitian, since the product  $\mathbf{x}\mathbf{p}$  is not.

An important remark is in order. By Hermitian metric one usually means  $g_{ij} = g_{ji}^*$ . This should *not* be confused with performing the Hermitian (adjoint) operation to each one of the entries inside the metric matrix  $g_{ij}(\mathbf{x}, \mathbf{p})$ . If the entries of the metric matrix are given by polynomials in the operators  $\mathbf{x}, \mathbf{p}$ , the Weyl ordering procedure will ensure that each one of the entries of the metric matrix will remain Hermitian. For example if  $g_{ij}(\mathbf{x}, \mathbf{p}) = g_{ij}(\mathbf{x}\mathbf{p})$ , a Weyl ordering yields  $g_{ij}(\frac{1}{2}(\mathbf{x}\mathbf{p} + \mathbf{p}\mathbf{x}))$  ensuring that the argument of the metric matrix is Hermitian. Similarly, by anti-Hermitian metric one usually means  $g_{ij} = -g_{ji}^*$ .

Since the commutator of two Hermitian operators is anti-Hermitian, one may postulate the following commutators below (in a fully relativistic phase space) given in terms of a *real* metric which has both symmetric  $g_{(\mu\nu)}$  and anti-symmetric components  $g_{[\mu\nu]}$  as follows ( $\hbar = c = 1$ )

$$[\mathbf{x}_\mu, \mathbf{p}_\nu] = i ( A g_{(\mu\nu)}(\mathbf{x}, \mathbf{p}) + B g_{[\mu\nu]}(\mathbf{x}, \mathbf{p}) ) \mathbf{1} \quad (16)$$

$A, B$  are real numerical coefficients.

$$[\mathbf{x}_\mu, \mathbf{x}_\nu] = i L_P^2 g_{[\mu\nu]}(\mathbf{x}, \mathbf{p}) \mathbf{1} \quad (17)$$

$$[\mathbf{p}_\mu, \mathbf{p}_\nu] = i (R_H)^{-2} g_{[\mu\nu]}(\mathbf{x}, \mathbf{p}) \mathbf{1} \quad (18)$$

the right hand sides are anti-Hermitian due to  $(i \mathbf{1})^\dagger = -i \mathbf{1}$ . A Taylor expansion of the metric components in powers of  $\mathbf{x}, \mathbf{p}$  must be followed by a Weyl ordering of *all* the  $\mathbf{x}, \mathbf{p}$  variables to ensure Hermiticity of the arguments of the metric. An UV (ultraviolet) cutoff is given by the Planck scale  $L_P$ ; an IR (infrared) cutoff is given by  $R_H$  (Hubble radius). The Jacobi identities will impose very strong constraints on the functional form of  $g_{(\mu\nu)}$  and  $g_{[\mu\nu]}$ . A complex Hermitian metric can be introduced by writing  $g_{\mu\nu} = g_{(\mu\nu)} + i g_{[\mu\nu]}$  such that  $g_{\mu\nu} = g_{\nu\mu}^*$ . This raises the possibility that complex Hermitian metrics might be relevant in Quantum Gravity.

It is at this point where the following Schrodinger-Robertson inequalities for  $2n$  observables  $A_1, A_2, \dots, A_{2n}$  will play an important role. They are given by the inequality of the determinants below involving the covariance  $\Sigma$  and commutator  $\mathbf{C}$  matrices [17]

$$\det \Sigma \geq \det \mathbf{C}, \quad \Sigma_{kl} = \text{cov}(A_k, A_l), \quad C_{kl} = -\frac{i}{2} \langle \Psi | [A_k, A_l] | \Psi \rangle \quad (19)$$

the covariance is defined as

$$\text{cov}(A_k, A_l) = \frac{1}{2} (\langle \Psi | A_k A_l | \Psi \rangle + \langle \Psi | A_l A_k | \Psi \rangle) - \langle \Psi | A_k | \Psi \rangle \langle \Psi | A_l | \Psi \rangle \quad (20)$$

uncorrelated variables have zero covariance. The uncertainty squared is  $(\Delta A)^2 = \text{cov}(A, A)$ .

For the  $2n$  phase space coordinates, the  $\Sigma$  and  $\mathbf{C}$  matrices are respectively given by

$$\Sigma = \begin{pmatrix} \text{cov}(\mathbf{x}_\mu, \mathbf{x}_\nu) & \text{cov}(\mathbf{x}_\mu, \mathbf{p}_\nu) \\ \text{cov}(\mathbf{p}_\mu, \mathbf{x}_\nu) & \text{cov}(\mathbf{p}_\mu, \mathbf{p}_\nu) \end{pmatrix} \quad (21)$$

$$\mathbf{C} = -\frac{i}{2} \begin{pmatrix} \langle \Psi | [\mathbf{x}_\mu, \mathbf{x}_\nu] | \Psi \rangle & \langle \Psi | [\mathbf{x}_\mu, \mathbf{p}_\nu] | \Psi \rangle \\ \langle \Psi | [\mathbf{p}_\mu, \mathbf{x}_\nu] | \Psi \rangle & \langle \Psi | [\mathbf{p}_\mu, \mathbf{p}_\nu] | \Psi \rangle \end{pmatrix} \quad (22)$$

Due to the nontrivial commutation relations (16-18), the Schrodinger-Robertson inequalities  $\det \Sigma \geq \det \mathbf{C}$  will lead to very complicated uncertainty relations. Furthermore, because the phase space coordinates are noncommutative [12], one must deal now with Noncommutative Quantum Mechanics; i.e. Quantum Mechanics on Noncommutative spacetimes which is the realm of Hopf algebras and Quantum Groups.

Closely related to the nontrivial commutation relations (16-18) is Yang's algebra in an  $8D$  Noncommutative phase space [14]

$$[\mathbf{x}_\mu, \mathbf{p}_\nu] = -i \frac{L_P}{R_H} \eta_{\mu\nu} \mathbf{J}_{56}, \quad \mathbf{J}_{56} = -\mathbf{J}_{65} \quad (23)$$

$$[\mathbf{x}_\mu, \mathbf{x}_\nu] = -i L_P^2 \eta_{55} \mathbf{J}_{\mu\nu}, \quad \mathbf{J}_{\mu\nu} = -\mathbf{J}_{\nu\mu} \quad (24)$$

$$[\mathbf{p}_\mu, \mathbf{p}_\nu] = -i (R_H)^{-2} \eta_{66} \mathbf{J}_{\mu\nu} \quad (25)$$

$$[\mathbf{J}_{\mu\nu}, \mathbf{x}_\rho] = i (\eta_{\nu\rho} \mathbf{x}_\mu - \eta_{\mu\rho} \mathbf{x}_\nu) \quad (26)$$

$$[\mathbf{J}_{\mu\nu}, \mathbf{p}_\rho] = i (\eta_{\nu\rho} \mathbf{p}_\mu - \eta_{\mu\rho} \mathbf{p}_\nu) \quad (27)$$

$$[\mathbf{J}_{56}, \mathbf{x}_\mu] = -i L_P R_H \eta_{55} \mathbf{p}_\mu, \quad [\mathbf{J}_{56}, \mathbf{p}_\mu] = -i (L_P R_H)^{-1} \eta_{66} \mathbf{x}_\mu \quad (28)$$

$$[\mathbf{J}_{\mu\nu}, \mathbf{J}_{56}] = 0, \quad [\mathbf{J}_{\mu\nu}, \mathbf{J}_{\rho\sigma}] = i \eta_{\nu\rho} \mathbf{J}_{\mu\sigma} \pm \dots \quad (29)$$

Yang's algebra can be obtained simply by replacing

$$\mathbf{x}_\mu \rightarrow L_P \mathbf{J}_{\mu 5}; \quad \mathbf{p}_\mu \rightarrow (R_H)^{-1} \mathbf{J}_{\mu 6}, \quad \mu, \nu = 1, 2, 3, 4 \quad (30)$$

and recurring to the angular momentum algebra in  $6D$ . The Jacobi identities are satisfied because the angular momentum algebra in  $6D$  obeys them. The noncommuting coordinates and momenta are just rotations/boosts involving the extra directions.  $\eta_{55}, \eta_{66}$  may be chosen to be  $\pm 1$ , depending on the signature of the extra two dimensions.  $\mathbf{J}_{56}$  is an exchange operator which exchanges  $\mathbf{x} \leftrightarrow \mathbf{p}$  in eq-(28). When  $L_P \rightarrow 0$  and/or  $R_H \rightarrow \infty$ ,  $[\mathbf{x}_\mu, \mathbf{p}_\nu] \rightarrow 0$ . Thus the classical commuting  $8D$  phase space is recovered when  $L_P \rightarrow 0$  and  $R_H \rightarrow \infty$ .

One may notice that Yang's algebra and the algebra of eqs-(16-18) bears a certain resemblance if one were to set the numerical coefficient  $B$  to zero;  $A g_{(\mu\nu)} \mathbf{1} \leftrightarrow (L_P/R_H) \eta_{\mu\nu} \mathbf{J}_{56}$ , and  $g_{[\mu\nu]} \mathbf{1} \leftrightarrow \mathbf{J}_{\mu\nu}$ . The Schrodinger-Robertson inequalities  $\det \Sigma \geq \det \mathbf{C}$  could be applied directly to the Yang's algebra commutators by taking the expectation values with respect to angular momentum eigenstates. If one interprets  $i [\mathbf{x}_\mu, \mathbf{x}_\nu] \sim \mathbf{J}_{\mu\nu}$  as a bivector  $\mathbf{x}_{\mu\nu}$  Hermitian operator, and which in turn can be seen as a geometric area operator  $\mathbf{A}_{\mu\nu}$ , then the norm of the spatial area operator would be

$$(\langle J | \mathbf{A}_{ij} \mathbf{A}^{ij} | J \rangle)^{\frac{1}{2}} = L_P^2 (\langle J | \mathbf{J}_{ij} \mathbf{J}^{ij} | J \rangle)^{\frac{1}{2}} = L_P^2 \sqrt{J(J+1)} \quad (31)$$

which bears a similarity to the results associated to the area operator obtained in Loop Quantum Gravity (LQG) and based on spin networks. The Planck area is the quantum of minimal area [16]. This deserves further investigation. Modified uncertainty relations also apply to the energy and time variables  $\Delta E, \Delta t$  as well. The *granularity* of spacetime has been interpreted from the principle of Born reciprocity by [17].

Symplectic geometry is the realm of phase spaces [7] where the symplectic form  $\omega_{\mu\nu}$  plays an essential role. Fedosov deformation quantization [5] and the generalized star products in *curved* phase spaces are tailor made for these generalized gravitational theories in curved phase spaces (cotangent bundle). The geometry of the cotangent bundle  $T^*M$  of spacetime has been rigorously studied by [4], among others. In particular, deformation quantization in Fedosov-Finsler spaces has been analyzed extensively by [6].

To conclude, we may add that non-geometric fluxes in string theory give rise to noncommutative/nonassociative structures. More recently, the differential geometry on the simplest nonassociative (phase) space arising for a constant non-geometric  $R$ -flux has been analyzed in [15]. This nonassociativity for a constant  $R$ -flux background in closed strings is captured by the commutation relations  $[x^\mu, x^\nu] = R^{\mu\nu\rho} p_\rho$ ,  $[x^\mu, p_\nu] = i\hbar\delta_\nu^\mu$ . These studies paves the road towards a Noncommutative and Nonassociative gravity which might be a key feature in the final theory of Quantum Gravity.

### Acknowledgements

We thank M. Bowers for very kind assistance.

## References

- [1] C. Castro, "Solutions to the Gravitational Field Equations in Curved Phase-Spaces" vixra.org : 1603.0164
- [2] D. Gross and P. Mende, Phys. Letts **B 197** (1) (1987) 129.  
D. Gross and P. Mende, Nuc. Phys. **B 303** (3), (1988) 407.
- [3] M. Born, Proc. Royal Society **A 165** (1938) 291.  
M. Born, Rev. Mod. Physics **21** (1949) 463.

- [4] R. Miron, D. Hrimiuc, H. Shimada and S. Sabau, *The Geometry of Hamilton and Lagrange Spaces* ( Kluwer Academic Publishers, Dordrecht, Boston, 2001 ).
- R. Miron, *Lagrangian and Hamiltonian geometries. Applications to Analytical Mechanics*, arXiv: 1203.4101 [math.DG].
- S. Vacaru, P. Stavrinou, E. Gaburov, and D. Gonta, "Clifford and Riemann-Finsler Structures in Geometric Mechanics and Gravity" (Geometry Balkan Press, 693 pages).
- [5] B. Fedosov, "A simple geometrical construction of deformation quantization" *J. Diff. Geom.* **40** (1994) 213.
- B. Fedosov, *Deformation Quantization and index Theory* (Academie Verlag, Berlin, 1996).
- [6] S. Vacaru, "Generalized Lagrange Transforms: Finsler Geometry Methods and Deformation Quantization of Gravity" *An. St. Univ. Al. I. Cuza din Iasi (S.N.), Matematica*, vol LIII, 2007, Supliment, 327-342 [arXiv : 0707.1526 (gr-qc)].
- [7] Jan J. Slawianowski, *Geometry of Phase Spaces* (John Wiley and Sons, 1990).
- [8] L. Nottale, L. Nottale, *Scale Relativity And Fractal Space-Time: A New Approach to Unifying Relativity and Quantum Mechanics* (2011 World Scientific Publishing Company).
- L. Nottale, *Fractal Space-Time and Micro-physics* (World Scientific, May 1993).
- Scale Relativity, [http://en.wikipedia.org/wiki/Laurent\\_Nottale](http://en.wikipedia.org/wiki/Laurent_Nottale)
- [9] G. Amelino-Camelia, L. Freidel, J. Kowalski-Glikman, and L. Smolin, "The principle of relative locality", *Phys. Rev.* **D84** (2011) 084010.
- G. Amelino-Camelia, L. Freidel, J. Kowalski-Glikman, and L. Smolin, "Relative locality: A deepening of the relativity principle", *Gen. Rel. Grav.* **43** (2011) 2547.
- [10] L. Barcaroli, L. Brunkhorst , G. Gubitosi, N. Lorent and C. Pfeifer, "Hamilton geometry: Phase space geometry from modified dispersion relations" arXiv : 1507.00922.
- [11] A. Kempf and G. Mangano, *Phys. Rev.* **D 55** (1997) 7909.
- A. Kempf, "Minimal Length Uncertainty Relation and Ultraviolet Regularisation", hep-th/9612084.
- [12] C. Castro, *Phys. Letts B* **668** (2008) 442.

- [13] C. Castro, “ On Clifford Space Relativity, Black Hole Entropy, Rainbow Metrics, Generalized Dispersion and Uncertainty Relations” *Found. Phys.* **44** (2014) 990.
- [14] C. Yang, *Phys. Rev.* vol. 72 (1947) 874.
- [15] R. Blumenhagen and M. Fuchs, “ Towards a Theory of Nonassociative Gravity” arXiv : 1604.03253  
G. Barnes, A. Schenkel, R. Szabo, “Working with Nonassociative Geometry and Field Theory” arXiv : 1602.07353.
- [16] C. Rovelli, *Quantum Gravity*, (Cambridge University Press, 2004)  
T. Thiemann, *Introduction to Modern Canonical Quantum General Relativity*, (Cambridge University Press 2007)
- [17] P. Jarvis and S. Morgan, “Born reciprocity and the granularity of Space-time” arXiv : math-ph/0508041.