Homogeneous ,
Reduction of Order Homogeneous
&
Inhomogeneous
Second Order Linear Ordinary Differential Equations
Solution
in one process

Recall from previous:

Corollary 1.5: If: $y = e^{\int s dx}$, for any function r ,
$$y'' + (-s + 2r)y' + (-s' - 2rs)y = 0$$

Unfortunately, the transformation:

$$\left.\begin{array}{|l|}\hline P = -s + 2r \\ \hline Q = -s' - 2rs \\ \hline \end{array}\right\} \Rightarrow \text{ yields Riccatis: } \Rightarrow \left\{\begin{array}{|l|}\hline s' + s^2 + sP = -Q \\ \hline r' + 2r^2 - rP = \frac{1}{2}(P' - Q) \\ \hline \end{array}\right.$$

However, it inspires a factorization of the HLODE & ILODE:

$$y'' + (-s + 2r)y' + (-s' - 2rs)y = W$$
$$\Rightarrow y'' - sy' + 2ry' - s'y - 2rsy = W$$
$$\Rightarrow y'' - sy' - s'y + 2ry' - 2rsy = W$$
$$\Rightarrow y'' - (sy)' + 2r(y' - sy) = W$$
$$\Rightarrow (y' - sy)' + 2r(y' - sy) = W$$
$$\Rightarrow e^{-\int 2rdx}\left((y' - sy)e^{\int 2rdx}\right)' = W$$
$$\Rightarrow e^{-\int 2rdx}\left(e^{\int sdx}\left(ye^{-\int sdx}\right)' e^{\int 2rdx}\right)' = W$$
$$\Rightarrow \left(e^{\int sdx}\left(ye^{-\int sdx}\right)' e^{\int 2rdx}\right)' = W e^{\int 2rdx}$$
$$\Rightarrow \left(ye^{-\int sdx}\right)' = k_1 e^{-\int (s+2r)dx} + e^{-\int (s+2r)dx}\int W e^{\int 2rdx} dx$$
$$\Rightarrow y = k_1 e^{\int sdx}\int e^{-\int (s+2r)dx} dx + k_2 e^{\int sdx} + e^{\int sdx}\int e^{-\int (s+2r)dx}\left(\int We^{\int 2rdx} dx\right) dx$$

Since: $P = -s + 2r \Rightarrow -2s - P = -(s + 2r)$ & $s + P = 2r$
From the ILODE formula:

$$y_{p1} = y_1 \int \left(\frac{1}{y_1^2} \int W y_1 e^{\int P dx} dx \right) e^{-\int P dx} dx$$

and reduction of order formula, this may be rewritten:

$$\Rightarrow y = k_1 e^{\int s dx} + k_2 e^{\int s dx} \int \frac{1}{\left(e^{\int s dx}\right)^2} e^{-\int P dx} dx +$$

$$+ e^{\int s dx} \int \frac{1}{\left(e^{\int s dx}\right)^2} \left(\int W \left(e^{\int s dx}\right) e^{\int P dx} dx \right) e^{-\int P dx} dx$$

Such a wonderful analysis of second order ordinary differential equations yielding homogeneous solution, reduction of order homogeneous solution and inhomogeneous solution in one process.

As illustrated in the following theorem.

Theorem #2: If $y'' + Py' + Qy = W$, then:

$$y = e^{\int s dx} \int e^{-2\int s dx} \left(\int e^{\int s dx} W e^{\int P dx} dx \right) e^{-\int P dx} dx + k_1 e^{\int s dx} \int e^{-\int 2s dx} e^{-\int P dx} dx + k_2 e^{\int s dx}$$

where:

$$s' + s^2 + Ps = -Q$$

Proof:

$$y'' + Py' + Qy = W$$
$$\Rightarrow y'' - sy' + sy' + Py' - s'y + s'y + Qy = W$$
$$\Rightarrow y'' - sy' - s'y + Py' + sy' + s'y + Qy = W$$
$$\Rightarrow y'' - (sy)' + Py' + sy' + s'y + Qy = W$$
$$\Rightarrow (y' - sy)' + (P + s)y' + (Q + s')y = W$$
$$\Rightarrow (y' - sy)' + (P + s)y' - (P + s)sy + (P + s)sy + (Q + s')y = W$$
$$\Rightarrow (y' - sy)' + (P + s)(y' - sy) + (P + s)sy + (Q + s')y = W$$
$$\Rightarrow (y' - sy)' + (P + s)(y' - sy) + (Ps + s^2 + s' + Q)y = W$$
$$\Rightarrow e^{-\int (P+s)dx} \left((y' - sy) e^{\int (P+s)dx} \right)' + (Ps + s^2 + s' + Q)y = W$$
$$\Rightarrow e^{-\int (P+s)dx} \left(e^{\int s dx} \left(y e^{-\int s dx} \right)' e^{\int (P+s)dx} \right)' + (Ps + s^2 + s' + Q)y = W$$
$$\Rightarrow \left(\left(y e^{-\int s dx} \right)' e^{\int (P+2s)dx} \right)' + e^{\int (P+s)dx} (Ps + s^2 + s' + Q)y = W e^{\int (P+s)dx}$$

Choosing s such that:

$Ps + s^2 + s' + Q = 0$:

$$\Rightarrow \left(\left(y e^{-\int s dx} \right)' e^{\int (P+2s)dx} \right)' = W e^{\int (P+s)dx}$$
$$\Rightarrow \left(y e^{-\int s dx} \right)' = e^{-\int (P+2s)dx} \int W e^{\int (P+s)dx} dx + k_1 e^{-\int (P+2s)dx}$$

$$\Rightarrow y = e^{\int sdx}\int e^{-\int (P+2s)dx}\left(\int We^{\int (P+s)dx}dx\right)dx + k_1 e^{\int sdx}\int e^{-\int (P+2s)dx}dx + k_2 e^{\int sdx}$$
$$\Rightarrow y = e^{\int sdx}\int e^{-2\int sdx}\left(\int e^{\int sdx}We^{\int Pdx}dx\right)e^{-\int Pdx}dx + k_1 e^{\int sdx}\int e^{-\int 2sdx}e^{-\int Pdx}dx + k_2 e^{\int sdx}$$

□

Similarly:

Theorem #3: If $y'' + Py' + Qy = W$, then:

$$y = e^{-\int (P+s)dx}\int e^{\int (P+2s)dx}\left(\int We^{-\int sdx}dx\right)dx + k_1 e^{-\int (P+s)dx}\int e^{\int (P+2s)dx}dx + k_2 e^{-\int (P+s)dx}$$

where:

$$s' - s^2 - Ps = Q - P'$$

Proof:

$$y'' + Py' + Qy = W$$
$$\Rightarrow y'' - sy' + sy' + Py' + Qy = W$$
$$\Rightarrow y'' - sy' + (P+s)y' + Qy = W$$
$$\Rightarrow e^{\int sdx}\left(y'e^{-\int sdx}\right)' + (P+s)y' + Qy = W$$
$$\Rightarrow e^{\int sdx}\left(y'e^{-\int sdx}\right)' + (P+s)\left(y' + \frac{Q}{P+s}y\right) = W$$
$$\Rightarrow e^{\int sdx}\left(y'e^{-\int sdx}\right)' + (P+s)e^{-\int\left(\frac{Q}{P+s}\right)dx}\left(ye^{\int\left(\frac{Q}{P+s}\right)dx}\right)' = W$$
$$\Rightarrow \left(y'e^{-\int sdx}\right)' + (P+s)e^{-\int\left(s+\frac{Q}{P+s}\right)dx}\left(ye^{\int\left(\frac{Q}{P+s}\right)dx}\right)' = We^{-\int sdx}$$

Choosing s such that:

$$(P+s)e^{-\int\left(s+\frac{Q}{P+s}\right)dx} = k_0 :$$
$$\Rightarrow (P+s)' - (P+s)s - Q = 0$$
$$\Rightarrow s' - s^2 - Ps = Q - P'$$
$$\Rightarrow \left(y'e^{-\int sdx}\right)' + k_0\left(ye^{\int\left(\frac{Q}{P+s}\right)dx}\right)' = We^{-\int sdx}$$
$$\Rightarrow y'e^{-\int sdx} + k_0 ye^{\int\left(\frac{Q}{P+s}\right)dx} = \int We^{-\int sdx}dx + k_1$$
$$\Rightarrow y'e^{-\int sdx} + (P+s)e^{-\int\left(s+\frac{Q}{P+s}\right)dx}ye^{\int\left(\frac{Q}{P+s}\right)dx} = \int We^{-\int sdx}dx + k_1$$
$$\Rightarrow y'e^{-\int sdx} + (P+s)e^{-\int sdx}y = \int We^{-\int sdx}dx + k_1$$
$$\Rightarrow y' + (P+s)y = e^{\int sdx}\int We^{-\int sdx}dx + k_1 e^{\int sdx}$$
$$\Rightarrow e^{-\int (P+s)dx}\left(ye^{\int (P+s)dx}\right)' = e^{\int sdx}\int We^{-\int sdx}dx + k_1 e^{\int sdx}$$

$$\Rightarrow \left(ye^{\int(P+s)dx}\right)' = e^{\int(P+2s)dx}\int We^{-\int sdx}dx + k_1e^{\int(P+2s)dx}$$

$$\Rightarrow y = e^{-\int(P+s)dx}\int e^{\int(P+2s)dx}\left(\int We^{-\int sdx}dx\right)dx + k_1e^{-\int(P+s)dx}\int e^{\int(P+2s)dx}dx + k_2e^{-\int(P+s)dx}$$

□

But NOTE:

$$y'' + (-s+2r)y' + (-s' - 2rs)y = W$$
$$\Rightarrow y'' - sy' + 2ry' - s'y - 2rsy = W$$
$$\Rightarrow y'' - sy' - s'y + 2ry' - 2rsy = W$$
$$\Rightarrow y'' - (sy)' + 2r(y' - sy) = W$$
$$\Rightarrow (y' - sy)' + 2r(y' - sy) = W$$
$$\Rightarrow (D+2r)(y' - sy) = W$$
$$\Rightarrow (D+2r)(D-s)y = y'' + (-s+2r)y' + (-s' - 2rs)y = W$$

is clearly only the special case linear differential operator factorization

where:

$$\begin{aligned} Ps + s^2 + s' + Q &= -(s+2r)s + s^2 + s' + Q \\ &= -s^2 + 2rs + s^2 + s' + Q = Q - (-s' - 2rs) = 0 \end{aligned}$$

Thus, two similar analyses yield homogeneous solution, reduction of order homogeneous solution and inhomogeneous solution in one process.