May 22, 2016

Editor-in-Chief
viXra.org

I hereby submit my paper, "*A Totally Ordered Set with Cardinality Strictly between Natural and Real Numbers*", to "*viXra.org'.* An earlier version was presented to a leading Set Theory journal last year. It was scan/read by several reviewers and nothing egregiously wrong found. It was then recommended to resubmit to a more specialized Set Theory journal.

This paper identifies a set with cardinality strictly between natural and real numbers. It is essentially an experimental finding, from unrelated research leading to patents on nonuniform data sampling and on self-stabilizing computer arithmetic. I am a Computer Science practitioner; the contents of the paper are at the level of a Mathematics or Computer Science undergraduate. More specifically, several straightforward, if not simple, enumeration proofs are presented. This paper has been prepared and written entirely by me, and has not received any previous academic credit at this or any other institution.

Thank you for your consideration. Please do not hesitate to contact me with any questions.

Regards,

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April 17, 2016

A Totally Ordered Set with Cardinality Strictly between Natural and Real Numbers

by

Philip Druck

Abstract

A totally ordered set is identified with cardinality strictly between natural (N) and real (R) numbers. This set, denoted DS, is essentially an experimental finding, identified in unrelated patented research on nonuniform data sampling and self-stabilizing computer arithmetic. Its theoretical validation here will provide concrete proof that the Continuum Hypothesis (CH) is false. Note that this is distinct from determining whether CH can or cannot be proven from current axioms of set theory, which is settled. Also note that the Generalized Continuum Hypothesis is not addressed. First, Cantor diagonalization is applied isomorphically to prove that DS has strictly more than Cardinality(N) points. Then three (3) distinct proofs are provided to show that DS contains strictly fewer than Cardinality(R) elements. Each proof relies on a distinct property of primes. It is surmised that the considerable research efforts to-date on CH missed this result due to over-generalization, by considering all $Aleph_i$ sets, i=0.., ∞. Those efforts thereby missed the impact of primes specifically on $Aleph_0$/$Aleph_1$ sets.

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April 17, 2016

A Totally Ordered Set with Cardinality Strictly between Natural and Real Numbers

By
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1. Introduction:

A totally ordered set is identified with cardinality strictly between natural (N) and real (R) numbers [1]. This set, denoted DS, is essentially an experimental finding, identified in unrelated patented research on nonuniform data sampling [2] and self-stabilizing computer arithmetic [3]. Its theoretical validation here will provide concrete proof that the Continuum Hypothesis (CH) [4, 5] is false. Note that this is distinct from determining whether CH can or cannot be proven from current axioms of set theory, which is settled [6]. Also note that the Generalized Continuum Hypothesis [4] is not addressed. For the following proofs are based fundamentally on several unique properties of prime numbers. A subsequent paper will present a countable number of distinct DS sets.

It is surmised that the considerable research efforts to-date on CH missed this result due to over-generalization, by considering all Aleph_i sets, i=0.., ∞. Those efforts thereby missed the impact of primes specifically on $\text{Aleph}_0/\text{Aleph}_1$ sets.

This set might be applied to physical measurements due to its attractive density. For it is far denser than integers, as required by experimental data. Yet it is not nearly as dense as real numbers, which must be approximated by finite, floating point, significant digits [2,3].

2. Summary of Proofs:

First, Cantor diagonalization [7] is applied isomorphically to prove that DS has strictly more than Cardinality(N) points. Then three (3) distinct proofs are provided to show that DS contains strictly fewer than Cardinality(R) elements. Each proof relies on a distinct property of primes. In no particular order, asymptotic analysis [8] is first applied to prove that asymptotically, DS has fewer numbers than R. It is based fundamentally on the asymptotic scarcity of prime numbers (from the Prime Number Theorem [9]). A second proof constructs a set with Cardinality(R), yet not intersecting DS, for which there is no 1-to-1 correspondence between it and DS. That is, DS will be shown to exhaustively map to that set, but it will have remaining points not mapped by

DS points. It is based on the incommensurability of reciprocals of primes [2]. A third proof constructs another set of Cardinality(R) for which there is no 1-of-1 mapping between it and DS. It is based on the dichotomy property of N whereby each integer in N is either composite or prime. It is also shown that if there is no 1-to-1 mapping between DS and one non-intersecting set of size R, then there is no such mapping to any other set of Cardinality(R). This solidifies the 2^{nd} and 3^{rd} proofs in not needing to examine any other Cardinality(R) sets.

This paper is organized as follows:

I. Introducing the DS Set
II. DS Set's Total Ordering Properties
III. Proof that Cardinality(N) < Cardinality(DS)
IV. 3 proofs that Cardinality(DS) < Cardinality(R)
   a. Based on asymptotic scarcity of primes.
   b. Based on incommensurate reciprocals of primes.
   c. Based on dichotomy of N into primes and composites.

I. <u>Introducing the DS Set</u>

1.1 Generation of the DS Set:

The DS set is generated in a manner isomorphic to the usual base number generation of real numbers (e.g., binary, decimal, octal, hexadecimal), with the twist that prime numbers are used instead. As such, each of its points has a "base prime number" representation. Only points within the half-inclusive unit interval [0,1) are considered for simplicity and without loss of generality.

Formally:

$$DS = \{ \sum_{k=0}^{\infty} a_k / \prod_{i=0}^{k} p_i \mid a_k \,\varepsilon\, \{0,1\} \;\; p_0=2,\, p_1=3,\; ..\text{ consecutive primes } ..\; \infty \} \qquad (1)$$

The numerator of each term in the infinite series, $a_{k,}$ is referred to as a 'digit', isomorphic to the usual digit in a base number representation.

Sample DS Point: 1/2 + 0/(2*3) + 0/(2*3*5) + 1/(2*3*5*7) + 1/(2*3*5*7*11) + …. (2)

1.2 Decimal-Like, Base-Prime Representation of DS Points:

Each point in DS can be mapped to a familiar decimal-like base-prime format as follows:

$$P == a_0/p_0 + a_1/(p_0{*}p_1) + \ldots + a_k/(p_0{*}..p_{k-1}{*}p_k) + \ldots..\infty \;\;\langle\text{-->}\;\; 0.b_0,\, b_1,\, \ldots\ldots,\, b_k \ldots..\,\infty \qquad (3)$$

where:

$$a_j/(p_0{*}p_1{*}\ldots{*}p_j) \;\langle\text{-->}\; b_j$$

For the sample point in (2):

$1/2 + 0/(2*3) + 0/(2*3*5) + 1/(2*3*5*7) + 1/(2*3*5*7*11) \ldots. == .10011\ldots$ (base prime)

A base prime representation is convertible to a base 10 representation by performing the usual arithmetic division on each DS digit as a real number, and then summing terms.

For the sample (2) above:

$$\begin{aligned}
.10011..\text{(base prime)} &== 1/2 + 0/(2*3) + 0/(2*3*5) + 1/(2*3*5*7) + 1/2*3*5*7*11) + \ldots. \\
&== 1/2 + 0 + 0 + 1/210 + 1/2310 + \ldots \\
&== 0.5 + 0 + 0 + 0.00476 + 0.000432 + \ldots \\
&== 0.505192\ldots \text{ (base 10)}
\end{aligned}$$

1.3 **Convergence of DS Points:**

Theorem: Each DS point converges to a number < 1. That is:

$$\text{DS Point } P = \lim_{n \to \infty} \sum_{k=0}^{n} a_k/(p_0*p_1..*p_k) < 1 \qquad (4)$$

where $a_k \varepsilon \{0,1\}$ $p_0=2, p_1=3, p_2=5, p_3=7, p_5=11$ .. consecutive primes.. ∞

Proof:

Since: $a_0/(p_0) <= 1/2$ and $a_k/(p_0*..p_{k-1}*p_k) < 1/2^{k+1}$ for k=1,..

Then: $P == a_0/p_0 + a_1/(p_0*p_1) + \ldots. + a_k/(p_0*..p_{k-1}*p_k) + \ldots..\infty < 1/2^1 + 1/2^2 + \ldots. + 1/2^{k+1} + .. \infty$

Upon adding and subtracting $1/2^0$ and 1 respectively, to the right-hand side:

$P == 1/2^0 + (1/2^1 + 1/2^2 + \ldots + 1/2^{k+1} + .. \infty) - 1$

The infinite series on the right side is a clearly recognized infinite geometric progression with limit: $a/(1-r)$ where a=1 and r=1/2

Therefore:

$P == 1/2^0 + (1/2^1 + \ldots + 1/2^{k+1} + .. \infty) - 1 = 1/(1-1/2) - 1 = 2/1-1 = 1$

Or: $P < 1$

II. <u>DS Set's Total Ordering Properties</u>

2.1 **Ordering of DS Numbers:**

2.1.1 <u>Key Ordering Theorem</u>:

Given 2 DS points, P_1 and P_2, such that their digits match up to and including the k^{th} position, where k+1= 0,.., then the point with the lesser $(k+1)^{st}$ digit value, is less than the other.

Formally, using base-prime notation:

If: $P_1 = 0.a_{0,} a_{1, \dots} a_{k,} a_{k+1}, \dots a_{k+n}, \dots.. \infty$

$P_2 = 0.a_{0,} a_{1, \dots} a_{k,} b_{k+1}, \dots b_{k+n}, \dots.. \infty$

where $a_i == b_i$ for $0 =< i =< k$, and $a_{k+1} < b_{k+1}$, then $P_1 < P_2$

Note for k+1=0:

$P_1 = 0.a_{0,} a_{1, \dots} a_{k,} a_{k+1}, \dots a_{k+}, \dots.. \infty$

$P_2 = 0.b_{0,} b_{1, \dots} b_{k,} b_{k+1}, \dots b_{k+n}, \dots.. \infty$

where $a_0 < b_0$

Note this ordering is valid regardless of respective digit values beyond the $(k+1)^{st}$. Also note that necessarily $a_{k+1}=0$ and $b_{k+1}=1$ when $a_{k+1} < b_{k+1.}$ For otherwise, $a_{k+1} = b_{k+1,}$ negating the premise.

Finally note that this theorem and proof are isomorphic to those for usual base-10 decimals. In that case, $a_k \varepsilon \{0,9\}$ and digit k, represents $a_k/10^k$. There is one exception, not applicable to base-prime, whereby .nnnn…n9999.. == .nnnn…,n+1,0000 for an arbitrary mix of digits n.

Proof:

First, revert to the actual value of each point. From (4) and that $a_i == b_i$ for $0 =< i =< k$:

$$P_1 = a_0/(p_0) + \dots + a_k/(p_0*p_1..*p_k) + a_{k+1}/(p_0*p_1..*p_k*p_{k+1}) + a_{k+2}/(p_0*p_1..*p_k*p_{k+1}*p_{k+2}) + \dots \infty \quad (5)$$

$$P_2 = a_0/(p_0) + \dots + a_k/(p_0*p_1..*p_k) + b_{k+1}/(p_0*p_1..*p_k*p_{k+1}) + b_{k+2}/(p_0*p_1..*p_k*p_{k+1}*p_{k+2}) + \dots \infty$$

Must show:

$$a_0/(p_0) + \dots + a_k/(p_0*p_1..*p_k) + a_{k+1}/(p_0*p_1..*p_k*p_{k+1}) + a_{k+2}/(p_0*p_1..*p_k*p_{k+1}*p_{k+2}) + \dots \infty \quad (6)$$

?<

$$a_0/(p_0) + \dots + a_k/(p_0*p_1..*p_k) + b_{k+1}/(p_0*p_1..*p_k*p_{k+1}) + b_{k+2}/(p_0*p_1..*p_k*p_{k+1}*p_{k+2}) + \dots \infty$$

A stronger inequality is proved, upon removing the b_j terms on the right side with terms j>k+1:

$$a_0/(p_0) + \dots + a_k/(p_0*p_1..*p_k) + a_{k+1}/(p_0*p_1..*p_k*p_{k+1}) + a_{k+2}/(p_0*p_1..*p_k*p_{k+1}*p_{k+2}) + \dots \infty \quad (7)$$

?<

$$a_0/(p_0) + \dots + a_k/(p_0*p_1..*p_k) + b_{k+1}/(p_0*p_1..*p_k*p_{k+1})$$

For if (7) is true, then (6) is certainly true, since its right side is even greater than (7).

Upon subtracting common terms from both sides:

$$a_{k+1}/(p_0*p_1..*p_k*p_{k+1}) + a_{k+2}/(p_0*p_1..*p_k*p_{k+1}*p_{k+2}) + \dots + a_{k+n}/(p_0*p_1..*p_k*p_{k+1}*p_{k+2}*.*p_{k+n}) + ..\infty \quad (8)$$

?<

$$b_{k+1}/(p_0*p_1..*p_k*p_{k+1})$$

Since $a_{k+1} < b_{k+1,}$ consider the closest case, without lose of generality, whereby $b_{k+1} == a_{k+1} + 1$.

Note that this can only occur if $a_{k+1} = 0$ and $b_{k+1} = 1$

Then: $b_{k+1}/(p_0*p_1..*p_k*p_{k+1}) == a_{k+1}/(p_0*p_1..*p_k*p_{k+1}) + 1/(p_0*p_1..*p_k*p_{k+1})$

Substituting that into (8) leads to:

$a_{k+1}/(p_0*p_1..*p_k*p_{k+1})+a_{k+2}/(p_0*p_1..*p_k*p_{k+1}*p_{k+2})+... +a_{k+n}/(p_0*p_1.*p_k*p_{k+1}*..*p_{k+n})+..\infty$ (9)

?<

$a_{k+1}/(p_0*p_1..*p_k*p_{k+1}) + 1/(p_0*p_1..*p_k*p_{k+1})$

Upon subtracting the common $a_{k+1}/(p_0*p_1..*p_k*p_{k+1})$ from both sides, the above becomes:

$a_{k+2}/(p_0*p_1..*p_k*p_{k+1} *p_{k+2}) +... +a_{k+n}/(p_0*p_1..*p_k*p_{k+1}*p_{k+2}*...*p_{k+n}) +.. \infty$ (10)

?<

$1/(p_0*p_1..*p_k*p_{k+1})$

After multiplying both sides by $(p_0*p_1 * ..*p_k*p_{k+1})$:

$a_{k+2}/p_{k+2} + ... a_{k+n}/(p_{k+2} *...* p_{k+n}) + ... \infty$ **?<** 1 (11)

Note that for each term in (11):

$a_{k+i}/(p_{k+2}*..*p_{k+i}) < 1/3^{i-1}$ for $i > 1$ (12)

For $a_j =< 1$ and $1/p_i < 1/3^1$ since $p_i >2$ for $i > 1$

Note for k+1=0, i=1:

$a_1/p_1 = a_1/3 <= 1/3$

Applying (12) to each term on the left side of (11) above:

$a_{k+2}/p_{k+2} + ... +a_{k+n}/(p_{k+2} *...* p_{k+n}) + ... \infty < (1/3^1 + ... + 1/3^{n-1} + .. \infty)$ (13)

The infinite series on the right hand side is a clearly recognized geometric progression with limit: (a/1-r) - 1, where a=1 and r=1/3

Therefore:

$a_{k+2}/p_{k+2} + ... a_{k+n}/(p_{k+2} *...*p_{k+n}) + ... \infty < (1/(1-1/3)) - 1 == 3/2-1 == 1/2$

Or:

$a_{k+2}/p_{k+2} + ... a_{k+n}/(p_{k+2} *p_{k+n}) + ... \infty < ½ < 1$ (14)

This proves inequality (6) by reversing the steps from (14) to (7) upward.

From (6), $P_1 < P_2$

2.1.2 Corollary: Distance between DS Points:

Given any 2 distinct DS points, P_1 and P_2 where their digits match up to and including the k^{th} position, where k+1 = 0,…, then their difference is greater than a position dependent value. More concretely:

Given:

$P_1 = 0.a_{0,} a_{1, \ldots} a_{k,} a_{k+1}, \ldots a_{k+n}, \ldots\ldots \infty$

$P_2 = 0.a_{0,} a_{1, \ldots} a_{k,} b_{k+1}, \ldots b_{k+n}, \ldots\ldots \infty$

where $a_i == b_i$ for 0 =< i =< k, and $a_{k+1} != b_{k+1}$

then:

$$| P_2 - P_1 | > 1/(p_0*p_1..*p_k*p_{k+1})*(½) \quad (15)$$

For from (10) above:

$P_1 < P_2 \rightarrow a_{k+2}/(p_0*p_1..*p_k*p_{k+1}*p_{k+2}) + \ldots +a_{k+n}/(p_0*p_1..*p_k*p_{k+1}*p_{k+2}*\ldots*p_{k+n}) +.. \infty$
$< 1/(p_0*p_1..*p_k*p_{k+1})$

Or: : $(P_2 – P_1) > 0 \rightarrow$

$$1/(p_0*p_1..*p_k*p_{k+1}) * (1 - (a_{k+2}/(p_{k+2}) + \ldots +a_{k+n}/(p_{k+2}* \ldots*p_{k+n}) +.. \infty)) > 0$$

By (14) above: $a_{k+2}/p_{k+2} + \ldots a_{k+n}/(p_{k+2}*p_{k+n}) + \ldots. \infty < ½$

Then: $P_1 < P_2 \rightarrow (P_2 - P_1) > 0 \rightarrow$

$$1/(p_0*p_1..*p_k*p_{k+1}) * (1 - (a_{k+2}/(p_{k+2}) + \ldots +a_{k+n}/(p_{k+2}* \ldots*p_{k+n}) +.. \infty)) >$$
$$1/(p_0*p_1..*p_k*p_{k+1}) *(1 - ½) = 1/(p_0*p_1..*p_k*p_{k+1})*(½) > 0$$

Similarly: $P_2 < P_1 \rightarrow (P_1 - P_2) > 1/(p_0*p_1..*p_k*p_{k+1})*(1 - ½) = 1/(p_0*p_1..*p_k*p_{k+1})*(½) > 0$

Or: $| P_2 - P_1 | > 1/(p_0*p_1..*p_k*p_{k+1})*(1/2)$

2.2 **Uniqueness of DS Points:**

Each point in DS is unique, from the following proof. This insures no double-counting of DS elements in the subsequent counting proofs.

Given two arbitrarily selected points in the DS set, P_1 and P_2, consider their base-prime representations:

$P_1 = a_{0,} a_{1, \ldots} a_{k,} a_{k+1}, \ldots a_{k+n}, \ldots\ldots$

$P_2 = b_{0,} b_{1, \ldots} b_{k,} b_{k+1}, \ldots b_{k+n}, \ldots\ldots$

a. If a_k==b_k for all k, then P_1==P_2

For then, using (1) for P_1 and P_2:

$$P_1 == \sum_{k=0}^{\infty} a_k/(p_0*p_1..*p_k) = \text{limit} \sum_{k=0}^{\infty} b_k/(p_0*p_1..*p_k) == P_2 \qquad (16)$$

b. If P_1==P_2 then a_k==b_k for all k.

The equivalent contrapositive statement is proved: If a_k != b_k for some k, then P_1!=P_2

For, consider the first k for which a_{k+1} != b_{k+1}. Then assume, without lose of generality, that $a_{k+1} < b_{k+1}$ for that k. From Section 2.1 above, that implies that $P_1 < P_2$ or P_1!=P_2

2.3 Converse of Ordering Theorem:

Given 2 DS points, P_1 and P_2, if $P_1 < P_2$ per section 1.2, then there is an index k in base-prime, such that their respective digits match up to k and differ at k+1 (with k+1= 0,1,..).

Or, if $P_1 < P_2$:

Then:

$P_1 = a_{0,} a_{1, \dots} a_{k,} a_{k+1}, \dots a_{k+n}, \dots.. \infty$

$P_2 = a_{0,} a_{1, \dots} a_{k,} b_{k+1}, \dots b_{k+n}, \dots.. \infty$

where $a_i == b_i$ for 0 =< i =< k, and $a_{k+1} < b_{k+1}$

Consider the possibility that there is no k such that respective indexes differ. That is, consider $a_k = b_k$ for all k. Then by uniqueness of DS points (Section 2.2), $P_1 = P_2$, contradicting premise.

2.4 ***Total Ordering Property of DS:***

DS is totally ordered with respect to the usual < operation in R, as proved here.

Note that total ordering = partial ordering + comparability property.

2.4.1 Partial Ordering of DS Set:

Partial Ordering requires these properties:

2.4.1.1 Reflexivity: P = P for all P in DS.

Proof: The decimal decomposition of each side necessarily leads to identical values.

2.4.1.2 Antisymmetry:

$P_1 < P_2$ and $P_2 < P_1$ → $P_1 = P_2$ for all points in DS:

Consider the base-prime representation of each point:

$P_1 = a_0, a_1, \ldots a_k, a_{k+1}, \ \ldots \ a_{k+n}, \ldots.. \infty$
$P_2 = b_0, b_1, \ldots b_k, b_{k+1}, \ \ldots \ b_{k+n}, \ldots.. \infty$

By Theorem 2.3, $P_1 < P_2 \rightarrow$ there exists a k+1 such that:
$a_i = b_i$ for $0 =< i =< k$, and $a_{k+1} < b_{k+1}$
Also by Theorem 2.3, $P_2 < P_1 \rightarrow$ there exists an index l+1 such that:
$a_i = b_i$ for $0 =< i =< l$, and $a_{l+1} > b_{l+1}$

If $l > k$, then $a_{k+1} = b_{k+1}$ contradicting the above, that $a_{k+1} < b_{k+1}$
If $l < k$, then $a_{l+1} = b_{l+1}$ contradicting the above, that $a_{l+1} < b_{l+1}$
Therefore $l == k$ and P_1,P_2 can be expressed as:
$P_1 = a_0, a_1, \ldots a_k, a_{k+1}, \ \ldots \ a_{k+n}, \ldots.. \infty$
$P_2 = a_0, a_1, \ldots a_k, b_{k+1}, \ \ldots \ b_{k+n}, \ldots.. \infty$

If $a_{k+1} < b_{k+1}$ then $P_1 < P_2$ by the Ordering Theorem in Sec. 2.1. But $P_1 > P_2$ by hypothesis,
Therefore $P_1 \, !< P_2$
If $a_{k+1} > b_{k+1}$ then $P_1 > P_2$ by the Ordering Theorem. But $P_2 > P_1$, by hypothesis.
Therefore $P_1 \, !> P_2$
Combining the above: $P_1 \, !< P_2$ & $P_1 \, !> P_2 \rightarrow P1 = P2$

2.4.1.3 Transitivity: $P_1 < P_2$ and $P_2 < P_3 \rightarrow P_1 < P_3$
Consider the base-prime representation of any three points in DS:
$P_1 = a_0, a_1, \ldots a_k, a_{k+1}, \ \ldots \ a_{k+n}, \ldots.. \infty$
$P_2 = b_0, b_1, \ldots b_k, b_{k+1}, \ \ldots \ b_{k+n}, \ldots.. \infty$
$P_3 = c_0, c_1, \ldots c_k, c_{k+1}, \ \ldots \ c_{k+n}, \ldots.. \infty$

$P_1 < P_2 \rightarrow$ there is an index k such that $a_i = b_i$ for i=0..k and $a_{k+1} < b_{k+1}$ Or:
$P_1 = a_0, a_1, \ldots a_k, a_{k+1}, \ \ldots \ a_{k+n}, \ldots.. \infty$
$P_2 = a_0, a_1, \ldots a_k, b_{k+1}, \ \ldots \ b_{k+n}, \ldots.. \infty$

$P_2 < P_3 \rightarrow$ there is an index l such that $b_i = c_i$ for i=0..l and $b_{l+1} < c_{l+1}$ Or:
$P_2 = b_0, b_1, \ldots b_l, b_{l+1}, \ \ldots \ b_{l+n}, \ldots.. \infty$
$P_3 = b_0, b_1, \ldots b_l, c_{l+1}, \ \ldots \ c_{l+n}, \ldots.. \infty$

If $l+1 < k+1$, then upon combining the above for $P_1 < P_2$ and $P_2 < P_3$:
$P_1 = a_0, a_1, \ldots a_l, a_{l+1}, \ldots, a_k, \ a_{k+1}, \ldots \ a_{k+n}, \ldots.. \infty$
$P_3 = a_0, a_1, \ldots a_l, c_{l+1}, \ldots, c_k, \ c_{k+1}, \ldots \ c_{l+n}, \ldots.. \infty$

$P_1 < P_2 \rightarrow a_{l+1} = b_{l+1} \rightarrow$ But $P_2 < P_3 \rightarrow b_{l+1} < c_{l+1}$. Therefore $a_{l+1} < c_{l+1} \rightarrow P_1 < P_3$

If l+1 > k+1, then combining the above for $P_1 < P_2$ and $P_2 < P_3$:

$P_1 = a_0, a_1, \ldots a_l, \ldots, a_k, a_{k+1}, \ldots a_{l+1}, \ldots.. \infty$

$P_3 = a_0, a_1, \ldots a_l, \ldots, a_k, c_{k+1} \ldots c_{l+1}, \ldots.. \infty$

where $a_{k+1} < c_{k+1}$ because $a_{k+1} < b_{k+1}$ and $b_{k+1} = c_{k+1}$

Then by the Ordering Theorem in Section 2.1, $P_1 < P_3$

If l+1 == k+1, then $a_{k+1} < b_{k+1}$ and $b_{k+1} < c_{k+1} \rightarrow a_{k+1} < c_{k+1} \rightarrow P_1 < P_3$

2.4.2 Comparability:

For any (P_1, P_2) pair in DS, either $P_1 =< P_2$ or $P_2 =< P_1$

Proof:

Given any two points in DS in base-prime representation:

$P_1 = a_0, a_1, \ldots a_k, a_{k+1}, \ldots a_{k+n}, \ldots.. \infty$

$P_2 = b_0, b_1, \ldots b_k, b_{k+1}, \ldots b_{k+n}, \ldots.. \infty$

At the first index, k+1 when $a_{k+1} \,!= b_{k+1}$:

If $a_{k+1} < b_{k+1}$ then by the Ordering Theorem in Sec. 2.1, $P_1 < P_2$

If $a_{k+1} > b_{k+1}$ then by the Ordering Theorem in Sec. 2.1, $P_1 > P_2$

If for all indexes, $a_{k+1} = b_{k+1}$, (i.e, all digits are identical) then by the uniqueness of DS points from Section 2.2, $P_1 = P_2$

III. Diagonalization Proof that Cardinality(N) < Cardinality(DS):

The following proof that Cardinality (N) < Cardinality (DS), is isomorphic to the standard Cantor diagonalization proof that Cardinality (N) < Cardinality (R) [7].

Assume that DS is only countable infinite. Listing its points effectively enumerates them, mapped from an implicit integer index. Consider any enumeration of its elements in base-prime(Section 1.2), as follows:

$DS_0 == 0.a_{0,0}, a_{0,1}, \ldots\ldots a_{0,k}, \ldots \infty$

.

.

$DS_m == 0.a_{m,0}, a_{m,1}, \ldots\ldots a_{m,k}, \ldots \infty$

.

.

Consider a DS point $0.b_0, b_1, \ldots, b_k, \ldots.$, constructed as follows:

If $a_{k,k}$ == 0 then set b_k=1. Otherwise If $a_{k,k}$ == 1, set b_k=0.

Then $0.b_0, b_1, \ldots, b_k, \ldots..$ necessarily differs from each of the purportedly enumerated elements of DS by at least one digit. Regardless of enumerated set, this diagonalization procedure will generate a point that is not in that enumerated set. Therefore there is at least one point in DS not in N. Hence Cardinality (N) < Cardinality (DS).

III. <u>**3 proofs that Cardinality (DS) < Cardinality (R)**</u>

4.1 **Asymptotic Proof that Cardinality(DS) > Cardinality(R):**

This proof relies, fundamentally, on the relative scarcity of primes vis-a-via all real numbers. Asymptotic analysis [8] is applied to compare the growth rates of DS and R points' base partial sums towards their respective limit points.

4.1.1 <u>Hierarchical Representation:</u>

A hierarchical tree representation of both R and DS is described, as an aid to visualize the growth rate and asymptotic evolution of each respective set's points' partial sums. Each level of the hierarchy, from the root, corresponds to a digit position in either a real point's binary or DS point's prime-base representation. The number of branches emanating from each node, at any level, corresponds to the number of possible digit values. This is 2 for both R and DS points, since their respective digits are in the inclusive integral range of [0,1]. Traversing any path from the root, is equivalent to, or in 1-to-1 correspondence with, the partial sum of a point in either set.

More concretely, consider the incremental increase in number of paths for a point in each set, as its depth level increments by 1:

Evolution of points in R:

From: $0.a_0, a_1, \ldots\ldots, a_k$ where a_k == $a_k/(2^{**k+1})$ at level n=k

To: $0.a_0, a_1, \ldots\ldots, a_k, a_{k+1}$ where a_{k+1} == $a_{k+1}/(2^{**k+2})$ at level n=k+1

Evolution of points in DS:

Let m=π (k) be the number of primes less than or equal to k.

From: $0.a_0, a_1, \ldots\ldots, a_m$ where a_m == $a_m/(p_0 * p_1 * \ldots * p_m)$ at level=m

To: $0.a_0, a_1, \ldots\ldots, a_m, a_{m+1}$ where a_{m+1} == $a_{m+1}/(p_0 * p_1 * \ldots * p_m * p_{m+1})$ at level=m+1

4.1.2 Counting DS vs. R Branches:

The limit of each partial path in either set's hierarchical representation, is a point in that set. This is equivalent to a point as the limit of a base series. Therefore counting the evolving branches will yield, asymptotically, the number of respective points in the R or DS sets. This counting is restricted to a unit interval, without loss of generality.

First consider the number of evolving partial paths in a unit interval in R. There are 2^{**n} partial paths at level n. These paths evolve into Cardinal(R) number of points as n →∞, because there are 2 choices, or branches, for each digit, a_j. Or:

$$\text{Cardinal(R)} = \lim_{n \to \infty} 2^{**n} \tag{17}$$

Next, consider the number of evolving partial DS paths in a unit interval. Each level corresponds to a prime. The number of prime levels less than or equal to index n is π(n).

By the Prime Number Theorem [9], π(n)~ n/ln(n). Therefore there are $2^{**\pi(n)}$ or $2^{**n/\ln(n)}$ partial paths indexed by n. These paths evolve into Cardinal(DS) number of points as n → ∞, because there are 2 choices, or branches, for each digit a_j. Or:

$$\text{Cardinal(DS)} = \lim_{n \to \infty} 2^{**n/\ln(n)} \tag{18}$$

4.1.3 Asymptotic Comparison:

Consider the relative rate of asymptotic path growth between (17) and (18), as n → ∞: Let '<~' denotes asymptotic inequality [9][10].

Claim that:

$$2^{**n/\ln(n)} \;\underline{\mathbf{?}}\!<\sim\; 2^{**n} \tag{19}$$

Start with the following asymptotic relation "<~" between two functions [10]:

$$f(n) <\sim g(n) \iff \lim_{n \to \infty} f(n)/g(n) = 0 \tag{20}$$

Then with: $f(n) = 2^{**n/\ln(n)}$ and $g(n) = 2^{**n}$

$$2^{**n/\ln(n)}/2^{**n} \to 2^{**(n/\ln(n) - n)} \to 2^{**n(1/\ln(n)-1)} \tag{21}$$

As n→∞, $2^{**n(1/\ln(n)-1)} \to 2^{**n(0-1)} \to 2^{**n(-1)} \to 1/2^{**n} \to 0$

Then, by (21), (19) is a valid asymptotic ordering.

Therefore:

Therefore: $\text{Cardinal(DS)} = \lim_{n\to\infty} 2^{**n/\ln(n)} <\sim \lim_{n\to\infty} 2^{**n} = \text{Cardinal(R)}$ (22)

4.1.4 <u>Discussion:</u>

The asymptotic inequality (22) provides a quantitative measure of just how asymptotically fewer evolving paths there are for DS vs. R. Thus the density of DS is asymptotically lower than R, with fewer DS points than reals, in a unit interval.

Fundamentally, this inequality is due to the unique nature of the distribution of prime numbers in N. They are (countably) infinitely abundant, yet too scarce asymptotically, to keep up with the R growth rate.

4.2 **Another Proof: Construction of an R Set > Cardinality(DS):**

A set A is constructed with cardinality R, yet with none of its elements in DS. It is then proved that there cannot be a 1-to-1 correspondence between it and DS. The proof is based fundamentally on the incommensurability of reciprocals of primes [2]. This necessarily creates gaps between reciprocals of primes, with each gap containing Cardinality(R) of real number [7].

4.2.1 <u>Any Cardinality(R) Set will do:</u>

Note that if there is no 1-1 mapping between DS →A, then there cannot be a 1-1 mapping between DS and any other set of cardinality(R).

For consider the possibility of a 1-1 mapping from set DS to another set B, also of cardinality R. Since sets A and B have the same cardinality(R), by definition, there must exist a 1-1 mapping between their respective members. Then there would necessarily be a 1-1 composite mapping between DS and set B via set A. That is:

DS<->B & B<->A → DS<->A

But that contradicts the premise that there is no 1-1 mapping between sets DS and A.

4.2.2 Construction of Set A:

Consider the following set A:

$$A = \bigcup_{k=0}^{k=\infty} \{ .a_0, a_1, \ldots\ \varepsilon_k \ldots, a_m \ldots.. \infty \mid a_m \in \{0,1\}\ \&\ \varepsilon_k = \delta_k/(p_0 * .. * p_k)\ \&\ 1/2 < \delta_k < 1 \} \quad (23)$$

Note: $0 < \delta_k < 1$ which differs from a DS digit: 0 or 1

δ_k can differ from one k to the next.

4.2.2.1 DS-to-A Mapping:

Each member of A can be generated from a corresponding point in DS, by just substituting $a_k \rightarrow \varepsilon_k$ as follows:

DS point: Pds → Set A point: Pa

$.a_0, a_1, \ldots\ a_k \ldots, a_m \ldots.. \infty \rightarrow .a_0, a_1, \ldots\ \varepsilon_k \ldots, a_m$

$$a_k = 0 \rightarrow \delta_{k0} \qquad 1/2 < \delta_{k0} < 1 \quad (24)$$

$$a_k = 1 \rightarrow \delta_{k1} \qquad 1/2 < \delta_{k1} < 1 \qquad \delta_{k0} \mathrel{!=} \delta_{k1}$$

Then $Pds - Pa = d_k = \Delta_k/(p_0 * .. * p_k)$ where: (25)

$$\Delta_k = -\delta_{k0} \quad \text{if } a_k = 0$$

$$= (1-\delta_{k1}) \quad \text{if } a_k = 1$$

Sets A and DS are distinct, not overlapping, as shown in a subsequent section.

4.2.3 Convergence:

Each point in A converges to a point in R. The proof is isomorphic to that in Section 1.3 with δ_k replacing a_k.

4.2.4 Counting Comparison between Sets A and DS:

4.2.4.1 *Set A Cardinality*:

The number of points in A is of Cardinality(R). For $0 < \delta_k < 1/3$ is a line segment which has R number of real points [7].

4.2.4.2 *Set DS vs. A Count:*

Note that Cantor's diagnalization proof cannot be applied to map corresponding points between DS and set A. For enumerations of DS points do not guarantee that all DS points are exhaustively indexed, since DS > N. Indeed, that approach was initially tried before realizing that diagonalization using DS points as the enumeration index, was not exhaustive. Therefore not all DS points would have mapped to set A.

The following alternate approach creates an exhaustive mapping of points in DS-to-A. It will then be shown that there is at least one point in A (and actually many more) not mapped into from DS. Hence DS < A → DS < R.

4.2.4.3 *Each point in set A is distinct*:

For consider any 2 points in set A:

$PA_1 = .a_0, a_1, \ldots\ \varepsilon_k, a_{k+1}, \ldots\ldots\ a_n, \ldots\ldots$

$PA_2 = b_0, b_1, \ldots\ b_k, b_{k+1}, \ldots\ldots \varepsilon_n, b_{n+1} \ldots$

Consider their corresponding DS points, with: $\varepsilon_k \to a_k$ and $\varepsilon_n \to b_n$

$Pds_1 = .a_0, a_1, \ldots\ a_k, a_{k+1}, \ldots\ldots\ a_n, \ldots\ldots$

$Pds_2 = b_0, b_1, \ldots\ b_k, b_{k+1}, \ldots\ldots\ b_n, b_{n+1} \ldots$

Let index i+1 be the first where their respective digits differ. Then, from (15), Sec 2.1.2,

$| Pds_1 - Pds_2 | > 1/(p_0*p_1..*p_i*p_{i+1})*(½)$

Then from (25) above:

$| PA_1 - PA_2 | == |Pds_1+d_k - Pds_2+d_n| == | Pds_1 - Pds_2 + d_k + d_n |$

$> 1/(p_0*p_1..*p_i*p_{i+1})*½ + d_k + d_n \ != 0$

Note: $1/(p_0*p_1..*p_i*p_{i+1})*½ + d_k + d_n \ != 0$ for any of the 4 combos of $d_{k,} d_n$ in (24).

Note the inequality cannot be 0, from the substitutions in (25). Therefore $PA_1 \ != PA_2$

Note that the logic above remains intact if both ε_k are in the same position:

$PA_1 = .a_0, a_1, \ldots\ \varepsilon_k, a_{k+1}, \ldots\ldots\ a_n, \ldots\ldots$

$PA_2 = .b_0, b_1, \ldots\ \varepsilon_k, b_{k+1}, \ldots\ldots\ b_n, \ldots\ldots$

For then:

$| PA_1 - PA_2 | == |Pds_1+d_k - Pds_2+d_k| == | Pds_1 - Pds_2 + d_k + d_k |$

$> 1/(p_0*p_1..*p_i*p_{i+1})*½ + d_k + d_k \ != 0$

where again: $1/(p_0*p_1..*p_i*p_{i+1})*½ + d_k + d_n \ != 0$ for any of the 4 combos of $d_{k,} d_n$ in (24).

4.2.4.4 *Equality of Points in Set A*:

Two points in set A are identical if and only if their corresponding digits are identical. Alternately, if any corresponding digits differ, then the two points differ.

For consider two arbitrary points, PA_1 and PA_2, in their base prime representations:

$PA_1 = .a_0, a_1, \ldots\ \varepsilon_k, a_{k+1}, \ldots\ a_n, \ldots \ldots$

$PA_2 = b_0, b_1, \ldots\ b_k, b_{k+1}, \ldots\ \varepsilon_n, b_{n+1} \ldots$

a. If $a_i == b_i$ for all i, and $\varepsilon_k = b_k$ and $a_n = \varepsilon_n$ then $PA_1 == PA_2$

The proof is isomorphic to Section 2.2a above.

b. If $PA_1 == PA_2$ then $a_i == b_i$ for all i, and $\varepsilon_k = b_k$ and $a_n = \varepsilon_n$.

The proof is isomorphic to Section 2.2b above.

This is why $1/2 < \delta_{ki} < 1$, to still satisfy the inequality of (14).

4.2.4.5 *Each distinct DS point maps to a unique point in A*:

Consider the mapping of any two distinct DS points. Distinct implies a difference of at least one digit in their respective base-prime representations.

DS Set: → Set A

$P_1 = .a_0, a_1, \ldots\ a_k, a_{k+1}, \ldots\ a_n, a_{n+1} \ldots \rightarrow .a_0, a_1, \ldots\ \varepsilon_k, a_{k+1}, \ldots a_n, \ldots \ldots$.

$P_2 = .b_0, b_1, \ldots\ b_k, b_{k+1}, \ldots b_n, b_{n+1} \ldots \rightarrow .b_0, b_1, \ldots\ \varepsilon_i, bi_{+1}, \ldots b_n, \ldots \ldots$

From 4.2.4.3 above, the two mapped points in set A are distinct.

4.2.4.6 *Set A has null intersection with set DS*:

Proof by Contradiction:

Given any point Pa in set A, assume it is equal to a point Pds in DS. Or in base-prime:

$Pa = .a_0, a_1, \ldots\ \varepsilon_k, a_{k+1}, \ldots a_n, \ldots \ldots$

$?== Pds = .b_0, b_1, \ldots\ b_k, b_{k+1}, \ldots\ b_n, b_{n+1} \ldots$

Consider the corresponding DS point of Pa with $\varepsilon_k \rightarrow a_k$, from the substitutions of (24):

$Pa\_ds = .a_0, a_1, \ldots\ a_k, a_{k+1}, \ldots a_n, \ldots \ldots$

From (25): $Pa - Pa\_ds = d_k$ or $Pa\_ds = Pa - d_k$

From (15): $Pds - Pa\_ds > \delta_k/(p_0 * .. * p_i) * 1/2$ where i is the first digit Pds & Pa_ds differ.

Upon adding d_k to both sides of the inequality above:

$Pds - Pa\_ds + d_k > \delta_k/(p_0 * .. * p_i) * 1/2 + d_k$

Upon substituting (25) into the inequality above:

$Pds - Pa > \delta_k/(p_0 * .. * p_i) * 1/2 + d_k$

But: $\delta_k/(p_0*..*p_i)*1/2 + d_k$!= 0 for all i, including k=i.

That contradicts the hypothesis that Pds = Pa.

Therefore Pds != Pa and there is null intersection between sets A and DS.

This is important for it assures that all points in DS exhaustively map to set A.

4.2.4.7 *Mapping of DS to A is exhaustive*:

By definition of the mapping in (24) and (25) in Section, 4.2.2.1, all DS points are mapped.

4.2.5 Theorem: DS < A

Will now identify points in set A not mapped from DS.

Consider any point Pa_1 in A mapped from DS. That is:

$Pa_1 = .a_0, a_1, \ldots\ \varepsilon_k, a_{k+1}, \ \ldots..\ \ a_n, \ldots.\ \ldots..$

where:

$k = 0, \ldots$ &

$\varepsilon_k = \delta_k/(p_0*..*p_k)$ &

$a_k = 0 \rightarrow \delta_{k0}$ &

$a_k = 1 \rightarrow \delta_{k1}$ &

$1/2 < \delta k_0,\ \ \delta_{k1} < 1$

Then consider:

$Pa = .a_0, a_1, \ldots\ \varepsilon_k + \Delta_k/(p_0*..*p_k), a_{k+1}, \ \ldots..\ \ a_n, \ldots.\ \ldots..$

where $1/2 < \Delta_k < 1/(p_0*..*p_k)$ and Δ_k != δ_{k0} or δ_{k1}

It is also a member of set A, for it satisfies the condition of (23) for membership. It was not mapped into by (26) since Δ_k != δ_{k0} or δ_{k1}.

There are a Cardinality(R) number of such values, none in set DS.

Hence Cardinality (DS) < Cardinality (A)

4.3 **Yet Another Proof: Construction of an R Set > DS:**

A set A is constructed with cardinality(R) and for which there is no 1-to-1 mapping between it and DS. That is, DS will exhaustively map to set A. But A will have remaining points not mapped, thereby showing Cardinality(R) > Cardinality (DS). The proof is based fundamentally on the dichotomy of integers into composite and prime numbers.

4.3.1 Description of Set A:

$$\text{Consider set } A = \{ a \mid a = \sum_{n=2}^{\infty} a_n/n! \ \text{ where } a_k \,\varepsilon\, \{0,1\} \} \qquad (27)$$

4.3.2 Decimal-Like Representation of Set A:

Each point can be represented in a "factorial-base" as:

$$= \sum_{n=2}^{\infty} a_n/n! = d_2, d_3, \ldots d_k, \ldots. d_n \ldots \infty \text{ where } a_k/k! \rightarrow d_k$$

Then a point d ε A is represented as:

$$d_2, d_3, \ldots d_k, \ldots. d_n \ldots == a_2/2! + a_3/3! + \ldots + a_k/k! + \ldots. + a_n/n! + \ldots \quad (28)$$

4.3.3 Cardinality of Set A:

Set A is shown to have Cardinality(R). First, a diagonalization proof shows it to be greater than Cardinality(N). Then a combinatorics argument shows it to be equal to R.

4.3.3.1 A vs. N:

Assume that set A is only countable infinite. Listing its points effectively enumerates them, mapped by an implicit integer index. Consider any enumeration of its elements in its factorial-base form, as follows:

$a_{0,2}, a_{0,3}, \ldots\ldots a_{0,k}, \ldots \infty$ where $i == i^{th}$ row and $a_{i,k} == a_{i,k}/k!$

.

.

$a_{i,2}, a_{i,3}, \ldots\ldots a_{i,k}, \ldots \infty$

.

.

Consider a point $b_0, b_1, \ldots, b_k, \ldots.$, constructed as follows:

If $a_{k,k}$ ==1 then set b_k=0. Otherwise set b_k=1

Then $b_0, b_1, \ldots, b_k, \ldots..$ necessarily differs from each of the purportedly enumerated elements of A by at least one digit. Regardless of enumerated set, this diagonalization procedure will generate a point that is not in that enumerated set. Therefore there is at least one point in A not in N, hence Cardinality (N) < Cardinality (A).

4.3.3.2 A vs. R:

Since each digit in A has 2 choices, there are then 2^{**n} points in set A. This is the same number as R, per (17). Alternately, there is a 1-to-1 correspondence between 1/n! and $1/10^n$ (in a decimal). Therefore Cardinality (A) == Cardinality (R)

4.3.4 Convergence of Set A:

Each point converges since:

$$\sum_{n=0}^{\infty} x/n! \text{ converges for all } x \rightarrow \sum_{n=2}^{\infty} x/n! \text{ converges for all } x$$

4.3.5 Ordering Theorem Generalization:

Note that the Ordering Theorem in Sec. 2.1 had denominators of the form: $\prod p_i$ whereas here, denominators have the form: n! Then consider this generalization:

Given 2 points in set A, P_1 and P_2, such that their digits match up to and including the k^{th}, where k+1= 0,.., then the point with the lesser $(k+1)^{st}$ value, is less than the other. Or:
If:

$P_1 = 0.a_0, a_1, \ldots a_k, a_{k+1}, \ldots a_{k+n}, \ldots.. \infty$

$P_2 = 0.a_0, a_1, \ldots a_k, b_{k+1}, \ldots b_{k+n}, \ldots.. \infty$

where $a_i == b_i$ for 0 =< i =< k, and $a_{k+1} < b_{k+1}$ then $P_1 < P_2$

The proof is identical to Section 2.1 above, by simply replacing: $\prod p_i \rightarrow n!$

4.3.6 Uniqueness of Points in Set A:

Given two arbitrarily selected points, P_1 and P2 in set A, in base-factorial:

$P_1 = a_0, a_1, \ldots a_k, a_{k+1}, \ldots a_{k+n}, \ldots..$

$P_2 = b_0, b_1, \ldots b_k, b_{k+1}, \ldots b_{k+n}, \ldots..$

a. If $a_k == b_k$ for all k, then $P_1 == P_2$
   The proof is isomorphic to Section 2.2a above.

b. If $P_1 == P_2$ then $a_k == b_k$ for all k.

The proof is isomorphic to Section 2.2b above.

Briefly, consider the first digit where they differ. Then by the Ordering Theorem in Section 2.1.1, either $P_1 < P_2$ or $P_1 > P_2$. Both results negate the premise that $P_1 == P_2$. Therefore there is no digit where they differ. Or $a_k == b_k$ for all k.

This uniqueness property of set A is needed below, when identifying set A points not mapped from DS, to insure that they are not in DS.

4.3.7 Refactoring Set A in DS-like Form:
Set A points will be refactored into a DS-like form. This will enable direct counting comparisons of set A vis-a-via set DS.

For instance, given:

$a = 1,0,0,1,1 \ldots == 1/2! + 0/3! + 0/4! + 1/5! + 1/6! + \ldots$

On collecting composite factors in each term's numerator:

$a = (1/1)/(2) + (0/1)/(2*3) + (0/1*4)/(2*3) + (1/1*4*6)/(2*3*5) + \ldots$

Then collecting terms with the same denominator (of primes only) yields:

$a = (1/2) + ((0/1)+0/1*4))/(2*3) + (1/1*4*6)/(2*3*5) + \ldots$

This point in set A now has the same format as a DS point, but with a rational numerator instead of 0/1. This reformatting will enable a direct comparison of DS vs. set A points.

More generally, from the dichotomy property of integers as either composite ('c') or primes ('p'), k! can be refactored into composites and primes. Thus, consider k! = 1*2*3…k for any $k=p_r$ for some r. When refactored, it becomes:

$k! = c_0 \ldots c_m, \ldots p_0 \ldots p_r$ (29)

In order to better associate composites with their respective primes, consider the following added subscript:

Let $c_{i,j}$ be the i^{th} composite associated with the j^{th} prime, appearing after the j-1^{st} prime. Then, after the m^{th} composite before p_k, the n composites after p_k, less than p_{k+1}, are:

$\ldots c_{m,k} * p_k \quad c_{m+1,k+1} \ldots c_{m+n-1,k+1}, c_{m+n,k+1}. \quad p_{k+1} \ldots\ldots$

Then each factorial inclusively between those primes p_k and p_{k+1}, can be refactored as:

$k! = c_{0,0} * \ldots * c_{m,k} * p_0 \ldots p_k$ (30)

$(k+1)! = c_{0,0} * \ldots * c_{m,k} * c_{m+1,k+1} * p_0 \ldots p_{k+1}$

…

$(k+n-1)! = c_{0,0} * \ldots * c_{m,k} * c_{m+1,k} * \ldots * c_{m+n-1,k+1} * p_0 \ldots p_{k+1}$

$(k+n)! = c_{0,0} * \ldots * c_{m,k} * c_{m+1,k} * \ldots * c_{m+n-1,k} * c_{m+n,k+1} * p_0 \ldots p_k \, p_{k+1}$

Substitute the above into (28) and place the composites in the numerator as follows:

$d_k = a_k/k! = (a_k/c_{0,0} \ldots c_{m,k}))/p_0 \ldots p_k$ (31)

……..

$d_{k+n-1} = a_{k+n-1}/(k+n-1)! = (a_{k+n-1}/c_{0,0} \ldots c_{m,k} \, c_{m+1,k+1} \ldots c_{m+n-1,k+1})/p_0 \ldots p_{k+1}$

$d_{k+n} = a_{k+n}/(k+n)! = (a_{k+n}/c_{0,0} \ldots c_{m,k} \, c_{m+1,k+1} \ldots c_{m+n-1,k+1}, c_{m+n,k+1})/p_0 \ldots p_k * p_{k+1}$

Then, from (28) and substituting terms from (31):

$$\sum_{k=2}^{\infty} a_k/k! = d_2, d_3, \dots d_k, \dots. d_n \dots \qquad (32)$$

$$= (a_2/1)/p_0 + \dots.. \qquad \text{where } p_0=2$$

$$\dots\dots. +$$

$$(a_{k-1}/c_{0,0}\dots c_{m-1,k})/p_0\dots p_k +$$

$$(a_k/c_{0,0}\dots c_{m,k})/p_0\dots p_k +$$

$$\dots\dots. +$$

$$(a_{k+n-1}/c_{0,0}\dots c_{m,k}\ c_{m+1,k+1}\ \dots\ c_{m+n-1,k+1})/p_0\dots p_{k+1} +$$

$$(a_{k+n}/c_{0,0}\dots c_{m,k}\ c_{m+1,k}\ \dots\ c_{m+n-1,k},\ c_{m+n,k+1})/p_0\dots p_k{*}p_{k+1} +$$

$$\dots\dots\dots$$

Consolidating partial sums with the same denominator, results in:

$$\sum_{k=2}^{\infty} a_k/k! = (\ (a_2/1\ )/p_0\ + \qquad \text{where } p_0=2 \qquad (33)$$

$$\dots\dots +$$

$$(\ (\dots.. + a_{k-1}{*}c_{m,k} + a_k)/(c_{0,0}\dots c_{m,k})\ )/p_0\dots p_k$$

$$+$$

$$(\ (\dots..+ a_{k+n-1}{*}\ c_{m+n,k+1} +\ a_{k+n})/(c_{0,0}\dots c_{m,k-1}c_{m+1,k}\ \dots\ c_{m+n-1,k},\ c_{m+n,k+1})\ )/p_0\dots p_k p_{k+1}\ +$$

$$\dots\dots\dots.$$

This summation is simplified as follows:

First simplify the numerator of each numerator of each summation term above. Let:

$$b_2 = a_2$$

$$\dots.$$

$$b_k = (\dots.. + a_k{*}c_{m+1,k})$$

$$b_{k+1} = (\dots..+ a_{k+n-1}{*}\ c_{m+n,k+1} +\ a_{k+n}) \qquad (34)$$

$$\dots..$$

Then simplify the numerator of each summation term in (33):

$$e_{2,} = b_2/1$$

$$\dots.$$

$$e_k = b_k/(c_{0,0}\dots c_{m,k}) \qquad (35)$$

$$e_{k+1} = b_{k+1}/(c_{0,0}\dots c_{m,k-1}c_{m+1,k}\ \dots\ c_{m+n-1,k},\ c_{m+n,k+1})$$

$$\dots.$$

The summation (33) is now more compactly expressed as:

$$\sum_{k=2}^{\infty} a_k/k! = e_2/p_0 + \ldots \; \ldots\ldots + \; e_k/p_0 \ldots p_k + \; e_{k+1}/p_0 \ldots p_k p_{k+1} + \; \ldots\ldots \quad (36)$$

Continue to simplify each summation term as follows: Let:

$f_2 = e_2/p_2$

....

$f_k = e_k/(p_0 \ldots \ldots p_k)$ (37)

$f_{k+1} = e_k/(p_0 \ldots \ldots p_k p_{k+1})$

....

Then:

$$\sum_{k=2}^{\infty} a_k/k! = \sum_{k=2}^{\infty} f_{k,}$$

Or, in the equivalent base-factorial notation:

$$\sum_{k=2}^{\infty} a_k/k! = f_2, f_3, \ldots\ldots f_{k, \ldots} \infty \quad (38)$$

Note that it does not matter for the proof whether each numerator < 1.

(38) shows how a point in set A, can be represented in a DS-like form, with the same respective denominators as a DS point.

4.3.8 <u>Set DS-to-Set A Mapping:</u>

Consider the following mapping from DS-to-A:

Set DS	Set A	
$h_0, h_1, \ldots\ldots h_{k, \ldots} \infty$ →	$f_2, f_3, \ldots\ldots f_{k, \ldots} \infty$	(39)

where:

$h_k = m_k/(p_0 \ldots\ldots\ldots p_k)$ → $f_{k,} = e_k/(p_0 \ldots \ldots p_k)$

and $m_k \varepsilon \{0,1\}$ and $e_{k,} = b_k/(c_{0,0} \ldots c_{m,k})$

and $b_k = \ldots.. + a_{k-1} * c_{m,k} + a_k$

and $a_i \varepsilon \{0,1\}$

Bringing it all together:

$$\sum_{k=0}^{\infty} m_k/(p_0 \ldots\ldots p_k) \rightarrow \sum_{k=2}^{\infty} e_k/(p_0 \ldots\ldots p_k) == \sum_{k=2}^{\infty} a_k/k! \quad (40)$$

Therefore $m_k \rightarrow \ldots.. + a_{k-1}*c_{m,k} + a_k$

Or, in simplified tuple form, since the c_k are fixed values:

$$m_k \rightarrow (\ldots.,a_{k-1,} a_k) \quad (41)$$

Within the mapping (39) above, each m_k digit will map to the first a_i of its associated tuple in (41). Then the remaining a_j associated with m_k are fixed at an arbitrary value, 0 for simplicity. That is:

$m_k = 0 \rightarrow (a_i =0, \ldots a_{k-1}=0_, a_k=0, \ldots)$

$m_k = 1 \rightarrow (a_i =1, \ldots a_{k-1}=0_, a_k=0, \ldots)$

$$\text{Therefore } m_k \rightarrow (m_{i\ldots\ldots}0_, 0) \quad (42)$$

Further, the respective denominators of DS vs. A in (41) are now identical per (40). This is needed for comparable counting of their respective like-formed points.

4.3.8.1 This mapping exhaustively maps all DS points into set A points:
From (39) and (42), there is at least one a_i in a set A point, for each m_k to map into. Therefore every DS point is mapped to a corresponding point in set A, to exhaustively map to set A.

4.3.8.2 This maps each DS point to a unique point in set A:
For, from (39) and (42), each m_k maps to a distinct sequence of $\{a_{i\ldots} a_k\}$.
Therefore each DS point maps into a distinct sequence of a point, $\{f_0 \ldots f_k \ldots.\}$, in set A point. Or:

$$h_0, h_1, \ldots\ldots h_k, \ldots \infty \rightarrow f_2, f_3, \ldots\ldots f_k, \ldots \infty \rightarrow \sum_{k=2}^{\infty} a_k/k! \quad (43)$$

But from Section 4.3.6, each sequence of $\{a_k\}$ is unique. Thus, a DS point maps to a unique point in set A.

4.3.8.3 Sets A and DS are non-intersecting:
For if there was a DS point in set A, then, from (39), its DS digit-by-digit representation should equal its set A digit-by-digit representation. But by (41) and (42), m_k != $(\ldots.,a_{k-1,} a_k)$ for any k. Therefore there is no point in DS that is also in set A. That is, they are non-intersecting.

4.3.8 <u>Proof that Cardinality(DS) < Cardinality(A):</u>

It was established above that:

- Sets DS and A are non-intersecting.
- DS is exhaustively mapped into A.
- Both DS and A points are unique.

It is now shown that set A has a least one point not mapped by DS (and actually at least a countable number).

From (42) again, to establish that all DS points are exhaustively mapped to set A:

Digit in a DS point: → Digit in a point in set A:

m_k → $(m_{i.......}0_, 0)$

More generally, for a m_i digit in a DS point, and ignoring the fixed composites $c_{i,}$ and identical denominators:

Digit in a DS point: → Digit in a point in set A:

m_i → $(m_{i.......}a_k \dots \;\; ..... + a_n)$ (44)

From (44), combining m_i of each DS point into a tuple:

DS point: → Set A point:

$(m_{0\,,\,....}m_k \dots)$ → $(m_0, a_1 \dots a_{i,\,....}\, a_{n,}\; \dots m_{k,\,........})$

This mapping leaves many degrees-of-freedom for values of the unmapped $a_{i\,.}$ Thus setting even one a_i =1 in $(m_0, a_1 \dots a_{i,\,....}\, a_{n,}\; \dots m_{k,\,........})$ yields a new point in set A that was not mapped into from DS. For example, this set A point could not have mapped into from DS, by definition of the mapping in (39) and (42):

$(m_0, 1 \dots 1_{,\,....}\, 1_,\; \dots m_{k,\,........})$

Therefore: Cardinality(DS) < Cardinality(A)

There are actually at least a countable number of a_i with varied values to generate new set A points not in DS. Each unmapped set A point not in DS is generated by new combinatorics of a_i where i > 0.

Then, mindful of the argument in Section 4.2.1 that if one mapping from DS-to-R is not 1-to-1, then no other mappings from DS-to-R are 1-to-1, this thereby proves that:

Cardinality(DS) < Cardinality(A).

5. **Conclusion:**

The proofs in the various sections above are combined as follows:

From Section III:

Cardinality(N) < Cardinality (DS)

From (22), where <~ denotes asymptotic inequality:

Cardinality (DS) <~ Cardinality (R)

From Sections 4.2 and 4.3:

Cardinality (DS) < Cardinality (R)

Therefore:

Cardinality(N) < Cardinality (DS) < Cardinality (R)

This result relies entirely on any of three unique properties of prime numbers; asymptotic scarcity, incommensurate reciprocals and the dichotomy of N into primes and composites. Hence it is not readily generalizable to higher order sets addressed by the Generalized Continuum Hypothesis [4]. Indeed, it is surmised that the research to-date on CH missed these conclusions, due to over-generalized arguments applied to all Alpha sets. That is, they did not account for primes specifically in Aleph$_0$/Aleph$_1$ sets.

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