The Geometrodynamic Foundation of Electrodynamics: A Brief Summary

Jay R. Yablon
910 Northumberland Drive
Schenectady, New York 12309-2814
yablon@alum.mit.edu

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Abstract: We summarize how the Lorentz Force motion observed in classical electrodynamics may be understood as geodesic motion derived by minimizing the variation of the proper time along the worldline of test charges in external potentials, while the spacetime metric remains invariant under, and all other fields in spacetime remain independent of, any rescaling of the charge-to-mass ratio q/m. In order for this to occur, time is dilated or contracted due to attractive and repulsive electromagnetic interactions respectively, in very much the same way that time is dilated due to relative motion in special relativity. As such, it becomes possible to lay an entirely geometrodynamic foundation for classical electrodynamics.

The equation of motion for a test particle along a geodesic line in curved spacetime as specified by the metric interval $c^2 d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu$ with metric tensor $g_{\mu\nu}$ was first obtained by Albert Einstein in §9 of his landmark 1916 paper [1] introducing the General Theory of Relativity. The infinitesimal linear element $d\tau = ds/c$ for the proper time is a scalar invariant that is independent of the chosen system of coordinates. Likewise the finite proper time $\tau = \int_A^B d\tau$ measured along the worldline of the test particle between two spacetime events A and B has an invariant meaning independent of the choice of coordinates. Specifically, the geodesic of motion is stationary, and satisfies the variational minimization equation

$$0 = \delta \int_A^B d\tau . \tag{1}$$

Simply put, a material particle goes from event A to event B in the physically shortest possible proper time. After carrying out the well-known calculation of [1] for which there is a very good online review at [2], this equation of motion is found to be:

$$\frac{d^2 x^\beta}{d\tau^2} = \frac{du^\beta}{d\tau} = -\Gamma^\beta{}_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = -\Gamma^\beta{}_{\mu\nu} u^\mu u^\nu , \tag{2}$$

with the Christoffel connections defined by $-\Gamma^\beta{}_{\mu\nu} \equiv \frac{1}{2} g^{\beta\alpha} \left(\partial_\alpha g_{\mu\nu} - \partial_\mu g_{\nu\alpha} - \partial_\nu g_{\alpha\mu} \right)$ and the relativistic four-velocity by $u^\mu \equiv dx^\mu / d\tau$. This geodesic (2) represents the path along which the proper time is minimized, again, the shortest proper time between two events.

If the test particle, to which we now ascribe a mass $m>0$, also has a non-zero net electrical charge $q \neq 0$ and the region of spacetime in which it subsists also has a nonzero electromagnetic field strength $F^{\beta\alpha} \neq 0$ defined by $F^{\beta\alpha} \equiv \partial^\beta A^\alpha - \partial^\alpha A^\beta$ in relation to the gauge potential four-vector A^α, with $F^{\beta\alpha}$ containing the electric and magnetic field bivectors **E** and **B**, then the equation of motion is no longer (2), but is supplemented by an additional term which contains the Lorentz force law, namely:

$$\frac{d^2 x^\beta}{d\tau^2} = \frac{du^\beta}{d\tau} = -\Gamma^\beta{}_{\mu\nu} \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} + \frac{q}{m} g_{\sigma\alpha} F^{\beta\alpha} \frac{dx^\sigma}{cd\tau} = -\Gamma^\beta{}_{\mu\nu} u^\mu u^\nu + \frac{q}{m} g_{\sigma\alpha} F^{\beta\alpha} \frac{u^\sigma}{c}, \tag{3}$$

The above is also well-known, observed, settled physics, see, e.g., the online review [3].

Given that the gravitational geodesic (2) specifies a path of minimized proper time (1), the question arises whether there is a way to obtain (3) from the same variation (1), thus revealing the electrodynamic motion to also be one of particles moving through spacetime along paths of minimized proper time. Philosophically, it cannot be argued other than that this would be a desirable state of affairs. But physically the difficulty rests in how to do this without ruining the integrity of the metric and the background fields in spacetime by making them a function of the charge-to-mass ratio q/m, because this ratio is and must remain a characteristic of the test particle alone. It is not and cannot be a characteristic of the metric $d\tau$ or the metric tensor $g_{\mu\nu}$ or the gauge field A^α or the field strength $F^{\beta\alpha}$ which define the field-theoretical spacetime background through which the test particle is moving. And, at bottom, this difficulty springs from the *inequivalence* of the "electrical mass" a.k.a. charge *q* and the inertial mass *m*, versus the Galilean equivalence of the gravitational and inertial mass. In (3), this is captured by the fact that *m* does *not* appear in the gravitational term $-\Gamma^\beta{}_{\mu\nu} u^\mu u^\nu$, while the q/m ratio *does* appear in the electrodynamic Lorentz force term that we rewrite as $(q/m) F^\beta{}_\sigma u^\sigma$ in natural $c=1$ units.

This may also be seen very simply if we compare Newton's law with Coulomb's law. In the former case we start with a force $F=-GMm/r^2$ (with the minus sign indicating that gravitation is attractive) and in the latter $F=-k_e Qq/r^2$ (for which we choose an attractive interaction), where G is Newton's gravitational constant and the analogous $k_e = 1/4\pi\varepsilon_0 = c^2\mu_0/4\pi$ is Coulomb's constant. If the gravitational field is taken to stem from *M* and the electrical field from *Q*, then the test particle in those fields has gravitational mass *m* and electrical mass *q*. But the Newtonian force $F=ma$ always contains the inertial mass *m*. So in the former case, because the gravitational and inertial mass are equivalent, the acceleration $a=Fm=-GMm/mr^2=-GM/r^2$ and these two masses cancel, hence $-\Gamma^\beta{}_{\mu\nu} u^\mu u^\nu$ without any mass in (3). But in the latter case the acceleration $a=F/m=-k_e Qq/mr^2=-(q/m)k_e Q/r^2$ because the electrical and inertial masses are not equivalent, hence $(q/m) F^\beta{}_\sigma u^\sigma$ containing this same ratio in (3), and the motion is distinctly dependent on the electrical and inertial masses *q* and *m* of the test particle, even though different charges *q* with different masses *m* may all be moving through the exact same background fields.

So, were we to pursue the philosophically-attractive goal of understanding electrodynamic motion as the result of particles moving through spacetime along paths of minimized proper time, with (1) applying to electrodynamic motion just as it does to gravitational motion, the metric element $d\tau$ would inescapably have to be a function $d\tau(q/m)$ of q/m. And this in turn would *appear* to violate the integrity of the metric element $d\tau$ as well as the metric tensor $g_{\mu\nu}$ in $c^2 d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu$, because these would all *seem to be* dependent upon the attributes *q* and *m* of the test particles that are moving through the spacetime background. Were this to be reality and not just seeming appearance, this would be is physically impermissible. Consequently, despite there being many known derivations of the Lorentz force law, there does not, to date, appear to be an acceptable rooting of the Lorentz force law in the variational equation $0=\delta\int_A^B d\tau$ which would reveal electrodynamic motion to be geodesic motion as is gravitational motion. And this is precisely because it is not understood how to do this while simultaneously maintaining the integrity of field theory such that the metric and the background fields do not depend upon the attributes of the test particles which may move through these fields. This, in turn, is because electrical mass is not equivalent to inertial mass in contrast to what is the Galilean equivalence of gravitational and inertial mass.

Nevertheless, it can be shown that we can in fact have a metric element $d\tau(q/m)$ which is a function of the electrical-to-inertial mass ratio q/m, from which the variational equation $0=\delta\int_A^B d\tau$ does yield the combined gravitational and electrodynamic equation of motion (3), yet for which the metric $d\tau$ and the metric tensor $g_{\mu\nu}$ and the gauge field A^α and the field strength $F^{\beta\alpha}$ are all independent of this q/m ratio. This seemingly-paradoxical result of having the metric be a *mathematical function of* q/m yet be *physically independent of* q/m reveals that when a first test particle with electrical mass *q* and inertial mass *m* is placed in a field $F^{\beta\alpha}$, and a second test particle with electrical mass q' and inertial mass m' and a different ratio $q'/m' \neq q/m$ is placed at equipotential in the same field $F^{\beta\alpha}$, the observably-different Lorentz force motions for these two different test particles even though they are at equipotential is the consequence of the fact that *time does not flow at the same rate for these two test particles in very much the same way that time does not flow at the same rate for two reference frames in special relativity which are in motion relative to one another*.

Specifically, it will be appreciated that the Lorentz motion in (3) also contains a set of coordinates x^μ, so that in the absence of gravitation with $g_{\mu\nu}=\eta_{\mu\nu}$ and $\Gamma^\beta{}_{\mu\nu}=0$, the first test particle will have a Lorentz motion given by:

$$\frac{d^2x^\beta}{d\tau^2}=\frac{q}{m}\eta_{\sigma\alpha}F^{\beta\alpha}\frac{dx^\sigma}{cd\tau}. \tag{4}$$

Ordinarily, it is assumed that for the second test particle, the motion is given by this same equation (4), merely with the substitution of $q\to q'$ and $m\to m'$, that is, by:

$$\frac{d^2x^\beta}{d\tau^2}=\frac{q'}{m'}\eta_{\sigma\alpha}F^{\beta\alpha}\frac{dx^\sigma}{cd\tau}. \tag{5}$$

The particular assumption is that there is no change in the rate at which time flows as between (4) and (5), and more generally the assumption is that the coordinate interval dx^σ in (4) is identical to the dx^σ in (5). Yet, it is impossible to have both (4) and (5) emerge through the variation $0=\delta\int_A^B d\tau$ from the same metric element $d\tau$, and simultaneously maintain field theory integrity, unless the coordinates are different, wherein dx^σ in (4) is *not identical* to what must now be $dx^\sigma \to dx'^\sigma \neq dx^\sigma$ in (5).

In fact, the very physics of having electric charges in electromagnetic fields induces a change in coordinates as between these two test charges with different $q'/m' \neq q/m$, very similar to the coordinate change via a Lorentz transformations induced by relative motion, whereby the electrodynamic motion of the second test charge is given not by (5), but by:

$$\frac{d^2x'^\beta}{d\tau^2}=\frac{q'}{m'}\eta_{\sigma\alpha}F^{\beta\alpha}\frac{dx'^\sigma}{cd\tau}. \tag{6}$$

Here, x^β and $x'^\beta \neq x^\beta$ are different sets of coordinates, yet they are interrelated by a definite transformation one to the other. Most importantly, this results in *time itself* being induced to flow differently as between these two sets of coordinates, making time dilation and contraction as fundamental an aspect of electrodynamics, as it already is of the special relativistic theory of motion and the general relativistic theory of gravitation. In fact, what is really happening – physically – is that the placement of a charge in an electromagnetic field is *inducing a physically-observable change of coordinates* $x^\beta(q/m) \to x'^\beta(q'/m')$ in the very same way that relative motion between the coordinate systems $x^\beta(v)$ and $x'^\beta(v')$ of two different reference frames with velocities v and v' induces a Lorentz transformation $x^\beta(v) \to x'^\beta(v')$ which with relates both coordinate systems to one another via $c^2d\tau^2=\eta_{\mu\nu}dx^\mu(v)dx^\nu(v)=\eta'_{\mu\nu}dx'^\mu(v')dx'^\nu(v')$ with the invariant metric element $c^2d\tau^2=c^2d\tau'^2$ and with the same metric tensor $\eta_{\mu\nu}=\eta'_{\mu\nu}$.

The metric which in fact yields (3) from (1) so as to include electrodynamic motion is:

$$c^2d\tau^2=g_{\mu\nu}\left(dx^\mu+\frac{q}{mc}d\tau A^\mu\right)\left(dx^\nu+\frac{q}{mc}d\tau A^\nu\right)=g_{\mu\nu}Dx^\mu Dx^\nu, \tag{7}$$

where we define a gauge-covariant coordinate interval $Dx^\mu \equiv dx^\mu+(q/mc)d\tau A^\mu$. And it will be seen that upon multiplying through by m^2c^2 and dividing through by $d\tau^2$ this becomes:

$$m^2c^4 = g_{\mu\nu}\left(mc\frac{dx^\mu}{d\tau}+qA^\mu\right)\left(mc\frac{dx^\nu}{d\tau}+qA^\nu\right) \tag{8}$$

for the squared rest energy of the invariant rest mass. This $Dx^\mu \equiv dx^\mu + (q/mc)d\tau A^\mu$ is a direct outgrowth of the gauge-covariant derivatives $D_\sigma \equiv \partial_\sigma + ieA_\sigma$ and canonical momenta $\pi^\sigma \equiv p^\sigma - eA^\sigma$ which emerge from gauge theory and in particular from the mandate for gauge (really, phase) symmetry.

Now, this metric (7) is clearly a function of q/m and so has the *appearance* of depending on the ratio q/m. But this is only appearance. For, when we now place the second test charge with the second ratio $q'/m' \neq q/m$ in the exact same metric measured by the invariant element $d\tau$ and moving through the exact same fields $g_{\mu\nu}$ and A^μ, this metric becomes:

$$c^2d\tau'^2 = c^2d\tau^2 = g_{\mu\nu}\left(dx'^\mu + \frac{q'}{m'c}d\tau A^\mu\right)\left(dx'^\nu + \frac{q'}{m'c}d\tau A^\nu\right) = g_{\mu\nu}Dx'^\mu Dx'^\nu . \tag{9}$$

So despite $d\tau$ being a *function of* the q/m ratio, this $d\tau = d\tau'$ as a measured proper time element is actually *invariant* with respect to the q/m ratio because *the differences between different q/m and q'/m' are entirely absorbed into the coordinate transformation $x^\mu \rightarrow x'^\mu$ which is wholly analogous to the Lorentz transformation of special relativity*. In fact, this transformation $x^\mu \rightarrow x'^\mu$ is *defined* so as to keep $d\tau = d\tau'$ and $g_{\mu\nu} = g'_{\mu\nu}$ and $A^\mu = A'^\mu$ and by implication the field strength bivector $F^{\beta\alpha} = F'^{\beta\alpha}$ all unchanged, just as Lorentz transformations are defined so as to maintain a constant speed of light for all inertial reference frames independently of their state of motion. So $d\tau = d\tau'$ is a function of charge q and mass m yet is invariant with respect to the same, and there is no paradox in having $d\tau = d\tau'$ be a function of, yet be invariant under, a rescaling of the q/m ratio. Likewise, the fields $g_{\mu\nu} = g'_{\mu\nu}$ and $A^\mu = A'^\mu$ are independent of the charge and the mass of the test particle, because again, *everything* emanating from the different ratios q/m and q'/m' is absorbed into a coordinate transformation $x^\mu \rightarrow x'^\mu$. Thus, while "gauge" is a historical misnomer for what is really invariance under local *phase* transformations $\psi \rightarrow \psi' = U\psi = e^{i\Lambda(t,\mathbf{x})}\psi$ applied to a wavefunction ψ, what we see contrasting (7) and (9) is that the metric truly is invariant under what can be genuinely called a *re-gauging* of the q/m ratio.

As a result, each and every different test particle carries its own coordinates all interrelated so as to keep $d\tau$ invariant and $g_{\mu\nu}$, A^μ and $F^{\beta\alpha}$ unchanged. The coordinate transformation interrelating all the test particles causes time to dilate for electrical attraction and to contact for repulsion, with a dimensionless ratio $dt/d\tau = dx^0/d\tau \equiv \gamma_{em}$ that integrally depends upon the magnitude of the likewise dimensionless ratio qA^μ/mc^2 of electromagnetic interaction energy qA^μ to the test particle's rest energy mc^2. This supplements the ratio $dt/d\tau = \gamma_v = 1/\sqrt{1-v^2/c^2}$

for motion in special relativity and $dt/d\tau = \gamma_g = 1/\sqrt{g_{00}}$ for a clock at rest in a gravitational field, and assembles them in the overall product combination $dt/d\tau = \gamma_{em}\gamma_g\gamma_v$ governing time dilation when all of motion and gravitation and electromagnetic interactions are present.

Operationally, the electromagnetic contribution γ_{em} to this time dilation or contraction would be measured in principle by comparing the rate at which time is kept by otherwise identical, synchronized geometrodynamic clocks or oscillators which are then electrically charged with different q/m ratios, and then placed at rest into a background potential $A^{\mu} = (\phi, \mathbf{A}) = (\phi_0, \mathbf{0})$ at equipotential, where φ_0 is the proper potential. More generally, this would be measured by electrically charging otherwise identical clocks and then placing them into the potential to have differing $qA^0/mc^2 = q\phi_0/mc^2$ ratios.

Empirically, for $q\phi_0/mc^2 \ll 1$, the interaction energies $E_{em} = \int F dr = +k_e Qq/r$ sans integration constant for an attractive Coulomb force $F = -k_e Qq/r^2$ are related to these electromagnetic time dilations in a fashion identical to how the kinetic energy in special relativity is observed to be the quantity $E_v = \frac{1}{2}mv^2$ in $mc^2\gamma_v = mc^2/\sqrt{1-v^2/c^2} \cong mc^2 + \frac{1}{2}mv^2$ for nonrelativistic velocities $v \ll c$. In fact, the actual expression for the electromagnetic contribution to the time dilation is $\gamma_{em} = 1 - q\phi_0/mc^2$ and for a Coulomb proper potential $\phi_0 = k_e Q/r$, this is $\gamma_{em} = 1 - k_e Qq/mc^2 r$ for an electrical interaction chosen to be attractive like gravitation. So the earlier-referenced combined time dilation $dt/d\tau = \gamma_{em}\gamma_g\gamma_v$, employing the gravitational factor $\gamma_g = 1/\sqrt{g_{00}(r)} \cong 1 + GM/c^2 r$ in the weak field Newtonian limit, produces an overall energy which, in the low velocity, weak-gravitational and electromagnetic interaction limit, is given by:

$$\begin{aligned} E &= mc^2 \frac{dt}{d\tau} = \gamma_{em}\gamma_g\gamma_v mc^2 = \frac{mc^2 + k_e Qq/r}{\sqrt{g_{00}}\sqrt{1-v^2/c^2}} \cong \left(1 + GM/c^2 r\right)\left(mc^2 + q\phi_{L0}\right)\left(1 + \frac{1}{2}\frac{v^2}{c^2}\right) \\ &= mc^2 + \frac{1}{2}mv^2 + \frac{k_e Qq}{r} + \frac{1}{2}\frac{k_e Qq}{c^2 r}v^2 + \frac{GMm}{r} + \frac{1}{2}\frac{GMm}{c^2 r}v^2 + \frac{GM}{r}\frac{k_e Qq}{c^2 r} + \frac{1}{2}\frac{GM}{c^2 r}\frac{k_e Qq}{c^2 r}v^2 \end{aligned} . \tag{10}$$

Above we see, in succession, 1) the rest energy mc^2, 2) the kinetic energy of the mass *m*, 3) the Coulomb interaction energy of the charged mass, 4) the kinetic energy of the Coulomb energy, 5) the gravitational interaction energy, 6) the kinetic energy of the gravitational energy, 7) the gravitational potential energy of the Coulomb potential energy and 8) the kinetic energy of the gravitational potential energy of the Coulomb potential energy. It is clear that this accords entirely with empirical observations of the linear limits of these same energies.

Importantly, unlike gravitational redshifts or blueshifts which are a consequence of spacetime curvatures, these electromagnetic time dilations *do not stem directly from curvature*, and they only affect curvature indirectly through any changes in energy to which they give rise because gravitation still "sees" all energy. Hermann Weyl's ill-fated attempt from 1918 until 1929 in [4], [5], [6] to base electrodynamics on *real* gravitational curvature rooted in invariance under a non-unitary local transformation $\psi \to \psi' = e^{\Lambda(t,\mathbf{x})}\psi$ which re-gauges the magnitude of a wavefunction, rather than under the correct transformation $\psi \to \psi' = U\psi = e^{i\Lambda(t,\mathbf{x})}\psi$ which simply redirects the phase, foreclosed this possibility, because the latter correct phase transformation is associated with an imaginary, not real, curvature that places a factor $i = \sqrt{-1}$ into the geodesic deviation $D^2\xi^\mu / D\tau^2$ when expressed in terms of the commutativity $\left[\partial_{;\mu}, \partial_{;\nu}\right]$ of spacetime derivatives. The alteration of time flow in electrodynamics, is therefore much more akin to the time dilation of special relativity, than it is to the gravitational redshifts and blueshifts of general relativity, and may transpire entirely in flat spacetime but to the degree that the electrodynamic energies associated with any particular time alterations as shown in the linear limit of (10), may reach sufficient magnitude to curve the nearby spacetime.

Also importantly, the similarity of the ratios $q\phi_0 / mc^2$ and v^2 / c^2 as the driving number in $\gamma_{em} = 1 - q\phi_0 / mc^2$ and $\gamma_v = 1/\sqrt{1 - v^2 / c^2}$ respectively, is more than just an analogy. Just as $v < c$ a.k.a. $mv^2 < mc^2$ is a fundamental limit on the motion of material subluminal particles, so too, when we develop the electrodynamic time dilations and contractions through to their logical conclusion, and if we likewise require that particle and antiparticle energies always be positive and that time always flows forward in accordance with Feynman-Stueckelberg and that the speed of light must remain the material limit that it is known to be, then it turns out that $q\phi_0 < mc^2$ is a material limit on the strength of the interaction energy between a test charge q with mass m interacting with the sources of the proper potential ϕ_0, just as is $mv^2 < mc^2$. And, it turns out that when $\phi_0 = k_e Q / r$ is the Coulomb potential whereby this limit becomes $k_e Qq / r < mc^2$ a.k.a. $r > k_e Qq / mc^2$, we find that there is a lower physical limit on how close two interacting charges can get to one another, thereby solving the long-standing problem of how to circumvent the $r = 0$ singularity in Coulombs law. To be sure, these electromagnetic time dilations are miniscule for everyday electromagnetic interactions, as are special relativistic time dilations for everyday motion. So testing of $dt / d\tau$ changes for electrodynamics may perhaps be best pursued with experimental approaches similar to those used to test relativistic time dilations.

In short, in order to be able to obtain equation (3) for gravitational and electrodynamic motion from the minimized proper time variation (1) in a way that preserves the integrity of the metric and the background fields independently of the q / m ratio for a given test charge and thereby achieves the philosophically-attractive goal of understanding electrodynamic motion to be geodesic motion just like gravitational motion, we are *required* to recognize that attractive electrodynamic interactions inherently dilate and repulsive interaction inherently contract time itself, *as an observable physical effect*. This is identical to how relative motion dilates time, and how gravitational fields dilate (redshift) or contract (blueshift) time. In this way, it becomes possible to have a spacetime metric which – although a function of the electrical charge and inertial mass of test particles – also remains invariant with respect to those charges and masses. This preserves the integrity of the field theory, and it establishes that electrodynamic motion is in fact geodesic motion that satisfies the minimized proper time variation $0 = \delta\int_A^B d\tau$ from (1). As a result, it becomes possible to lay an entirely geometrodynamic foundation for classical electrodynamics.

References

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