The Geometrodynamic Foundation of Classical Electrodynamics

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Abstract: We summarize how the Lorentz Force motion observed in classical electrodynamics may be understood as geodesic motion derived by minimizing the variation of the proper time along the worldline of test charges in external potentials, while the spacetime metric remains invariant under, and all other fields in spacetime remain independent of, any rescaling of the charge-to-mass ratio q/m. In order for this to occur, time is dilated or contracted due to attractive and repulsive electromagnetic interactions respectively, in very much the same way that time is dilated due to relative motion in special relativity, without contradicting the latter's well-corroborated experimental content. As such, it becomes possible to lay an entirely geometrodynamic foundation for classical electrodynamics in four spacetime dimensions.

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1. Motivation and Purpose

The equation of motion for a test particle along a geodesic line in curved spacetime as specified by the metric interval $c^2 d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu$ with metric tensor $g_{\mu\nu}$ was first obtained by Albert Einstein in §9 of his landmark 1915 paper [1] introducing the General Theory of Relativity. The infinitesimal linear element $d\tau = ds/c$ for the proper time is a scalar invariant which is independent of the chosen system of coordinates. Likewise the finite proper time $\tau = \int_A^B d\tau$ measured along the worldline of the test particle between two spacetime events *A* and *B* has an invariant meaning independent of the choice of coordinates. Specifically, the geodesic of motion is stationary, and results from a minimization of the variational equation

$$0 = \delta \int_A^B d\tau \, . \tag{1.1}$$

Simply put, a material particle goes from event *A* to event *B* in the physically-shortest possible proper time. After carrying out the well-known calculation originally given by Einstein in [1], the particle's equation of motion is found to be:

$$\frac{d^2 x^\beta}{d\tau^2} = \frac{du^\beta}{d\tau} = -\Gamma^\beta{}_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = -\Gamma^\beta{}_{\mu\nu} u^\mu u^\nu \, , \tag{1.2}$$

with the Christoffel connection defined by $-\Gamma^{\beta}{}_{\mu\nu} \equiv \frac{1}{2} g^{\beta\alpha} \left(\partial_\alpha g_{\mu\nu} - \partial_\mu g_{\nu\alpha} - \partial_\nu g_{\alpha\mu} \right)$ and the relativistic four-velocity given by $u^\mu \equiv dx^\mu / d\tau$. The geodesic given by (1.2), again, represents the shortest proper time between two events. Motivated by the geodesic nature of gravitational motion, the purpose of this paper is to summarize how electrodynamic Lorentz Force motion is likewise geodesic motion, as a consequence of heretofore unrecognized time dilations and contractions which occur any time two material bodies are electromagnetically interacting.

2. Geometro-electrodynamics and Time Dilations and Contractions: An Overview

To begin, if the test particle, to which we now ascribe a mass $m > 0$, also has a non-zero net electrical charge $q \neq 0$ and the region of spacetime in which it subsists also has a nonzero electromagnetic field strength $F^{\beta\alpha} \neq 0$ (defined as usual by $F^{\beta\alpha} \equiv \partial^\beta A^\alpha - \partial^\alpha A^\beta$ in relation to the gauge potential four-vector A^α, with $F^{\beta\alpha}$ containing the electric and magnetic field bivectors **E** and **B)**, then the equation of motion is no longer given by (1.2), but is supplemented by an additional term which contains the Lorentz Force law, namely:

$$\frac{d^2 x^\beta}{d\tau^2} = \frac{du^\beta}{d\tau} = -\Gamma^{\beta}{}_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} + \frac{q}{m} g_{\sigma\alpha} F^{\beta\alpha} \frac{dx^\sigma}{c d\tau} = -\Gamma^{\beta}{}_{\mu\nu} u^\mu u^\nu + \frac{q}{m} g_{\sigma\alpha} F^{\beta\alpha} \frac{u^\sigma}{c}. \tag{2.1}$$

The above force law is of course a well-known, well-corroborated, well-established law of physics.

Given that the gravitational geodesic (1.2) specifies a path of minimized proper time (1.1), the question arises whether there is a way to obtain (2.1) from the same variation as in (1.1), thus revealing the electrodynamic motion to also entail particles moving through spacetime along paths of minimized proper time in four spacetime dimensions. Conceptually, it cannot be argued other than that this would be a desirable state of affairs. But physically the difficulty rests in how to accomplish this without ruining the integrity of the metric and the background fields in spacetime by making them a function of the charge-to-mass ratio q/m. This ratio is and must remain a characteristic of the test particle alone. It is not and cannot be a characteristic of the line element $d\tau$, or the metric tensor $g_{\mu\nu}$, or the gauge field A^α, or the field strength $F^{\beta\alpha}$ which define the field-theoretical spacetime background through which the test particle is moving. And, at bottom, this difficulty springs from the *inequivalence* of the "electrical mass" (a.k.a. charge) q and the inertial mass m, versus the Newtonian equivalence of gravitational and inertial mass. In (2.1), this is captured by the fact that m does *not* appear in the gravitational term $-\Gamma^{\beta}{}_{\mu\nu} u^\mu u^\nu$, while the q/m ratio *does* appear in the electrodynamic Lorentz Force term that we rewrite as $(q/m) F^{\beta}{}_{\sigma} u^\sigma$ in natural units with $c = 1$.

This may also be seen very simply if we compare Newton's law with Coulomb's law. In the former case we start with a force $F = -GMm/r^2$ (with the minus sign indicating that gravitation is attractive) and in the latter $F = -k_e Qq/r^2$ (for which we choose an attractive interaction to provide a direct comparison to gravitation), where G is Newton's gravitational constant and the analogous $k_e = 1/4\pi\varepsilon_0 = c^2 \mu_0 / 4\pi$ is Coulomb's constant. If the gravitational field is taken to stem from *M* and the electrical field from *Q*, then the test particle in those fields has gravitational mass *m* and electrical mass *q*. But the Newtonian force $F = ma$ always contains the inertial mass *m*. So in the former case, because the gravitational and inertial mass are equivalent, the acceleration $a = F/m = -GMm/mr^2 = -GM/r^2$ and these two masses cancel, giving $-\Gamma^{\beta}{}_{\mu\nu}u^{\mu}u^{\nu}$ without any mass in (2.1). But in the latter case the acceleration $a = F/m = -k_e Qq/mr^2 = -(q/m)k_e Q/r^2$ because the electrical and inertial masses are not equivalent, hence $(q/m)F^{\beta}{}_{\sigma}u^{\sigma}$ containing this same ratio in (2.1). Here, the motion is distinctly dependent on the electrical and inertial masses *q* and *m* of the test particle, even though different charges *q* with different masses *m* may all be moving through the exact same background fields.

So, were we to pursue the conceptually-attractive goal of understanding electrodynamic motion as the result of particles moving through spacetime along paths of minimized proper time, with (1.1) applying to electrodynamic motion just as it does to gravitational motion, the line element $d\tau$ would inescapably have to be a function $d\tau(q/m)$ of q/m. And this in turn would *appear* to violate the integrity of the line element $d\tau$ as well as the metric tensor $g_{\mu\nu}$ in $c^2 d\tau^2 = g_{\mu\nu}dx^{\mu}dx^{\nu}$, because these would all *seem to be* dependent upon the attributes *q* and *m* of the test particles that are moving through the spacetime background. Were this to be reality and not just seeming appearance, this would be physically impermissible.

Consequently, despite there being many known derivations of the Lorentz Force law, there does not, to date, appear to be an acceptable rooting of the Lorentz Force law in the variational equation $0 = \delta\int_A^B d\tau$ which would reveal electrodynamic motion to be geodesic motion just like the familiar gravitational motion. And this is because it has not been understood how to obtain electrodynamic motion from a minimized variation while simultaneously maintaining the integrity of field theory such that the metric and the background fields do not depend upon the attributes of the test particles which may move through these fields. This, in turn, is because electrical mass is not equivalent to the inertial mass, which causes different test particles to move differently even when in the exact same background fields, in contrast to the Newtonian equivalence of the gravitational and inertial masses from which all particles respond alike in the same background.

Given that when a first test particle with electrical mass *q* and inertial mass *m* is placed in a field $F^{\beta\alpha}$, and a second test particle with electrical mass q' and inertial mass m' of a different ratio $q'/m' \neq q/m$ is placed at equipotential in the same field $F^{\beta\alpha}$, there are observably-different Lorentz Force motions for these two different test particles even though they are at equipotential, having the line element $d\tau$ be a mathematical function of q/m yet be physically independent of q/m may seem paradoxical. Nevertheless, it is possible to have a line element $d\tau(q/m)$ which is a function of the electrical-to-inertial mass ratio q/m, from which the variational equation $0 = \delta\int_A^B d\tau$ does yield the combined gravitational and electrodynamic equation of motion (2.1), yet for which the line element $d\tau$, the metric tensor $g_{\mu\nu}$, the gauge field A^{α}, and the electromagnetic field strength $F^{\beta\alpha}$ are all independent of this q/m ratio. Specifically, close study reveals that this paradox may be resolved by recognizing that *time does not flow at the same rate for these two test particles in very much the same way that time does not flow at the same rate for two reference frames in special relativity which are in motion relative to one another*.

In the absence of gravitation with $g_{\mu\nu} = \eta_{\mu\nu}$ and $\Gamma^{\beta}{}_{\mu\nu} = 0$, the first test particle will have a Lorentz motion given by:

$$\frac{d^2 x^{\beta}}{d\tau^2} = \frac{q}{m}\eta_{\sigma\alpha}F^{\beta\alpha}\frac{dx^{\sigma}}{cd\tau}. \tag{2.2}$$

Note that this Lorentz motion also contains a set of coordinates x^{μ}. Now usually it is assumed that for the second test particle the motion is given by this same equation (2.2), merely with the substitution of $q \to q'$ and $m \to m'$; that is, by:

$$\frac{d^2 x^{\beta}}{d\tau^2} = \frac{q'}{m'}\eta_{\sigma\alpha}F^{\beta\alpha}\frac{dx^{\sigma}}{cd\tau}. \tag{2.3}$$

The particular assumption here is that there is no change in the rate at which time flows when (2.2) is replaced with (2.3); and more generally the assumption is that the coordinate interval dx^{σ} in (2.2) is identical to the dx^{σ} in (2.3). Yet, it is impossible to have both (2.2) and (2.3) emerge through the variation $0 = \delta\int_A^B d\tau$ from the same metric element $d\tau$, and simultaneously maintain the integrity of the field theory, unless the coordinates are different, wherein dx^{σ} in (2.2) is *not identical* to what must now be $dx^{\sigma} \to dx'^{\sigma} \neq dx^{\sigma}$ in (2.3).

In fact, the very physics of having electric charges in electromagnetic fields induces a change in coordinates as between these two test charges with different $q'/m' \neq q/m$, very similar to the coordinate change via Lorentz transformations induced by relative motion. As a result, the electrodynamic motion of the second test charge is given, not by (2.3), but by:

$$\frac{d^2 x'^{\beta}}{d\tau^2} = \frac{q'}{m'}\eta_{\sigma\alpha}F^{\beta\alpha}\frac{dx'^{\sigma}}{cd\tau}. \tag{2.4}$$

Here, x^{β} in (2.2) and $x'^{\beta} \neq x^{\beta}$ in (2.4), respectively, are two different sets of coordinates, yet they are interrelated by a definite transformation. Most importantly, this results in *time itself* being induced to flow differently as between these two sets of coordinates, making time dilation and contraction as fundamental an aspect of electrodynamics, as it already is of the special relativistic theory of motion and the general relativistic theory of gravitation. In fact, what is really happening – physically – is that the placement of a charge in an electromagnetic field is *inducing a physically-observable change of coordinates* $x^{\beta}(q/m) \to x'^{\beta}(q'/m')$ in the very same way that relative motion between the coordinate systems $x^{\beta}(v)$ and $x'^{\beta}(v')$ of two different inertial reference frames with velocities v and v' induces a Lorentz transformation $x^{\beta}(v) \to x'^{\beta}(v')$ that relates the two coordinate systems to one another via $c^2d\tau^2 = \eta_{\mu\nu}dx^\mu(v)dx^\nu(v) = \eta'_{\mu\nu}dx'^\mu(v')dx'^\nu(v')$, with the invariant line element $d\tau^2 = d\tau'^2$ and the same metric tensor $\eta_{\mu\nu} = \eta'_{\mu\nu}$.

As it turns out, the line element that yields (2.1) from (1.1), including electrodynamic motion is:

$$c^2d\tau^2 = g_{\mu\nu}\left(dx^\mu + \frac{q}{mc}d\tau A^\mu\right)\left(dx^\nu + \frac{q}{mc}d\tau A^\nu\right) = g_{\mu\nu}Dx^\mu Dx^\nu\,, \tag{2.5}$$

where we have defined a gauge-covariant coordinate interval $Dx^\mu \equiv dx^\mu + (q/mc)d\tau A^\mu$. And it will be seen that upon multiplying through by m^2 and dividing through by $d\tau^2$ this becomes:

$$m^2c^2 = g_{\mu\nu}\left(m\frac{dx^\mu}{d\tau} + \frac{q}{c}A^\mu\right)\left(m\frac{dx^\nu}{d\tau} + \frac{q}{c}A^\nu\right) = g_{\mu\nu}\pi^\mu\pi^\nu\,, \tag{2.6}$$

which is the usual relationship between rest mass m and canonical energy-momentum $\pi^\mu \equiv mdx^\mu/d\tau + qA^\mu/c = p^\mu + qA^\mu/c$, where ordinary mechanical / kinetic energy-momentum is $p^\mu = mdx^\mu/d\tau$. This gauge interval $Dx^\mu \equiv dx^\mu + (q/mc)d\tau A^\mu$ is indeed merely a restatement of the gauge-covariant derivatives $D_\sigma \equiv \partial_\sigma - iqA_\sigma$ and canonical momenta $\pi^\mu \equiv p^\mu + qA^\mu/c$ which emerge from gauge theory via $i\partial_\sigma \Leftrightarrow p_\sigma$ and $iD_\sigma \Leftrightarrow \pi_\sigma$, and in particular from the mandate for gauge (really, phase) symmetry. Some authors continue to use p^μ to denote the canonical momentum; we find it preferable to employ the different symbol π^μ to avert confusion.

Now, the line element (2.5) is clearly a function of q/m and so has the *appearance* of depending on the ratio q/m. But this is only appearance. For, when we now place the second test charge with the second ratio $q'/m' \neq q/m$ in the exact same metric measured by the invariant line element $d\tau$ and moving through the exact same fields $g_{\mu\nu}$ and A^μ, this metric gives:

$$c^2d\tau'^2 = c^2d\tau^2 = g_{\mu\nu}\left(dx'^\mu + \frac{q'}{m'c}d\tau A^\mu\right)\left(dx'^\nu + \frac{q'}{m'c}d\tau A^\nu\right) = g_{\mu\nu}Dx'^\mu Dx'^\nu\,. \tag{2.7}$$

So despite $d\tau$ being a function of the q/m ratio, this $d\tau = d\tau'$ as a measured proper time element is actually *invariant* with respect to the q/m ratio because *the differences between different q/m and q'/m' are entirely absorbed into the coordinate transformation $x^\mu \to x'^\mu$, which is quite analogous to the Lorentz transformation of special relativity*. The counterpart to (2.6) now becomes:

$$m'^2c^2 = g_{\mu\nu}\left(m'\frac{dx'^\mu}{d\tau} + \frac{q'}{c}A^\mu\right)\left(m'\frac{dx'^\nu}{d\tau} + \frac{q'}{c}A^\nu\right) = g_{\mu\nu}\pi'^\mu\pi'^\nu\,, \tag{2.8}$$

with an invariant $d\tau$ and unchanged background fields $g_{\mu\nu}$ and A^{μ}.

In fact, this transformation $x^{\mu} \to x'^{\mu}$ is *defined* so as to keep $d\tau = d\tau'$, $g_{\mu\nu} = g'_{\mu\nu}$, and $A^{\mu} = A'^{\mu}$, and by implication the field strength bivector $F^{\beta\alpha} = F'^{\beta\alpha}$, all unchanged, just as Lorentz transformations are defined so as to maintain a constant speed of light for all inertial reference frames independently of their state of motion. That is, combining (2.5) and (2.7), this transformation $x^{\mu} \to x'^{\mu}$ which results in time dilations and contractions is defined by:

$$c^2 d\tau^2 = g_{\mu\nu}\left(dx^{\mu} + \frac{q}{mc} d\tau A^{\mu}\right)\left(dx^{\nu} + \frac{q}{mc} d\tau A^{\nu}\right) \equiv g_{\mu\nu}\left(dx'^{\mu} + \frac{q'}{m'c} d\tau A^{\mu}\right)\left(dx'^{\nu} + \frac{q'}{m'c} d\tau A^{\nu}\right). \tag{2.9}$$

Consequently, $d\tau = d\tau'$ is a function of charge *q* and mass *m* yet is invariant with respect to the same, and there is no inconsistency in having $d\tau = d\tau'$ be a function of, yet be invariant under, a rescaling of the q/m ratio. Likewise, the fields $g_{\mu\nu} = g'_{\mu\nu}$ and $A^{\mu} = A'^{\mu}$ are independent of the charge and the mass of the test particle, because again, everything steming from the different ratios q/m and q'/m' is absorbed into a coordinate transformation $x^{\mu} \to x'^{\mu}$. Thus, while "gauge" is a historical misnomer for what is really invariance under local *phase* transformations $\psi \to \psi' = U\psi = e^{i\Lambda(t,\mathbf{x})}\psi$ applied to a wavefunction ψ, we see in (2.9) that the line element $d\tau$ truly is invariant under what can be genuinely called a *re-gauging* of the q/m ratio. And from (2.6) and (2.8), we see that this symmetry is really not new. It is merely a restatement of the usual relationship $m^2c^2 = g_{\mu\nu}\pi^{\mu}\pi^{\nu}$ between rest mass and canonical momentum.

As a result, each and every different test particle carries its own coordinates, all interrelated so as to keep $d\tau$ invariant, and $g_{\mu\nu}$, A^{μ} and $F^{\beta\alpha}$ unchanged. The coordinate transformation interrelating all the test particles causes time to dilate for electrical attraction and to contact for repulsion, with a dimensionless ratio $dt/d\tau = dx^0/d\tau \equiv \gamma_{em}$ that integrally depends upon the magnitude of the likewise-dimensionless ratio qA^{μ}/mc^2 of electromagnetic interaction energy qA^{μ} to the test particle's rest energy mc^2. This in turn supplements the ratio $dt/d\tau = \gamma_v = 1/\sqrt{1-v^2/c^2}$ for motion in special relativity and $dt/d\tau = \gamma_g = 1/\sqrt{g_{00}}$ for a clock at rest in a gravitational field, and assembles them in the overall product combination $dt/d\tau = \gamma_{em}\gamma_g\gamma_v$ governing time dilation when all of motion and gravitation and electromagnetic interactions are present.

Operationally, the electromagnetic contribution γ_{em} to this time dilation or contraction would be measured in principle by comparing the rate at which time is kept by otherwise identical, synchronized geometrodynamic clocks or oscillators which are then electrically charged with different q/m ratios, and then placed at rest into a background potential $A^{\mu} = (\phi, \mathbf{A}) = (\phi_0, \mathbf{0})$ at equipotential, where φ_0 is the proper potential. Or more generally, this would be measured by electrically charging otherwise identical clocks and then placing them into the potential to have differing dimensionless $qA^0/mc^2 = q\phi_0/mc^2$ ratios.

Empirically, for $q\phi_0/mc^2 << 1$, the interaction energies $E_{em} = \int F dr = +k_e Qq/r$ plus integration constant for an attractive Coulomb force $F = -k_e Qq/r^2$ are related to these electromagnetic time dilations in a manner identical to how the kinetic energy $E_v = \frac{1}{2}mv^2$ is contained in $mc^2\gamma_v = mc^2/\sqrt{1-v^2/c^2} \cong mc^2 + \frac{1}{2}mv^2$ for nonrelativistic velocities $v << c$ in special relativity. In fact, the actual expression for the electromagnetic contribution to the time dilation for $q\phi_0/mc^2 << 1$ interactions is $\gamma_{em} = 1 - q\phi_0/mc^2$. And for a Coulomb proper potential $\phi_0 = -k_e Q/r$ for an electrical interaction chosen to be attractive like gravitation, this is $\gamma_{em} = 1 + k_e Qq/mc^2 r$. So the combined time dilation $dt/d\tau = \gamma_{em}\gamma_g\gamma_v$ mentioned earlier, employing the gravitational factor $\gamma_g = 1/\sqrt{g_{00}(r)} \cong 1 + GM/c^2 r$ in the weak field Newtonian limit (where the Reissner–Nordström metric term $Gk_e Q^2/c^4 r^2$ may clearly be neglected), produces an overall energy which, in the low velocity, weak-gravitational and weak-electromagnetic interaction limit, is given by:

$$\begin{aligned} E &= mc^2 \frac{dt}{d\tau} = mc^2 \gamma_{em}\gamma_g\gamma_v = mc^2 \frac{1 + k_e Qq/mc^2 r}{\sqrt{g_{00}}\sqrt{1-v^2/c^2}} \cong mc^2\left(1+\frac{GM}{c^2 r}\right)\left(1+\frac{k_e Qq}{mc^2 r}\right)\left(1+\frac{1}{2}\frac{v^2}{c^2}\right) \\ &= mc^2 + \frac{1}{2}mv^2 + \frac{k_e Qq}{r} + \frac{1}{2}\frac{k_e Qq}{c^2 r}v^2 + \frac{GMm}{r} + \frac{1}{2}\frac{GMm}{c^2 r}v^2 + \frac{GM}{r}\frac{k_e Qq}{c^2 r} + \frac{1}{2}\frac{GM}{c^2 r}\frac{k_e Qq}{c^2 r}v^2 \end{aligned} . \quad (2.10)$$

What we see here, in succession, are 1) the rest energy mc^2, 2) the kinetic energy of the mass *m*, 3) the Coulomb interaction energy of the charged mass, 4) the kinetic energy of the Coulomb energy, 5) the gravitational interaction energy of the mass, 6) the kinetic energy of the gravitational energy, 7) the gravitational energy of the Coulomb energy and 8) the kinetic energy of the gravitational energy of the Coulomb energy. It is clear that this accords entirely with empirical observations of the linear limits of these same energies.

Importantly, unlike gravitational redshifts or blueshifts which are a consequence of spacetime curvatures, these electromagnetic time dilations *do not stem directly from curvature*, and they only affect curvature indirectly through any changes in energy to which they give rise because gravitation still "sees" all energy. Hermann Weyl's ill-fated attempt from 1918 until 1929 in [2], [3], [4] to base electrodynamics on *real* gravitational curvature foreclosed any such real curvature explanation. This is because Weyl's attempt was rooted in invariance under a non-unitary local transformation $\psi \to \psi' = e^{\Lambda(t,\mathbf{x})}\psi$ which re-gauges the magnitude of a wavefunction, rather than under the correct transformation $\psi \to \psi' = U\psi = e^{i\Lambda(t,\mathbf{x})}\psi$ with an imaginary exponent that simply redirects the phase. Specifically, the latter correct phase transformation is associated with an *imaginary*, not real, curvature that places a factor $i = \sqrt{-1}$ into the geodesic deviation $D^2\xi^\mu/D\tau^2$ when expressed in terms of the commutativity $\left[\partial_{;\mu}, \partial_{;\nu}\right]$ of spacetime derivatives, so at best, electrodynamics can be understood on the basis of mathematically-imaginary spacetime curvature. The alteration of time flow in electrodynamics we suggest here, is therefore much more akin to the time dilation of special relativity than it is to the gravitational redshifts and blueshifts of general relativity. It may transpire entirely in flat spacetime, and real spacetime curvature only becomes implicated when the energies added to mc^2 reach sufficient magnitude beyond their linear limits shown in (2.10) to curve the nearby spacetime.

Also importantly, the similarity of the ratios $q\phi_0 / mc^2$ and v^2 / c^2 as the driving number in $\gamma_{em} = 1 - q\phi_0 / mc^2$ and $\gamma_v = 1/\sqrt{1 - v^2 / c^2}$, respectively, is more than just an analogy. Just as $v < c$ (a.k.a. $mv^2 < mc^2$) is a fundamental limit on the motion of material subluminal particles, so too, it turns out that $q\phi_0 < mc^2$ is a material limit on the strength of the interaction energy between a test charge *q* with mass *m* interacting with the sources of the proper potential ϕ_0. This transpires when we develop the electrodynamic time dilations and contractions through to their logical conclusion, by requiring particle and antiparticle energies to always be positive and time to always flow forward in accordance with Feynman-Stueckelberg, and by maintaining the speed of light as the material limit which it is known to be. Further, it turns out that when $\phi_0 = k_e Q / r$ is the Coulomb potential whereby this limit becomes $k_e Qq / r < mc^2$ (a.k.a. $r > k_e Qq / mc^2$), we find that there is a lower physical limit on how close two interacting charges can get to one another, thereby solving the long-standing problem of how to circumvent the $r = 0$ singularity in Coulomb's law.

To be sure, these electromagnetic time dilations are miniscule for everyday electromagnetic interactions, as are special relativistic time dilations for everyday motion. So testing of $dt / d\tau$ changes for electrodynamics may perhaps be best pursued with experimental approaches similar to those used to test relativistic time dilations. As a very simple example to establish a numeric benchmark, consider two bodies with charges $Q = q = 1\,\mathrm{C}$ (Coulomb) separated by $r = 1\,\mathrm{m}$ (meter). In this event, the Coulomb interaction energy has a magnitude $k_e Qq / r = k_e = 1/4\pi\varepsilon_0 = 8.897 \times 10^9\ \mathrm{J}$ (Joules). Yet, if the test particle which we take to have the charge q has a rest mass $m = 1\,\mathrm{kg}$ (kilogram), then the electrodynamic time dilation factor contained in (2.10) is $\gamma_{em} = 1 + k_e / c^2 = 1 + \mu_0 / 4\pi = 1 + 10^{-7} = 1.0000001$. This is a very tiny time dilation for a tremendously energetic interaction. The release of this much energy per second would yield a power of approximately 8.99 GW (gigawatts), which roughly approximates seven or eight nuclear power plants, or roughly four times the power of the Hoover Dam, or the power output of a single space shuttle launch, or the power of about seventy five jet engines, or that of a single lightning bolt. For a special relativistic comparison, consider an airplane which flies one mile in five seconds, versus light which travels about one million miles in five seconds. Here, $v / c \cong 10^{-6}$ and the time dilation is $\gamma_v = 1/\sqrt{1 - v^2 / c^2} \cong 1.0000000000005$. So in fact the exemplary electrodynamic time dilation is substantially less miniscule than this exemplary special relativistic dilation. However in daily experience where one encounters watts and kilowatts not gigawatts, these time dilations would be of similar magnitude.

In short, in order to be able to obtain equation (2.1) for gravitational and electrodynamic motion from the minimized proper time variation (1.1) in a way that preserves the integrity of the metric and the background fields independently of the q / m ratio for a given test charge and thereby achieves the conceptually-attractive goal of understanding electrodynamic motion to be geodesic motion just like gravitational motion, we are forced to recognize that attractive electrodynamic interactions inherently dilate and repulsive interactions inherently contract time itself, *as an observable physical effect*. This is identical to how relative motion dilates time, and to how gravitational fields dilate (redshift) or contract (blueshift) time. In this way, it becomes possible to have a spacetime metric which – although a function of the electrical charge and inertial mass of test particles – also remains invariant with respect to those charges and masses and particularly with respect to a re-gauging of the charge-to-mass ratio. This preserves the integrity of the field theory, and establishes that electrodynamic motion is in fact geodesic motion which satisfies the minimized proper time variation $0=\delta\int_A^B d\tau$ from (1.1). As a result, it becomes possible to lay an entirely geometrodynamic foundation for classical electrodynamics in four spacetime dimensions.

3. Derivation of Lorentz Force Geodesic Motion from Variation Minimization

The foundational calculation to derive (2.1) including the Lorentz force from the minimized variation (1.1) begins with the spacetime metric $c^2 d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu$ which is multiplied through by m and turned into the free particle energy-momentum relation $m^2c^2 = g_{\mu\nu} p^\mu p^\nu$ containing the mechanical momentum $p^\mu = m dx^\mu / d\tau$. This in turn is readily turned into Dirac's $\left(i\gamma^\mu \partial_\mu - m\right)\psi = 0$ for a free electron in flat spacetime making use of $\eta^{\mu\nu} = \tfrac{1}{2}\left\{\gamma^\mu, \gamma^\nu\right\}$. Then, we simply use Weyl's well-known gauge prescription [4] which transforms the mechanical momentum to the canonical momentum $p^\mu \to \pi^\mu \equiv p^\mu + qA^\mu / c$ thus the energy-momentum relation to $m^2c^2 = g_{\mu\nu}\pi^\mu\pi^\nu$ in (2.6), and the ordinary derivatives to gauge-covariant derivatives $\partial_\sigma \to D_\sigma \equiv \partial_\sigma - iqA_\sigma$ and thus Dirac's equation to $\left(i\gamma^\mu D_\mu - m\right)\psi = 0$ for interacting particles. All of this emerges by requiring "gauge" symmetry under the local phase transformation $\varphi \to \varphi' = U\varphi = e^{i\Lambda(t,\mathbf{x})}\varphi$ acting generally on the scalar fields $\varphi = \phi$ of the Klein-Gordon equation and the fermion fields $\varphi = \psi$ of Dirac's equation, redirecting phase but preserving magnitude. This is all well-known, so it is not necessary to detail this further. The point is that the relation $m^2c^2 = g_{\mu\nu}\pi^\mu\pi^\nu$ in (2.6) is easily derived from the metric $m^2c^2 = g_{\mu\nu} p^\mu p^\nu$ using local gauge symmetry, and that nothing more is needed to furnish the starting point to minimize the variation and arrive at the combined gravitation and electrodynamic motion (2.1).

Starting with (2.6) and dividing through by m^2c^2, we form the number 1 as such:

$$1 = g_{\mu\nu}\left(\frac{dx^\mu}{cd\tau} + \frac{q}{mc^2}A^\mu\right)\left(\frac{dx^\nu}{cd\tau} + \frac{q}{mc^2}A^\nu\right) = g_{\mu\nu}\left(\frac{u^\mu}{c} + \frac{q}{mc^2}A^\mu\right)\left(\frac{u^\nu}{c} + \frac{q}{mc^2}A^\nu\right) = g_{\mu\nu}\frac{U^\mu}{c}\frac{U^\nu}{c}, \tag{3.1}$$

which will be useful in a variety of circumstances. The above includes the mechanical four-velocity $u^\mu \equiv dx^\mu / d\tau$ and the canonical four-velocity $U^\mu \equiv u^\mu + qA^\mu / mc$. From here, we shall work in natural units $c = 1$ and use dimensional rebalancing to restore c only after a final result.

The first place that "1" above will be useful is in (1.1), where, distributing the expression after the first equality while absorbing $g_{\mu\nu}$ into the electrodynamic term indices, we write:

$$0=\delta\int_A^B d\tau(1)=\delta\int_A^B d\tau\left(g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}+2\frac{q}{m}A_\sigma\frac{dx^\sigma}{d\tau}+\frac{q^2}{m^2}A_\sigma A^\sigma\right)^{.5}. \tag{3.2}$$

From here, we carry out the variational calculation, which deductively culminates in:

$$0=\delta\int_A^B d\tau=\int_A^B \delta x^\alpha d\tau\begin{pmatrix}-g_{\alpha\nu}\dfrac{d^2x^\nu}{d\tau^2}+\dfrac{1}{2}\left(\partial_\alpha g_{\mu\nu}-\partial_\mu g_{\nu\alpha}-\partial_\nu g_{\alpha\mu}\right)\dfrac{dx^\mu}{d\tau}\dfrac{dx^\nu}{d\tau}\\ +\dfrac{q}{m}\left(\partial_\alpha A_\sigma-\partial_\sigma A_\alpha\right)\dfrac{dx^\sigma}{d\tau}+\dfrac{1}{2}\dfrac{q^2}{m^2}\partial_\alpha\left(A_\sigma A^\sigma\right)\end{pmatrix}. \tag{3.3}$$

Going from (3.2) to (3.3) is straightforward. The top line contains the same result always obtained for gravitational geodesics, i.e., the result of setting $q=0$ in (3.2). This is the calculation Einstein first presented in §9 of [1], and does not need to be reviewed further. The terms on the bottom line emerge as a direct and immediate consequence of starting with the canonical $m^2c^2=g_{\mu\nu}\pi^\mu\pi^\nu$ rather than the ordinary mechanical $m^2c^2=g_{\mu\nu}p^\mu p^\nu$ energy-momentum relation, which is to say, the bottom line is a result merely of mandating local gauge symmetry. Some specific guides to note when performing the detailed calculation include: a) we assume no variation in the charge-to-mass ratio, i.e., that $\delta(e/m)=0$, over the path from *A* to *B*; b) applied to gauge field terms, the variations are $\delta A_\sigma=\delta x^\alpha\partial_\alpha A_\sigma$ and $\delta\left(A_\sigma A^\sigma\right)=\delta x^\alpha\partial_\alpha\left(A_\sigma A^\sigma\right)$; c) we also use $dA_\sigma/d\tau=\partial_\alpha A_\sigma dx^\alpha/d\tau$; and d) there is an integration-by-parts in the calculation. This integration-by-parts produces a boundary term $\int_A^B d\left(A_\sigma\delta x^\sigma\right)=\left(A_\sigma\delta x^\sigma\right)\Big|_A^B=0$ that can be eliminated, and for the remaining term causes the sign reversal appearing in $\partial_\alpha A_\sigma-\partial_\sigma A_\alpha$.

The proper time $d\tau\neq 0$ for material worldlines, and between the boundaries at *A* and *B* the variation $\delta x^\sigma\neq 0$. So the large parenthetical expression in (3.3) must be zero. The connection $-\Gamma^\beta{}_{\mu\nu}=\frac{1}{2}g^{\alpha\beta}\left(\partial_\alpha g_{\mu\nu}-\partial_\mu g_{\nu\alpha}-\partial_\nu g_{\alpha\mu}\right)$ and field strength $F_{\alpha\sigma}=\partial_\alpha A_\sigma-\partial_\sigma A_\alpha=\partial_{;\alpha}A_\sigma-\partial_{;\sigma}A_\alpha$ (the expression with gravitationally-covariant derivatives meaning this result can be applied in curved spacetime), with *c* restored, enable us to extract:

$$\frac{d^2x^\beta}{d\tau^2}=-\Gamma^\beta{}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}+\frac{q}{m}F^\beta{}_\sigma\frac{dx^\sigma}{cd\tau}+\frac{1}{2}\frac{q^2}{m^2c^2}\partial^\beta\left(A_\sigma A^\sigma\right). \tag{3.4}$$

This clearly reproduces (2.1) and includes the Lorentz force alongside the gravitational geodesic, all obtained from the minimized variation (3.2). Therefore, (3.4) does represent geodesic motion. However, this also contains an additional term with $\partial^\beta\left(A_\sigma A^\sigma\right)$ that does not appear and is not observed in the Lorentz force. But unlike all the other terms in (3.4), neither is this term invariant under the gauge transformation $eA^{\mu} \to eA'^{\mu} = eA^{\mu} + \partial^{\mu}\Lambda$, so in fact it is unobservable. Thus, we can and should always choose $\Lambda(t,\mathbf{x})$ so as to gauge this term out of (3.4). Doing so, we now fix the gauge by *imposing* the gauge condition:

$$\partial_{\beta}\left(A_{\sigma}A^{\sigma}\right) \equiv 0 . \tag{3.5}$$

This gauge condition is imposed for one empirical reason and two theoretical reasons: The empirical reason is that this term needs to be removed from (3.4) to match the well-established, well-corroborated Lorentz Force law (2.1). The first theoretical reason is that the motion cannot depend upon a term $\partial_{\beta}\left(A_{\sigma}A^{\sigma}\right)$ which in turn depends upon and changes as a function of the unobservable local phase $\Lambda(t,\mathbf{x})$. This would leave the observable motion ambiguous. The second theoretical reason is that by removing this term, (3.4) now does fully describe the Lorentz motion as geodesic motion, which is conceptually attractive. Because the gauge condition (3.5) causes (3.4) derived from (3.2) a.k.a. (1.1) to become synonymous with (2.1) and reveal the Lorentz force motion to be geodesic motion emanating from the metric line element (2.5), we shall refer to (3.5) as the "*geodesic gauge*." In this geodesic gauge, the combined gravitational plus Lorentz motion is geodesic motion.

In the next section we shall examine the geodesic gauge condition (3.5) in further detail. But first it is important to see just how Lorentz motion is now merely a consequence of local gauge symmetry: It is well-known how imposing gauge symmetry spawns the heuristic rules $\partial_{\sigma} \to D_{\sigma} \equiv \partial_{\sigma} - iqA_{\sigma}$ and $p^{\mu} \to \pi^{\mu} \equiv p^{\mu} + qA^{\mu}/c$ for gauge-covariant derivatives and canonical momentum, and $m^2c^2 = g_{\mu\nu}p^{\mu}p^{\nu} \to m^2c^2 = g_{\mu\nu}\pi^{\mu}\pi^{\nu}$ for the energy momentum relation. Here, we see another heuristic rule which emerges in lockstep with these others, namely:

$$\frac{du^{\beta}}{d\tau} = -\Gamma^{\beta}{}_{\mu\nu}u^{\mu}u^{\nu} \to \frac{Du^{\beta}}{D\tau} = -\Gamma^{\beta}{}_{\mu\nu}u^{\mu}u^{\nu} + \frac{q}{mc}F^{\beta}{}_{\sigma}u^{\sigma}, \tag{3.6}$$

where $Du^{\beta}/D\tau$ symbolizes the gauge-covariant or *canonical acceleration*. This is tied to the further heuristic $dx^{\mu} \to Dx^{\mu} \equiv dx^{\mu} + (q/mc)\,d\tau A^{\mu}$ defined in (2.5). To avoid notational confusion, note that this is not a "derivative along the curve" defined using gravitationally-covariant derivatives $\partial_{;\nu}B^{\beta}$ for a given four-vector B^{β} by $DB^{\beta}/D\tau \equiv \left(\partial x^{\nu}/\partial\tau\right)\partial_{;\nu}B^{\beta}$. But they are closely related, because the latter yields $Du^{\beta}/D\tau = \left(\partial x^{\nu}/\partial\tau\right)\partial_{;\nu}u^{\beta} = du^{\beta}/d\tau + \Gamma^{\beta}{}_{\mu\nu}u^{\mu}u^{\nu}$ for $B^{\beta} = u^{\beta}$.

In fact, if we use $\mathrm{A}^{\beta} \equiv \mathfrak{D}u^{\beta}/\mathfrak{D}\tau$ to denote the *gravitationally* <u>*and*</u> *gauge-covariant* acceleration and thus remove any notational ambiguity, we may combine what is in the preceding paragraph with (3.4) and (3.5) and the usual gravitational $Du^{\beta}/D\tau = \left(\partial x^{\nu}/\partial\tau\right)\partial_{;\nu}u^{\beta}$ to write:

$$\mathrm{A}^{\beta} \equiv \frac{\mathfrak{D}u^{\beta}}{\mathfrak{D}\tau} \equiv \frac{Du^{\beta}}{D\tau} - \frac{q}{m}F^{\beta}{}_{\sigma}u^{\sigma} = \frac{du^{\beta}}{d\tau} + \Gamma^{\beta}{}_{\mu\nu}u^{\mu}u^{\nu} - \frac{q}{m}F^{\beta}{}_{\sigma}u^{\sigma} = 0. \tag{3.7}$$

Here, $\mathrm{A}^{\beta} \equiv \mathfrak{D}u^{\beta} / \mathfrak{D}\tau \equiv 0$ is another way of representing the variation $0 = \delta\int_A^B d\tau$ considered in (1.1). It states that acceleration generally is gravitationally-covariant and gauge-covariant, which is why $\mathrm{A}^{\beta} \equiv \mathfrak{D}u^{\beta} / \mathfrak{D}\tau$ is equal to zero; yet when shown in terms of mechanical four-velocities $u^{\mu} = dx^{\mu} / d\tau$, this acceleration contains the geodesic motion of gravitation and the Lorentz force motion of electrodynamics. In absence of any charge or electromagnetic potential the above reverts back to $Du^{\beta} / D\tau = du^{\beta} / d\tau + \Gamma^{\beta}{}_{\mu\nu}u^{\mu}u^{\nu} = 0$ for gravitationally-covariant motion. In absence of gravitation, $\mathfrak{D}u^{\beta} / \mathfrak{D}\tau = du^{\beta} / d\tau - (q/m)F^{\beta}{}_{\sigma}u^{\sigma} = 0$ for the Lorentz force alone. And in the absence of both gravitation and electromagnetism what remains is merely $du^{\beta} / d\tau = 0$ for the Newtonian inertial motion governed by special relativity alone. From this view, all motion is inertial because $\mathfrak{D}u^{\beta} / \mathfrak{D}\tau \equiv 0$; it is simply *covariantly-inertial* with any gravitational curvature and any canonical gauge elements.

4. The Geodesic Gauge and the Action Gauge: Intrinsic Effects on Gauge fields

Now let us study closely the geodesic gauge condition $\partial_{\beta}\left(A_{\sigma}A^{\sigma}\right) = 0$ specified in (3.5). We begin with Maxwell's equation $J^{\beta} = \partial_{\alpha}F^{\alpha\beta}$ for the electric charge density which we rewrite via the usual expression for the field strength $F^{\alpha\beta} = \partial^{\alpha}A^{\beta} - \partial^{\beta}A^{\alpha}$ in terms of the gauge fields as $J^{\beta} - \partial_{\alpha}\partial^{\alpha}A^{\beta} + \partial^{\beta}\partial_{\alpha}A^{\alpha} = 0$. But we do *not* impose the Lorenz gauge $\partial_{\alpha}A^{\alpha} = 0$; rather for now we leave this term as is. We then multiply this Maxwell equation through by A_{β}, thus writing the scalar equation:

$$A_{\beta}J^{\beta} - A_{\beta}\partial_{\alpha}\partial^{\alpha}A^{\beta} + A_{\beta}\partial^{\beta}\partial_{\alpha}A^{\alpha} = 0. \tag{4.1}$$

For the second term above we have $-A_{\beta}\partial_{\alpha}\partial^{\alpha}A^{\beta} = \partial_{\alpha}A_{\beta}\partial^{\alpha}A^{\beta} - \partial_{\alpha}\left(A_{\beta}\partial^{\alpha}A^{\beta}\right)$ using the product rule. We may also form the identity $A_{\beta}\partial^{\alpha}A^{\beta} = \tfrac{1}{2}\partial^{\alpha}\left(A_{\beta}A^{\beta}\right)$. Using both of these in (4.1) yields:

$$A_{\beta}J^{\beta} + \partial_{\alpha}A_{\beta}\partial^{\alpha}A^{\beta} - \tfrac{1}{2}\partial_{\alpha}\partial^{\alpha}\left(A_{\beta}A^{\beta}\right) + A_{\beta}\partial^{\beta}\partial_{\alpha}A^{\alpha} = 0. \tag{4.2}$$

The second term $\partial_{\alpha}A_{\beta}\partial^{\alpha}A^{\beta} = \tfrac{1}{4}F_{\alpha\beta}F^{\alpha\beta}$, and with this, the first two terms are equivalent to minus the electrodynamic Lagrangian density, $A_{\beta}J^{\beta} + \tfrac{1}{4}F_{\alpha\beta}F^{\alpha\beta} = -\mathcal{L}_{em}$. Therefore, (4.2) is simply:

$$-\tfrac{1}{2}\partial_{\alpha}\partial^{\alpha}\left(A_{\beta}A^{\beta}\right) + A_{\beta}\partial^{\beta}\partial_{\alpha}A^{\alpha} = \mathcal{L}_{em}. \tag{4.3}$$

Until now we have done nothing to remove any of the gauge redundancy that arises when the four-vector A^{α} with four degrees of freedom is used to represent a field of massless photons, each of which has only two degrees of transverse polarization freedom. However, the first term in the above contains $\partial^\alpha\left(A_\beta A^\beta\right)$, which, imposing the geodesic gauge condition (3.5), must be set to zero. This leaves us merely with:

$$A_\beta\partial^\beta\partial_\alpha A^\alpha = \mathcal{L}_{em} . \tag{4.4}$$

This term contains the expression $\partial_\alpha A^\alpha$, which, if we were to employ the Lorenz gauge $\partial_\alpha A^\alpha = 0$ in addition to the geodesic gauge, would over-determine the gauge freedom and leave us with $\mathcal{L}_{em} = 0$. But because the photon is massless, we are not *required* to use $\partial_\alpha A^\alpha = 0$ as we would be if photons were massive with a third longitudinal degree of freedom. Therefore we leave (4.4) as is, and impose this as a further gauge condition, which we shall refer to as the "*Lagrangian gauge*" for obvious reasons. With the Lagrangian gauge (4.4) and the geodesic gauge (3.5) both imposed, all redundant freedom is removed from the massless gauge field and the physical equations become unambiguous.

It is also very useful to write the above directly in term of the electrodynamic action $S_{em} = \int d^4x\mathcal{L}_{em}$, via the product rule $A_\beta\partial^\beta\partial_\alpha A^\alpha = \partial^\beta\left(A_\beta\partial_\alpha A^\alpha\right) - \partial_\beta A^\beta\partial_\alpha A^\alpha$. But within the action we may set $\int d^4x\partial^\beta\left(A_\beta\partial_\alpha A^\alpha\right) = 0$ via the boundary condition $A_\beta(t,\mathbf{x}) = 0$ at the extremum $t,\mathbf{x} = \pm\infty$. What we then end up with, is a very simple action:

$$S_{em} = \int d^4x\mathcal{L}_{em} = -\int d^4x\left(\partial_\beta A^\beta\partial_\alpha A^\alpha\right) = -\int d^4x\left(\partial_\alpha A^\alpha\right)^2 . \tag{4.5}$$

We shall refer to this as the "*action gauge*" condition, and it clearly fixes the term $\partial_\alpha A^\alpha$, which otherwise becomes zero in the Lorenz gauge. But rather than fix the gauge with an auxiliary condition $\partial_\alpha A^\alpha = 0$, we instead fix $\partial_\alpha A^\alpha = \partial\phi/\partial t + \nabla\cdot\mathbf{A}$ to the physical Lagrangian density and the physical action. It will be seen that (4.5) is a cousin of the R_ξ gauge conditions, which are ordinarily written as $\delta\mathcal{L} = -\left(\partial_\alpha A^\alpha\right)^2/2\xi$. Once we are working with the action, we are but a step away from Quantum Electrodynamics, which is generated through the path integration $Z_{em} = \int DA^\alpha \exp\left(iS_{em}/\hbar\right)$. As usual, we may start with $A_\beta J^\beta + \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} = -\mathcal{L}_{em}$ to obtain the action $S_{em} = \int d^4x\left(\frac{1}{2}A_\mu\left(g^{\mu\nu}\partial_\sigma\partial^\sigma - \partial^\mu\partial^\nu\right)A_\nu - J_\mu A^\mu\right)$ and then use this via Gaussian integration to path integrate. But the upshot of (4.5) is to tell us, alternatively, that:

$$S_{em} = \int d^4x\left(\frac{1}{2}A_\mu\left(g^{\mu\nu}\partial_\alpha\partial^\alpha - \partial^\mu\partial^\nu\right)A_\nu - J_\alpha A^\alpha\right) = -\int d^4x\left(\partial_\alpha A^\alpha\right)^2 . \tag{4.6}$$

The foregoing are all consequences of using the geodesic gauge (3.5), which *intrinsically* affect the freedom of the gauge field itself. In particular, using the geodesic gauge requires us to also fix the fields to the Lagrangian gauge of (4.4).

5. The Geodesic Gauge and the Action Gauge: Extrinsic Effects and the Power Equation

Next, we study the *extrinsic* effects on the canonical energy-momentum relation (2.6). We first return to (2.6), which we write as $m^2 = \pi_\sigma \pi^\sigma = \left(p_\sigma + qA_\sigma\right)\left(p^\sigma + qA^\sigma\right)$. The mass is invariant, so its four-gradient $\partial_\beta m = 0$. Therefore, if we distribute the terms in (2.6) and then take the gradient of both sides of (2.6), then after reduction we obtain:

$$0 = p_\sigma \partial_{;\beta} p^\sigma + q\partial_{;\beta} A_\sigma p^\sigma + qA_\sigma \partial_{;\beta} p^\sigma + \tfrac{1}{2} q^2 \partial_{;\beta}\left(A_\sigma A^\sigma\right). \tag{5.1}$$

Here again we apply the geodesic gauge (3.5), so the last term $\partial_{;\beta}\left(A_\sigma A^\sigma\right) = \partial_\beta\left(A_\sigma A^\sigma\right) = 0$ is removed. We may also use the field strength to replace $\partial_{;\beta} A_\sigma = F_{\beta\sigma} + \partial_{;\sigma} A_\beta$. Additionally, $p^\sigma = mu^\sigma$ is the ordinary mechanical momentum, so we can divide out *m*, whereby $p^\sigma \to u^\sigma$ throughout the remaining terms in the above. Thus, lowering the free index and segregating the field strength term on the left, (5.1) becomes, in geodesic and action gauges:

$$qF_{\beta\sigma} u^\sigma = -p_\sigma \partial_{;\beta} u^\sigma - qA_\sigma \partial_{;\beta} u^\sigma - q\partial_{;\sigma} A_\beta u^\sigma . \tag{5.2}$$

We of course recognize $qF_{\beta\sigma} u^\sigma$ as a variant of the Lorentz force term in (2.1).

Now, we wish to express the terms on the right in relation to the passage of proper time, that is, as derivatives along the curve, see (3.6) and (3.7). For the last term in (5.2) we may substitute $\partial_{;\sigma} A_\beta u^\sigma = dA_\beta / d\tau - \Gamma^\tau{}_{\sigma\beta} A_\tau u^\sigma$ derived using the gravitationally-covariant derivative and the chain rule. So (5.2) advances to:

$$qF_{\beta\sigma} u^\sigma = -p_\sigma \partial_{;\beta} u^\sigma - qA_\sigma \partial_{;\beta} u^\sigma - q\frac{dA_\beta}{d\tau} + q\Gamma^\tau{}_{\sigma\beta} A_\tau u^\sigma . \tag{5.3}$$

As to the remaining terms, we now multiply by $u^\beta = dx^\beta / d\tau$ throughout, giving us a $u^\beta \partial_{;\beta} u^\sigma$ in the first two terms after the equality. Then we may similarly derive and then substitute $u^\beta \partial_{;\beta} u^\sigma = du^\sigma / d\tau + \Gamma^\sigma{}_{\beta\tau} u^\beta u^\tau$. Also writing $p_\sigma = mu_\sigma$ for the remaining mechanical momentum, and seeing that the terms with $\Gamma^\tau{}_{\sigma\beta} A_\tau u^\beta u^\sigma$ cancel identically, with renamed indices and *c* restored, we now have:

$$\frac{q}{c} F_{\mu\nu} u^\mu u^\nu = -\left(mu_\sigma + \frac{q}{c} A_\sigma\right)\frac{du^\sigma}{d\tau} - \frac{q}{c} u^\sigma \frac{dA_\sigma}{d\tau} - m\Gamma^\sigma{}_{\mu\nu} u_\sigma u^\mu u^\nu . \tag{5.4}$$

This $\left(q/c\right) F_{\mu\nu} u^\mu u^\nu$ is a scalar number, and it has dimensions of power. So this is an expression for electrodynamic power. However, because $F_{\mu\nu}$ is an antisymmetric tensor, the term on the left vanishes identically. Therefore, moving all of the mechanical and gravitational terms to the left and keeping the electrodynamic terms on the right, we may consolidate to:

$$mu_\sigma\left(\frac{du^\sigma}{d\tau}+\Gamma^\sigma{}_{\mu\nu}u^\mu u^\nu\right)=-\frac{q}{c}\frac{d}{d\tau}\left(A_\sigma u^\sigma\right). \tag{5.5}$$

It is easily see that when the right hand side becomes zero in the absence of electrodynamics, the left hand side contains the gravitational geodesic motion (1.1). In terms of spacetime coordinates with all terms expanded, and isolating all the acceleration terms on the left, another way to express this is:

$$\left(m\frac{dx_\sigma}{d\tau}+\frac{q}{c}A_\sigma\right)\frac{d^2x^\sigma}{d\tau^2}=-\left(m\Gamma^\sigma{}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}+\frac{q}{c}\frac{dA^\sigma}{d\tau}\right)\frac{dx_\sigma}{d\tau}. \tag{5.6}$$

In the absence of gravitation, we merely set $\Gamma^\sigma{}_{\mu\nu}=0$. It is important to keep in mind that (5.5), (5.6) is fixed to the geodesic gauge $\partial_\beta\left(A_\sigma A^\sigma\right)=0$ of (3.5), thus also to the action gauge $S_{em}=-\int d^4x\left(\partial_\alpha A^\alpha\right)^2$ of (4.5).

6. Electrodynamic Time Dilation and Contraction

As noted earlier, the number "1" constructed in (3.1) is useful in a variety of circumstances. Another such circumstance is to explicitly introduce the Lorentz contraction factor $\gamma_v=1/\sqrt{1-v^2/c^2}$ and the ordinary four-velocity $v^\mu/c=\left(1,\mathbf{v}/c\right)$. With $g_{\mu\nu}=\eta_{\mu\nu}$, it is easily shown and well-known that $\eta_{\mu\nu}\left(\gamma_v v^\mu\right)\left(\gamma_v v^\nu\right)/c^2=1$, which is another "1." So if we write (3.1) in flat spacetime as $\eta_{\mu\nu}U^\mu U^\nu/c^2=1$, we see that the canonical velocity U^μ, not the mechanical velocity u^μ, is related expressly to γ_v and v^μ by:

$$U^\mu=\gamma_v v^\mu. \tag{6.1}$$

This may then be generalized into curved spacetime. Additionally, we may ascertain from the final equality in (3.1), which we then combine with (6.1), that:

$$U^\mu=u^\mu+\frac{q}{mc}A^\mu=\frac{dx^\mu}{d\tau}+\frac{q}{mc}A^\mu=\gamma_v v^\mu. \tag{6.2}$$

This may be conversely rewritten in terms of the ordinary mechanical velocity as:

$$u^\mu=\frac{dx^\mu}{d\tau}=U^\mu-\frac{q}{mc}A^\mu=\gamma_v v^\mu-\frac{q}{mc}A^\mu. \tag{6.3}$$

With these relationships, we return to (2.9), which states that the metric line element $d\tau$ must be invariant, and the metric tensor $g_{\mu\nu}$ and the gauge field A^μ (the latter now subject to the geodesic and action gauge conditions (3.5) and (4.5)) must be unchanged under a rescaling of $q/m \to q'/m'$. Thus, it is (2.9) which *defines* the coordinate transformation $x^\mu \to x'^\mu$ leading to electrodynamic time dilation and contraction. Now we show exactly how this occurs.

Generally, we will wish to compare the rate at which time flows for a massive body which has a net charge of zero and so is neutral, in relation to a material body with a nonzero net charge. We assume for now that there is no gravitation. Via (2.9), this means that we shall set $q=0$ (neutrality) and leave q' as is (charged). Therefore, (2.9) becomes:

$$c^2 d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{\mu\nu}\left(dx'^\mu + \frac{q'}{m'c} d\tau A^\mu\right)\left(dx'^\nu + \frac{q'}{m'c} d\tau A^\nu\right). \tag{6.4}$$

From this, we can immediately extract the coordinate transformation:

$$dx'^\mu = dx^\mu - \frac{q'}{m'c} d\tau A^\mu . \tag{6.5}$$

Because the coordinates x^μ are associated with a neutral net charge, as a notational convenience we shall drop the primes from the mass and charge and write this as $dx'^\mu = dx^\mu - (q/mc)\, d\tau A^\mu$. Thus, dx'^μ represents the coordinates of the body with q/m and dx^μ the coordinates of the neutral body. With this notational adjustment, and dividing through by $d\tau$, we obtain the relation:

$$u'^\mu = \frac{dx'^\mu}{d\tau} = \frac{dx^\mu}{d\tau} - \frac{q}{mc} A^\mu = u^\mu - \frac{q}{mc} A^\mu . \tag{6.6}$$

The time component of this with $x^\mu = (ct, \mathbf{x})$ and $A^\mu = (\phi, \mathbf{A})$ is easily seen to be:

$$\frac{dt'}{d\tau} = \frac{dt}{d\tau} - \frac{q\phi}{mc^2} . \tag{6.7}$$

So in the rest frame where $dt/d\tau = 1$ for the neutral body (because we have posited no gravitation for now) and $A^\mu = (\phi_0, 0)$ with ϕ_0 being the proper scalar potential, this becomes:

$$\gamma_{em} \equiv \frac{dt'}{d\tau} = 1 - \frac{q\phi_0}{mc^2} . \tag{6.8}$$

This is where we define the factor γ_{em}, first introduced between (2.9) and (2.10), to be the rate of time flow for a net-charged body q in a proper potential ϕ_0, in relation to the rate of time flow for a net-neutral body, all at relative rest. As obtained from (6.4), the above (6.8) is what allows the

Lorentz force motion (2.1) to be deduced from the minimized variation (1.1) without compromising the integrity of the background fields.

Now, because $A^\mu = (\phi_0, \mathbf{0})$ at rest, the question also arises how to specify A^μ generally when there is motion. Specifically, the choice would be between $A^\mu = \phi_0 U^\mu / c$ using the canonical velocity or $A^\mu = \phi_0 u^\mu / c$ using the mechanical velocity. But we see from $U^\mu = \gamma_v v^\mu$ in (6.1) that $A^\mu = \phi_0 U^\mu / c$ is the proper choice, that is:

$$A^\mu = \phi_0 U^\mu / c = \phi_0 \gamma_v v^\mu / c, \tag{6.9}$$

because at rest $\gamma_v v^\mu / c = (1, \mathbf{0})$, and this yields the correct result that $A^\mu = (\phi_0, \mathbf{0})$ at rest.

With (6.9) we may now obtain several other important results. Using this in (6.3) yields:

$$u^\mu = \frac{dx^\mu}{d\tau} = \left(1 - \frac{q\phi_0}{mc^2}\right)\gamma_v v^\mu = \gamma_{em}\gamma_v v^\mu = \gamma_{em} U^\mu . \tag{6.10}$$

So we see that the mechanical velocity u^μ is related to the canonical velocity U^μ through a multiplicative factor given by γ_{em}. The inverse result $U^\mu = u^\mu / \gamma_{em}$ can be combined with (6.2) with everything multiplied through by *m* to also obtain:

$$mU^\mu = \frac{1}{\gamma_{em}} mu^\mu = \frac{1}{\gamma_{em}} p^\mu = mu^\mu + \frac{q}{c} A^\mu = \pi^\mu . \tag{6.11}$$

This contains the relationship $p^\mu = \gamma_{em}\pi^\mu$ between the mechanical and canonical momentum, mirroring $u^\mu = \gamma_{em} U^\mu$ in (6.10). Then, we may multiply (6.10) through by mc to obtain the energy-dimensioned four vector, and also use (6.11), to write:

$$cp^\mu = mcu^\mu = mc\frac{dx^\mu}{d\tau} = mc\gamma_{em}\gamma_v v^\mu = mc\gamma_{em} U^\mu = c\gamma_{em}\pi^\mu . \tag{6.12}$$

All of this finally leads us to take the time component in the non-relativistic limit, namely:

$$E = cp^0 = mc^2\gamma_{em}\gamma_v = mc^2 \frac{1 - \frac{q\phi_0}{mc^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \cong mc^2\left(1 - \frac{q\phi_0}{mc^2}\right)\left(1 + \frac{1}{2}\frac{v^2}{c^2}\right) = mc^2 + \frac{1}{2}mv^2 - q\phi_0 - \frac{1}{2}\frac{q\phi_0}{c^2}v^2 . \tag{6.13}$$

This is how the key energy relationship (2.10) originates. Here, in succession, we see 1) the rest energy mc^2, 2) the kinetic energy of the mass *m*, 3) the electrical interaction energy of the charged mass, 4) the kinetic energy of the electrical energy. If we then choose a Coulomb proper potential $\phi_0 = -k_e Q / r$ so that the charges have *opposite signs* and so are attracting in the same way that gravitation attracts, then we arrive precisely at the first four terms of (2.10).

Then to add gravitation, it is convenient to start with the metric (2.5) in the form $c^2 d\tau^2 = g_{\mu\nu} Dx'^{\mu} Dx'^{\nu}$ for the charged mass which has the x'^{μ} coordinates, which mass is taken to be at rest in the gravitational field so that $d\tau^2 = g_{00} Dt'^2$ a.k.a. $Dt' / d\tau = 1/\sqrt{g_{00}}$. If we write this out using $Dx'^{\mu} = dx'^{\mu} + (q/mc) d\tau A^{\mu}$, also using (6.8) in the form $dt'/d\tau = dt/d\tau - q\phi_0 / mc^2$ because setting $dt/d\tau = 1$ was appropriate for a neutral body with no gravitation but $dt/d\tau$ cannot be summarily set to 1 once there is gravitation, then we have

$$\frac{1}{\sqrt{g_{00}}} = \frac{Dt'}{d\tau} = \frac{dt'}{d\tau} + \frac{q\phi_0}{mc^2} = \frac{dt}{d\tau} - \frac{q\phi_0}{mc^2} + \frac{q\phi_0}{mc^2} = \frac{dt}{d\tau}. \tag{6.14}$$

The electrodynamic terms cancel, leaving the usual relationship $dt/d\tau = 1/\sqrt{g_{00}} \equiv \gamma_g$ for time dilation or contraction for a particle at rest in a gravitational field. This then supplements $\gamma_{em}\gamma_v \to \gamma_g \gamma_{em} \gamma_v$ in (6.10), (6.12) and (6.13). Particularly, (6.13) becomes $E = cp^0 = mc^2 \gamma_g \gamma_{em} \gamma_v$, which is synonymous with (2.10), and it then becomes possible to simultaneously represent the combined effects of gravitation, electrodynamics and motion, upon time and energy.

7. Conclusion

The energy relation $E = cp^0 = mc^2 \gamma_g \gamma_{em} \gamma_v$ shown in (2.10) , results from combining (6.13) and (6.14). The fact that (2.10) correctly reproduces widely-corroborated, well-established energy relations, is an important point of validation that the geometro-electrodynamic viewpoint which has been presented here is empirically correct. However, the mainspring which enables everything to fit together without contradiction is the time flow relationship

$$\frac{dt}{d\tau} = \gamma_g \gamma_{em} \gamma_v = \frac{1 + q\phi_0 / mc^2}{\sqrt{g_{00}}\sqrt{1 - v^2/c^2}} \cong \left(1 + \frac{GM}{c^2 r}\right)\left(1 + \frac{q\phi_0}{mc^2}\right)\left(1 + \frac{1}{2}\frac{v^2}{c^2}\right) \tag{7.1}$$

contained within (6.13) when supplemented by (6.14) and applied to gravitation in the Newtonian limit. Consequently, it becomes most important to perform experimental tests of these predicted time flow changes for charged bodies in electromagnetic fields. Although these time flow relations (7.1) go hand-in-hand with the energy relations (2.10), it is (7.1) which nevertheless is the theoretical foundation of the energy relations (2.10). That is, the widely-corroborated energy relations (2.10) are rooted in geometrodynamic measurement of space lengths and the flow rates of time. Experimental observation of a change in the rate at which time flows for charged bodies in electromagnetic fields an accordance with (7.1) would therefore confirm this geometrodynamic foundation for classical electrodynamics in four spacetime dimensions.

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