ALGEBRAIC POINCARÉ DUALITY 1

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Abstract. This paper includes two main chapters, §2 and §3. Each deals with one type of algebraic Poincaré duality (APD) on linear spaces originated from algebraic cycles. Two types of APD confirm the following conjectures:
(1) the Griffiths' conjecture on the incidence equivalence versus Abel-Jacobi equivalence.
(2) the standard conjectures including the "D" conjecture over $\mathbb{C}$.

1 Introduction

Let X be a smooth projective variety of dimension n over $\mathbb{C}$. There is Poincaré duality: the non degenerate intersection pairing $\mathcal{B}_X$ (or simply $\mathcal{B}$),

$$H^i(X;\mathbb{Q}) \times H^{2n-i}(X;\mathbb{Q}) \quad \rightarrow \quad \mathbb{Q}. \tag{1.1}$$

Furthermore, one can ask about the non-degeneracy of the restriction $\mathcal{B}|_{G_2\times G_1}$ to the subgroups

$$G_2 \times G_1 \subset H^i(X;\mathbb{Q}) \times H^{2n-i}(X;\mathbb{Q}) \tag{1.2}$$

We call the non-degeneracy of $\mathcal{B}|_{G_2\times G_1}$, algebraic Poincaré duality in the title or APD for the abbreviation. We are only interested in structural subgroups G_i in the cohomology.

Algebraic Poincaré duality in many specific cases is known. Among the known cases, the consequences and the reasons will vary with its structure on the underline variety (see [12]). They range from the topological structure to the complex Kähler structure. But non of them involve the algebro-complex structure (that is an algebraic structure, or a Kähler structure with the rational Kähler form). In this paper we discuss following two types of subspaces, which arise from topological and algebro-complex structure:

THEOREM 1.1. *(on cycles of odd dimensions) Let $G_2\times G_1$ be the "algebraic part" of cohomology (see definition 2.1 below),*

$$H_a^{2p+1}(X;\mathbb{Q}) \times H_a^{2n-2p-1}(X;\mathbb{Q}) \subset H^{2p+1}(X;\mathbb{Q}) \times H^{2n-2p-1}(X;\mathbb{Q}) \tag{1.3}$$

where the "algebraic part" $H_a^{2p+1}(X;\mathbb{Q})\otimes C$ is also called "cohomology lying on a subvariety of codimension at least p", denoted by $Filt'^p H^{2p+1}(X;\mathbb{C})$. In this case, APD holds.

THEOREM 1.2. *(on cycles of even dimensions) Let $G_i, i = 1, 2$ be the subspaces $A^p(X), A^{n-p}(X)$ of the rational cohomology groups, spanned by all* *rational algebraic cycles of corresponding codimensions $p, n-p$ respectively. So*

$$G_2 \times G_1 = A^p(X) \times A^{n-p}(X) \subset H^{2p}(X;\mathbb{Q}) \times H^{2n-2p}(X;\mathbb{Q}). \tag{1.4}$$

In this case, APD holds.

It follows from theorem 1.1 that

COROLLARY 1.3. *Griffiths' conjecture on incidence equivalence versus Abel-Jacobi equivalence is correct.*

It follows from theorem 1.2 that

COROLLARY 1.4. *The standard conjectures and "D" conjecture over $\mathbb{C}$ are correct.*

Notations:
(1) $(\bullet)^*$ denotes the dual of a vector space if $\bullet$ is a vector space or a vector.
(2) $(\bullet)^*$ also denotes a pullback from the cohomology or differential forms if $\bullet$ is a map.
(3) $(\bullet)_*$ denotes a pushforwad of the homology, or cycles, or currents if $\bullet$ is a map.
(4) $\overline{\bullet}$ denotes the complex conjugation on a complex vector space.
(5) $[a]$ denotes an equivalence class of the element a.
(6) CH denotes the Chow group, CH_{alg} denotes the subgroup of cycles algebraically equivalent to zero.
(7) J denotes the intermediate Jacobian or the Jacobian.
(8) $\mathcal{T}$ denotes the complex torus.
(9) All homology and cohomology are groups modulo their torsions.

Our idea of the proof is as follows.

It suffices to show theorems 1.1, 1.2 for cohomologies with complex coefficients. So throughout we change the coefficients to the complex numbers. First we prove the theorem 1.1. It has two parts: (1) Switch the APD from linear spaces to Abelian varieties, then break it into APDs on a finite and selected Abelian subvarieties. (2) Use both Lefschetz theorems to show APDs on the selected Abelian subvarieties hold. Let's see the detail.

(1). "Breaking the whole duality into sub-dualities". Notice that this is the APD on the classes of cycles of odd degrees on X. The proof uses the intermediate Jacobians over the complex numbers. For each fixed odd degree $2r-1$, the topological classes of degree $2r-1$ form a complex torus called "intermediate Jacobian $J^r(X)$". Subsequently the partially algebraic cycles form a subtorus $J_a^r(X) \subset J^r(X)$ which is Abelian inside of the intermediate Jacobian. Then the APD on the two subgroups of cohomology of complementary degrees becomes a particular non-degeneracy property for the Poincaré line bundle—the induced map $\mathcal{P}$ from one Abelian variety to the dual of another Abelian variety

$$\begin{array}{ccc} J_a^{p+1}(X) & \stackrel{\mathcal{P}}{\to} & Pic^0(J_a^{n-p}(X)) \\ (J_a^{n-p}(X) & \stackrel{\mathcal{P}'}{\to} & Pic^0(J_a^{p+1}(X))) \end{array}$$

is an isogeny. We call the isogeny, the Saito's duality between Abelian varieties

$$J_a^{p+1}(X) \quad and \ J_a^{n-p}(X)$$

(with map $\mathcal{P}$). To extend it, we call any isogeny between two Abelian subvarieties a Saito's duality. In [9], Saito did not prove, but speculated such a duality between $J_a^{p+1}(X)$ and $J_a^{n-p}(X)$.[1] Our APD is equivalent to the Saito's duality between

$$J_a^{p+1}(X) \quad and \quad J_a^{n-p}(X).$$

To prove it, we break the duality into a multiple smaller Saito's dualities between Abelian subvarieties that cover $J_a^{p+1}(X)$ and $J_a^{n-p}(X)$. We call it a curve-like sub Abelian structure. Let's see the detailed description. Let

$$C_2 \stackrel{\rho}{\to} J_a^{p+1}(X) \tag{1.5}$$

be a regular map from a smooth projective curve to the Abelian variety. Then it is known that the Jacobian $J(C_2)$ of C_2 is mapped to $J_a^{p+1}(X)$, whose image denoted by $\mathcal{T}_{C_2}$ is isomorphic to a quotient of the Jacobian $J(C_2)$. By the Jacobi inversion, if C_2 goes through 0, the image of C_2 is contained in $\mathcal{T}_{C_2}$. Now such Abelian varieties $\mathcal{T}_{C_2}$ will cover the entire $J_a^{p+1}(X)$ (One selected C_2 is enough). We call the set of $\mathcal{T}_{C_2}$ the curve-like sub Abelian structure of $J_a^{p+1}(X)$. Then similarly the other part of the dual,

$$Pic^0(J_a^{n-p}(X)) \simeq J_a^{n-p}(X) \tag{1.6}$$

also has the curve-like sub Abelian structure, whose Abelian subvarieties are denoted by $\mathcal{T}_{C_2^1}$ for smooth projective curves C_2^1. It is quite natural to speculate that the Saito's duality (which is induced by an isogeny between two Abelian varieties) between

$$J_a^{p+1}(X) \quad and \quad J_a^{n-p}(X)$$

is equivalent to the Saito's duality between

$$\begin{array}{ccc} \mathcal{T}_{C_2} & and & \mathcal{T}_{C_2^1} \\ \bigcap & & \bigcap \\ J_a^{p+1}(X) & & J_a^{n-p}(X) \end{array}$$

[1] This duality is equivalent to the Griffiths' conjecture on the incidence equivalence, and it was one of main topics in [9]. But the Saito's proof was not complete because of lacking of a rigorous argument for a Griffiths' announcement.

for all C_2.[2] The speculation is correct and the proof of it is presented in section 6, [12]. Unfortunately the proof over there was not complete because for general p, it assumed the Lefschetz standard conjecture. To avoid the Lefschetz standard conjecture, in this paper we use the curve-like sub Abelian structures again. It reveals a fact that a sufficient condition for Saito's duality between

$$J_a^{p+1}(X) \quad and \quad J_a^{n-p}(X)$$

turns out to be the Saito's dualities between

$$\mathcal{T}_{C_2} \quad and \quad \mathcal{T}_{C_2^1}$$

for some selective curves C_2 (not ALL of them), and the only requirement for such curves C_2 is that they cover $J_a^{p+1}(X)$ and all contain 0. Above steps were completed in the first paper [12] without presence of geometric X.

(2) "Chow motivic correspondence". The last step, which is the main topic of this paper, is to show that the Saito's duality between two Abelian subvarieties

$$\mathcal{T}_{C_2} \quad and \quad \mathcal{T}_{C_2^1}$$

with selective C_2 holds. To do this, we need to make a connection between intermediate Jacobians and Chow motivic correspondences. By a "Chow motivic correspondence" we mean the correspondence is an algebraic cycle within its rational equivalence class. This is done by using Abel-Jacobi maps. The ambient Abelian variety $J_a^{p+1}(X)$ in general has another interpretation as the image of the Abel-Jacobi map on the parameter space of algebraic cycles of degree 1 less. Abel-Jacobi map is compatible with a correspondence between two varieties, which means that the curve C_2 to the Abelian variety $J_a^{p+1}(X)$ gives a rise to a Chow motivic correspondence Z of degree $-(n-p-1)$ from X to C_2. Using this interpretation, when restricted to a curve C_2, above duality becomes the equality of two sub linear spaces of $H^{1,0}(C_2)$ of the curve C_2 (i.e. equality of two Abelian subvarieties inside of the Jacobian $J(C_2)$), obtained by pulling the cohomologies on X back to C_2 through Z. A proof of the equality is the key to reveal a deep relation on the topological cycles on X. The difficulty of it is apparent. However the difficulty turns into a quasi-Lefschetz-hyperplane-theorem (see footnote 5 after formula (2.62)) once the curve C_2, i.e. the correspondence Z is chosen to meet the criterion above. Thus the flexibility in choosing the correspondence Z, which only requires C_2 goes through two points, is the key.[3] This is all done in section 2.3, step 1.

[2] Such a duality between Abelian subvarieties is exactly a type of algebraic Poincaré duality in the title, and it naturally spreads into the Abelian varieties. For instance, the dual curve C_2^1 of C_2 is $\mathcal{P}(C_2)$.

[3] In an interpretation of a classical approach by J. Murre [8] (also see section 6, [12]), the correspondence Z can't be altered, i.e. such a correspondence Z must be arbitrary to insure the duality on the entire Abelian variety. Then the Lefschetz standard conjecture must be used in dealing with arbitary Z for higher codimensions. Please see section 6, [12] for the detail.

This is the proof of theorem 1.1

After theorem 1.1, we prove theorem 1.2. Through a product with an elliptic curve, the APD on partially algebraic part in odd degree cohomology descends to that on algebraic cycles in even degree cohomology.

So in a short summary we first prove APD for the cycles of odd degrees because there are intermediate Jacobians in these degrees. Then we use a Cartesian product with a curve to transform the APD from odd degrees to even degrees. Therefore the main technique is the full theory of intermediate Jacobian over $\mathbb{C}$. The theory is indispensable to our proof.

We organize the paper as follows. In §2, we apply the result in [12] to prove theorem 1.1. The main work in this section is the verification for all conditions in the theorem 4.5, [12]. In the second half of the section, subsection 2.4, we introduce the Griffiths' conjecture on the incidence equivalence and show it is a corollary of theorem 1.1. In §3, using the product with an elliptic curve, we deduce theorem 1.2. In the second half of the section we show the standard conjectures as a corollary.

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2 APD on the cohomology groups of odd degrees

2.1 Definitions

In this section we prove theorem 1.1. Throughout we use p, q to denote two whole numbers satisfying $p + q = n - 1$. We first recall some definitions.

We define a 1-parameter family of r-cycles in the following set-up: there are a smooth projective curve T and an algebraic cycle

$$Z \in Z^{n-r}(T \times X), \tag{2.1}$$

such that,

(1) the support $|Z|$ of Z is projected onto the smooth T,

(2) the projection of $|Z| \to X$ is generically finite to one,

(3) Z intersects $\{t\} \times X$ properly.

We denote

$$Z(t) = (Pr_X)_*(Z \cdot (\{t\} \times X))$$

and Pr_X is the projection from $T \times X$ to X.

Definition 2.1.

(1) For any smooth 1-parameter space T of p-cycles in X, we let

$$H^T_{2p+1}(X;\mathbb{Z}) \quad (or \quad H^Z_{2p+1}(X;\mathbb{Z})) \tag{2.2}$$

be the subgroup of $H_{2p+1}(X;\mathbb{Z})$ defined to be the image of the topological homomorphism

$$\begin{array}{rcc} \nu_T : H_1(T;\mathbb{Z}) & \to & H_{2q+1}(X;\mathbb{Z}) \\ \gamma & \to & (Pr_X)_*((\gamma \times [X]) \cup [Z]). \end{array} \tag{2.3}$$

Let

$$H^T_{2p+1}(X;\mathbb{A}) = H^T_{2p+1}(X;\mathbb{Z}) \otimes \mathbb{A} \tag{2.4}$$

and

$$H_T^{2q+1}(X;\mathbb{A}) \quad (or \quad H_Z^{2q+1}(X;\mathbb{A})) \tag{2.5}$$

be its Poincaé dual, for $\mathbb{A} = \mathbb{Q}, \mathbb{R}, \mathbb{C}$.

(2) Let $H^a_{2p+1}(X;\mathbb{Z})$ be the subgroup of $H_{2p+1}(X;\mathbb{Z})$ generated by all

$$H^T_{2p+1}(X;\mathbb{Z}).$$

Similarly let

$$H^a_{2p+1}(X;\mathbb{A}) = H^a_{2p+1}(X;\mathbb{Z}) \otimes \mathbb{A} \tag{2.6}$$

and

$$H_a^{2q+1}(X;\mathbb{A}) \tag{2.7}$$

be its Poincaé dual, for $\mathbb{A} = \mathbb{Q}, \mathbb{R}, \mathbb{C}$. The subspace

$$H_a^{2q+1}(X;\mathbb{C}) \tag{2.8}$$

is called the "algebraic part" of cohomology. This coincides with the definition given by Murre in [8], and agrees with the Grothendieck's definition $Filt'^q H^{2q+1}(X)$.

Remark We skip the proofs of the coincidence in part (2). For its complete proofs please see [8].

It was proved that the algebraic part $H_a^{2p+1}(X;\mathbb{C})$ is a sub Hodge structure ([8]). Thus we define

DEFINITION 2.2.

$$\begin{aligned} F^{r+1}H_a^{2r+1}(X;\mathbb{C}) &= F^{r+1}H^{2r+1}(X;\mathbb{C}) \cap H_a^{2r+1}(X;\mathbb{C}). \\ \overline{F^{r+1}H_a^{2r+1}(X;\mathbb{C})} &= \overline{F^{r+1}H^{2r+1}(X;\mathbb{C})} \cap H_a^{2r+1}(X;\mathbb{C}). \end{aligned} \tag{2.9}$$

2.2 Review of the generalized APD for application

Theorem 1.1 uses another theorem, theorem 4.5 in [12]. So let's recall it. This theorem has a general setting, which does not require the presence of the projective variety X. Let V_1, V_2 be two isomorphic vector spaces over $\mathbb{C}$, equipped with a complex, non-degenerate, bilinear form $\mathcal{B}$ on $V_2 \times V_1$. Let Λ_1, Λ_2 be lattices of V_1, V_2, i.e. they are discrete subgroups of V_i such that

$$\Lambda_i \simeq \mathbb{Z}^{2dim(V_i)} \ and \ span_{\mathbb{R}}(\Lambda_i) = V_i.$$

Assume

$$\Lambda_i^a = \Lambda_i \cup V_i^a \tag{2.10}$$

are lattices of V_i^a. Let

$$\mathcal{B}_2 : V_2^a \xrightarrow{\mathcal{B}|_{V_2^a}} (V_1)^* \xrightarrow{restriction} (V_1^a)^*$$

$$\mathcal{B}_1 : V_1^a \xrightarrow{\mathcal{B}|_{V_1^a}} (V_2)^* \xrightarrow{restriction} (V_2^a)^*.$$

Also we assume the $\mathcal{B}$ satisfies

$$\Lambda_2 \times \Lambda_1 \to \mathbb{Z}.$$

We defined the lattice of the dual of the vector space to be the

$$hom(lattice, \mathbb{Z}). \tag{2.11}$$

We obtain the complex tori

$$\mathcal{T}(V_i^a) = \frac{V_i^a}{\Lambda_i^a}. \tag{2.12}$$

Let $C \stackrel{\rho}{\to} \mathcal{T}(V_i^a)$ be a regular map from a smooth complex curve C (not necessarily contains the origin).

Define Λ_C to be the image

$$H_1(C; \mathbb{Z}) \ \to \ H_1(\mathcal{T}(V_i^a)) = \Lambda_i^a. \tag{2.13}$$

Furthermore we denote $span_{\mathbb{C}}(\Lambda_C)$ by

$$H_C.$$

By [12], Λ_C is an integral lattice of H_C.

Define

$$\frac{H_C}{\Lambda_C} = \mathcal{T}_C, \tag{2.14}$$

to be the sub-torus of $\mathcal{T}(V_i^a)$. We call it a curve-like sub-torus.

DEFINITION 2.3. *(curve-like sub-Abelian structure)*

We say the complex torus $\mathcal{T}(V_i^a)$ has a curve-like sub-Abelian structure if there are a projective variety M_i and a SURJECTIVE group homomorphism from the Chow group of 0-cycles of M_i, that are algebraically equivalent to zero, to the complex torus,

$$\phi_i : CH_{alg}^{dim(M_i)}(M_i) \quad \to \quad \mathcal{T}(V_i^a) \tag{2.15}$$

for $i = 1, 2$, satisfying:

(1) it is regular, i.e. for any projective family T_i of algebraic 0-cycles of M_i, the composition map

$$\psi_{T_i} : T_i \quad \xrightarrow{(T_i)_t - m_i} \quad CH_{alg}^{dim(M_i)}(M_i) \quad \xrightarrow{\phi_i} \quad \mathcal{T}(V_i^a) \tag{2.16}$$

is regular, where m_i is a fixed point in the same irreducible component of M_i as T_i.

(2) If T_i is a smooth projective curve parametrizing 0-cycles on M_i, we'll denote the composition map

$$CH_{alg}^1(T_i) \quad \to \quad CH_{alg}^{dim(M_i)}(M_i) \quad \xrightarrow{\phi_i} \quad \mathcal{T}(V_i^a) \tag{2.17}$$

also by ϕ_i. Then for $\psi_{T_i}(T_i) \in \mathcal{T}(V_i^a)$,

$$\phi_i(CH_{alg}^1(T_i)) = \mathcal{T}_{\psi_{T_i}(T_i)} \tag{2.18}$$

and it is Abelian.

Let C_2 be a smooth curve on M_2, and $\psi_{C_2} : C_2 \to \mathcal{T}(V_2^a)$ be a morphism from definition 2.3. Denote the composition map

$$C_2 \quad \to \quad \mathcal{T}(V_2^a) \quad \overset{\tilde{\mathcal{B}}_R}{\to} \quad \mathcal{T}((V_1)^*) \tag{2.19}$$

by ρ_1, and

$$C_2 \quad \to \quad \mathcal{T}(V_2^a) \quad \overset{\tilde{\mathcal{B}}_2}{\to} \quad \mathcal{T}((V_1^a)^*) \tag{2.20}$$

by ρ_1^a.

DEFINITION 2.4. *We define*

$$\begin{aligned} E_1 &= (\rho_1)^* \Big(H^{1,0}(\mathcal{T}((V_1)^*)) \Big) \\ E_1^a &= (\rho_1^a)^* \Big(H^{1,0}(\mathcal{T}((V_1^a)^*)) \Big), \end{aligned} \tag{2.21}$$

and

$$\sigma(C_2) = dim(E_1) - dim(E_1^a). \tag{2.22}$$

Similarly we define $\sigma(C_1)$ for $C_1 \subset M_1$.

Then the theorem 4.5 in [12] says

THEOREM 2.5. *Let $\beta \in V_2^a, \beta' \in V_1^a$. Assume*
(1) $\mathcal{T}(V_i^a)$ both are Abelian and both have curve-like sub-Abelian structures,
(2) there exists a curve C_2 such that C_2 is through $\pi_2(\beta), 0$ with

$$\sigma(C_2) = 0. \tag{2.23}$$

and the same is true in opposite direction, i.e., there exists a curve C_1 such that C_1 is through $\pi_1(\beta'), 0$ for given $\beta' \in V_1^a$ with

$$\sigma(C_1) = 0. \tag{2.24}$$

Then APD holds.

2.3 Geometric application

The content above from [12] is abstract. Next we give the geometric setting, then apply it. This section contains the last geometric step as introduced in section 1.

Let

$$\begin{aligned} V_2 &= \overline{F^{p+1}H^{2p+1}(X;\mathbb{C})} \\ V_1 &= F^{q+1}H^{2q+1}(X;\mathbb{C}) \end{aligned} \tag{2.25}$$

Let

$$\begin{aligned} V_2^a &= \overline{F^{p+1}H_a^{2p+1}(X;\mathbb{C})} \\ V_1^a &= F^{q+1}H_a^{2q+1}(X;\mathbb{C}) \end{aligned} \tag{2.26}$$

Now due to the sub Hodge structures of algebraic parts, they both have lattices denoted by

$$\overline{F^{p+1}H_a^{2p+1}(X;\mathbb{Z})}, F^{q+1}H_a^{2q+1}(X;\mathbb{Z}) \tag{2.27}$$

respectively. Now it is known that the complex tori obtained from $V_i^a, i = 1, 2$ are just the images

$$J_a^\bullet(X)$$

of Abel-Jacobi maps, and it is well-known that they are Abelian. Thus

$$\begin{aligned} \mathcal{T}(V_2^a) &\simeq J_a^{p+1}(X) \\ \mathcal{T}(V_1^a) &\simeq J_a^{q+1}(X). \end{aligned} \tag{2.28}$$

Now let M_1, M_2 be projective varieties parametrizing the q, p cycles of large degrees. Next ϕ_1, ϕ_2 are the Abel-Jacobi maps. There are multiple versions of such M_i. In the following we describe the one we'll use that satisfies definition 2.3. Let's start with a general description. Let Q be a smooth projective variety (irreducible) and let

$$\mathcal{Q} \subset Q \times X \tag{2.29}$$

be a flat family of codimension r schemes in X. This defines a family of $n-r$ cycles by taking

$$\mathcal{Q}_q = (Pr_X)_*[\mathcal{Q} \cap (\{q\} \times X)]. \tag{2.30}$$

for $q \in Q$. Then there is a holomorphic map into the intermediate Jacobian ψ_Q:

$$\begin{array}{ccc} Q & \stackrel{\psi_Q}{\rightarrow} & J^r(X) = \frac{(F^{n-r+1}H^{2n-2r+1}(X))^*}{H_{2n-2r+1}(X;\mathbb{Z})} \\ q & \rightarrow & \frac{\int_{\Gamma_q}(\cdot)}{H_{2n-2r+1}(X;\mathbb{Z})}, \end{array} \tag{2.31}$$

where Γ_q is a singular chain on X satisfying $\partial\Gamma_q = \mathcal{Q}_q - \mathcal{Q}_{q_0}$ where $q_0 \in Q$ is fixed. If $\mathcal{Q}_q$ is rationally equivalent to $\mathcal{Q}_{q_0}$, then $\psi_Q(q) = 0$. This allows us to have a well-defined map ϕ:

$$\begin{array}{ccc} CH^{n-r}_{alg}(X) & \stackrel{\phi}{\rightarrow} & J^r(X) \\ B & \rightarrow & \phi(B) \end{array} \tag{2.32}$$

where $CH^{n-r}_{alg}(X)$ denotes the subgroup of the Chow group consists of r-dimensional cycles algebraically equivalent to zero, the map is denoted by ϕ, still called Abel-Jacobi map. By the proposition 1.2, [9], there exist an Abelian variety M and a cycle class

$$\mathcal{M} \subset CH^r(M \times X) \tag{2.33}$$

such that the composition map ϕ_M

$$M \xrightarrow{\mathcal{M}_m - \mathcal{M}_0} CH^r_{alg}(X) \xrightarrow{\phi(\mathcal{M}_m - \mathcal{M}_0)} J^r_a(X) \tag{2.34}$$

is an isogeny.

Coming back to our situation we let M_1, M_2 be the above Abelian varieties for the tori $\mathcal{T}(V_i^a), i = 1, 2$ and ϕ_1, ϕ_2 be the Abel-Jacobi maps induced from ϕ_{M_i},

$$\begin{array}{ccc} CH^{dim(M_1)}_{alg}(M_1) & \stackrel{\phi_1}{\rightarrow} & J^{q+1}_a(X) \\ CH^{dim(M_2)}_{alg}(M_2) & \stackrel{\phi_2}{\rightarrow} & J^{p+1}_a(X). \end{array} \tag{2.35}$$

Next we would like to show the intermediate Jacobians have curve-like sub Abelian structures defined in [12]. It suffices to verify assumptions 1 and 2 of theorem 2.5. Let's start with assumption 1. Let T be a smooth projective variety representing a 1 parameter family of q-cycles in X as in section 2.1.

Define

$$J_T^{p+1}(X) = \frac{\overline{F^{p+1}H_T^{2p+1}(X;\mathbb{C})}}{H_T^{2p+1}(X;\mathbb{Z})}. \tag{2.36}$$

Note $J_T^{p+1}(X)$ is a subtorus of the intermediate Jacobian $J^{p+1}(X)$.

Denote the composition map

$$CH^1_{alg}(T) \ \rightarrow \ CH^{p+1}_{alg}(X) \ \stackrel{\phi}{\rightarrow} \ J_a^{p+1}(X). \tag{2.37}$$

also by ϕ, where the first map is the correspondence map.

Regarding T as a curve on M_i, the following theorem verifies the assumption 1 of theorem 2.5.

Theorem 2.6.

Let T be a family of algebraic cycles of codimension $p+1$. Then

$$J_T^{p+1}(X) = \phi(CH^1_{alg}(T)). \tag{2.38}$$

and it is an Abelian variety.

Proof. Let

$$\begin{array}{ccc} & Z \in CH^{p+1}(T\times X) & \\ & \swarrow p_1 \quad p_2 \searrow & \\ T & & X, \end{array} \tag{2.39}$$

be the correspondence for T. First we consider the commutative diagram

$$\begin{array}{ccccc} H_1(T;\mathbb{C}) & \stackrel{\nu_T}{\longrightarrow} & H_{2q+1}(X;\mathbb{C}) & & \leftarrow\text{-- } \textit{homology} \\ {\scriptstyle P}\downarrow\uparrow{\scriptstyle P^{-1}} & & {\scriptstyle P}\downarrow\uparrow{\scriptstyle P^{-1}} & & \\ H^1(T;\mathbb{C}) & \stackrel{Z^*}{\longleftarrow} & H^{2p+1}(X;\mathbb{C}) & & \leftarrow\text{-- } \textit{cohomology} \end{array} \tag{2.40}$$

where Z^* is induced from the pushforward of currents,

$$Z \rightarrow T$$

of C^∞-forms on $T\times X$ restricted to Z, and P is the Poincaré duality. Through the Poincaré duality, the cokernel of ν_T is the kernel of Z^* and vice versa. In the spaces of the diagram (2.40), we remove all cokernels and kernels. Then obtain the diagram

$$\begin{array}{ccc} W_1 & \xrightarrow{\nu_T} & V_{2q+1} \\ P \Big\downarrow \Big\uparrow P^{-1} & & P \Big\downarrow \Big\uparrow P^{-1} \\ W^1 & \xleftarrow{Z^*} & V^{2p+1} \end{array} \tag{2.41}$$

where W_1 is the direct sum complement of the kernel of ν_T, V_{2q+1} is the image of ν_T, W^1 is the image of Z^* and V^{2p+1} is a direct sum complement of the kernel of Z^*. All homomorphism in (2.41) are isomorphisms. Inside of these spaces there are spanning integral lattices denoted by $W_1^{\mathbb{Z}}$, $W^1_{\mathbb{Z}}$, $V_{2q+1}^{\mathbb{Z}}$ and $V_{\mathbb{Z}}^{2p+1}$ respectively. Using the Hodge decomposition to the bottom row in (2.41), we obtain

$$(W^1)' \xleftarrow{Z^*} (V^{2p+1})' \tag{2.42}$$

where

$$(W^1)' = W^1 \cup H^{0,1}(X;\mathbb{C})$$
$$(V^{2p+1})' = V^{2p+1} \cup \overline{F^{p+1}H^{2p+1}(X;\mathbb{C})}.$$

Let

$$(W^1_{\mathbb{Z}})'$$
$$(V^{2p+1}_{\mathbb{Z}})'.$$

be their integral lattices. The existence of the integral lattices is guaranteed by the induced, sub-Hodge structure of $H_T^{2p+1}(X;\mathbb{C})$. Then by the isomorphisms in (2.41), we obtain that

$$\frac{(W^1)'}{(W^1_{\mathbb{Z}})'} \simeq \frac{(V^{2p+1})'}{(V^{2p+1}_{\mathbb{Z}})'}. \tag{2.43}$$

By the definition of the right side of (2.43)

$$\frac{(V^{2p+1})'}{(V^{2p+1}_{\mathbb{Z}})'}$$

is

$$J_T^{p+1}(X).$$

To understand the left side of (2.43), we need to understand the Jacobian $J(T)$ of the curve T. There is a projection map

$$J(T) \quad \to \quad \frac{(W^1)'}{(W^1_{\mathbb{Z}})'} \tag{2.44}$$

induced from the linear space projection ($(W^1)'$ is a subspace of $H^{0,1}(T;\mathbb{C})$),

$$H^{0,1}(T;\mathbb{C}) \to (W^1)'.$$

Let

$$\begin{array}{rcl}\phi' : CH^1_{alg}(T) & \to & \frac{(W^1)'}{(W^1_{\mathbb{Z}})'} \\ t & \to & \int_{\Gamma_t} \theta\end{array} \tag{2.45}$$

be the composition, where θ is a $(1,0)$ closed form and Γ_t is the real singular chain such that

$$[\partial \Gamma_t] = [t] \ \textit{in cohomology}.$$

Then applying the diagram (3.40) to the maps ϕ, ϕ', we obtain that

$$\phi(CH^1_{alg}(T)) = \phi'(CH^1_{alg}(T)). \tag{2.46}$$

We conclude this with a commutative diagram

$$\begin{array}{ccc} CH^1_{alg}(T) & = & CH^1_{alg}(T) \\ \phi' \Big\downarrow & & \phi \Big\downarrow \\ \frac{(W^1)'}{(W^1_{\mathbb{Z}})'} & \stackrel{\simeq}{\to} & \frac{(V^{2p+1})'}{(V^{2p+1}_{\mathbb{Z}})'}. \end{array} \tag{2.47}$$

For the Abel-Jacobi map on curves, there is the Jacobi inversion. So

$$\phi(CH^1_{alg}(T)) = J(T). \tag{2.48}$$

Hence ϕ' is onto, i.e.

$$\frac{(W^1)'}{(W^1_{\mathbb{Z}})'} = \phi'(CH^1_{alg}(T)). \tag{2.49}$$

So is ϕ. This completes the proof.
□

Remark This theorem is a version of Jacobi inversion.

We prove theorem 1.1 in two steps.

Step 1: We assume $dim(H^1(X;\mathbb{Q})) \neq 0$. It suffices to verify assumption 2 of theorem 2.5.

Let $u \in H^2(X;\mathbb{Z})$ be the class of a hyperplane section. Then for the vector space over $\mathbb{Q}$, we can decompose

$$\overline{F^{p+1}H^{2p+1}_a(X;\mathbb{Q})} = Z_1 \oplus Z_0, \tag{2.50}$$

where $\overline{F^{p+1}H^{2p+1}_a(X;\mathbb{Q})}$ is the projection of $H^{2p+1}_a(X;\mathbb{Q})$ to $\overline{F^{p+1}H^{2p+1}_a(X;\mathbb{C})}$, and

$$Z_0 = \{\alpha \in \overline{F^{p+1}H^{2p+1}_a(X;\mathbb{Q})} : \alpha \cup u^q = 0\}. \tag{2.51}$$

(note $\alpha \cup u^q$ is a class of degree $2n-1$). Thus $\alpha \cup u^q \neq 0$ for all non-zero $\alpha \in Z_1$. By the assumption

$$H^{1,0}(X;\mathbb{Q}) \cup u^p \subset \overline{F^{p+1}H_a^{2p+1}(X;\mathbb{Q})} \tag{2.52}$$

is non zero. Hence Z_1 is non-zero whose dimension is at least $dim(H^{1,0}(X;\mathbb{Q}))$. Multiplying by an integer, we obtain the same result on lattices. Thus upto a finite index, we may assume the lattice also has a decomposition

$$\overline{F^{p+1}H_a^{2p+1}(X;\mathbb{Z})} = L_1 \oplus L_0, \tag{2.53}$$

where $\overline{F^{p+1}H_a^{2p+1}(X;\mathbb{Z})}$ is the projection of $H_a^{2p+1}(X;\mathbb{Z})$ to $\overline{F^{p+1}H_a^{2p+1}(X;\mathbb{C})}$, and

$$L_0 = \{\alpha \in \overline{F^{p+1}H_a^{2p+1}(X;\mathbb{Z})} : \alpha \cup u^q = 0\}. \tag{2.54}$$

and for non-zero $\alpha \in L_1$, $\alpha \cup u^q \neq 0$ (such L_1 is not unique). We denote

$$K_1 = span_{\mathbb{C}}(L_1), K_0 = span_{\mathbb{C}}(L_0).$$

By the counting the dimensions, we obtain that $L_i, i = 0, 1$ are lattices of $K_i, i = 0, 1$. Then

$$\overline{F^{p+1}H_a^{2p+1}(X;\mathbb{C})} = K_1 \oplus K_0, \tag{2.55}$$

such that K_0, K_1 have lattices L_0, L_1 respectively. Note $\alpha \cup u^q \neq 0$ for all non-zero $\alpha \in K_1$. After shifting by the integral lattice and choosing an appropriate direct sum complement K_1, we may assume $\beta \in K_1$(the new β is the sum of old β and an integral class l_β in L_1). Without loosing generosity, we may assume

$$J_a^{p+1}(X) = \mathcal{T}(K_1) \oplus \mathcal{T}(K_0) \tag{2.56}$$

and $\pi_2(\beta) \in \mathcal{T}(K_1)$, i.e.

$$\pi_2(\beta) \in \mathcal{T}(K_1) \oplus \{0\}.$$

Since the complex torus $\mathcal{T}(K_1)$ is a non-empty Abelian subvariety, we can choose a curve R_2' on $\mathcal{T}(K_1)$ through the points $\pi_2(\beta), 0$. Let $R_2 = R_2' \times \{0\}$. Using the surjectivity of the Abel-Jacobi map ϕ_2, we obtain a smooth curve C_2 and a correspondence

$$Z \subset CH^{p+1}(C_2 \times X) \tag{2.57}$$

such that $\psi_{C_2}(C_2) = R_2$. Then it is automatic that R_2 on $J_a^{p+1}(X)$ goes through points $(\pi_2)|_{H_{C_2}}(\beta), 0$.

Next we verify the key assumption $\sigma(C_2)=0$.[4]

Let $\omega_Z\in H^{2p+2}(C_2\times X;\mathbb{C})$ be Poincaré dual to Z. Using Künneth decomposition

$$\omega_Z=\alpha_Z^1+\alpha_Z^2+\alpha_Z^0 \tag{2.58}$$

for

$$\alpha_Z^i\in H^i(C_2)\otimes H^{2p+2-i}(X), i=0,1,2.$$

Let

$$\alpha_Z^1=\sum_i l_i\otimes\omega_i \tag{2.59}$$

where

$$l_i\in H^1(C_2;\mathbb{C}),\omega_i\in H^{2p+1}(X;\mathbb{C}).$$

By the definitions, we immediately have

$$\begin{array}{c} span(\bar{l}_i)_i=(\rho_1)^*(H^{1,0}(\mathcal{T}((V_1)^*))) \\ span_i(\omega_i)=H_Z^{2p+1}(X;\mathbb{C}). \end{array} \tag{2.60}$$

where $\bar{l}_i$ is the projection of l_i to $H^{1,0}(C_2;\mathbb{C})$ in the Hodge decomposition. See definition 2.1 for $H_Z^{2p+1}(X;\mathbb{C})$. Next we apply the consequence of the construction Z (This is the key step). By the construction of Z,

$$\overline{F^{p+1}H_Z^{2p+1}(X;\mathbb{Z})}\subset L_1. \tag{2.61}$$

Since $\alpha\cup u^q\neq 0$ in

$$H^{2p+1}(X;\mathbb{C})$$

for all non-zero $\alpha\in K_1$, the restriction $\alpha|_Y\neq 0$ in

$$H^{2p+1}(Y;\mathbb{C})$$

for all non-zero $\alpha\in K_1$, where Y is the complete q codimensional hyperplane intersection of X. Thus the restriction map

$$\begin{array}{ccc} \overline{F^{p+1}H_Z^{2p+1}(X;\mathbb{C})} & \to & \overline{F^{p+1}H^{2p+1}(Y;\mathbb{C})} \\ \alpha & \to & \alpha|_Y \end{array} \tag{2.62}$$

[4] This assumption is the Griffiths' conjecture on the incidence equivalence versus Abel-Jacobi equivalence. This is because that

$$(\rho_1)^*(H^{1,0}(\mathcal{T}((V_1)^*)))$$

is the space for the Abel-Jacobi equivalence and

$$(\rho_1^a)^*(H^{1,0}(\mathcal{T}((V_1^a)^*)))$$

is that for the incidence equivalence.

is injective.[5] Therefore the dual map

$$\begin{array}{ccc} (\overline{F^{p+1}H^{2p+1}(Y;\mathbb{C})})^* & \to & (\overline{F^{p+1}H_Z^{2p+1}(X;\mathbb{C})})^* \\ \theta & \to & \theta(\alpha|_Y) \end{array} \tag{2.63}$$

is surjective. By the Lefschetz hyperplane theorem, the restriction map

$$\begin{array}{ccc} H^{1,0}(X;\mathbb{C}) & \to & H^{1,0}(Y;\mathbb{C}) \\ l & \to & l|_Y \end{array} \tag{2.64}$$

is an isomorphism. Using the Poincaré duality of Y, the composition map

$$\begin{array}{ccccc} H^{1,0}(X;\mathbb{C}) & \to & H^{1,0}(Y;\mathbb{C}) \simeq (\overline{F^{p+1}H^{2p+1}(Y;\mathbb{C})})^* & \to & (\overline{F^{p+1}H_Z^{2p+1}(X;\mathbb{C})})^* \\ l & \to & l \cup u^q & \to & (l \cup u^q)^* \end{array} \tag{2.65}$$

is surjective. This implies

$$\begin{array}{ccccc} H^{1,0}(X;\mathbb{C}) & \to & (\overline{F^{p+1}H_Z^{2p+1}(X;\mathbb{C})})^* & \to & span_i(\bar{l}_i) \\ l & \to & (l \cup u^q, \sum l_i \otimes w_i) & \to & \sum_i \bar{l}_i(l \cup u^q, w_i) \end{array} \tag{2.66}$$

is surjective, where the second map is the evaluation map

$$(F^{p+1}H_Z^{2p+1}(X;\mathbb{C}))^* \otimes H^1(C_2;\mathbb{C}) \otimes F^{p+1}H_Z^{2p+1}(X;\mathbb{C}) \quad \to \quad H^1(C_2;\mathbb{C}). \tag{2.67}$$

The final statement is equivalent to saying

$$\begin{array}{ccc} H^{1,0}(X;\mathbb{C}) & \to & span_i(\bar{l}_i) \\ l & \to & (\pi_1)_*(l \cup u^q \cup \omega_Z) \end{array} \tag{2.68}$$

is surjective where $\pi_1 : C_2 \times X \to C_2$ is the projection.

On the other hand, the image of the map

$$\begin{array}{ccc} F^{q+1}H_a^{2q+1}(X;\mathbb{C}) & \to & H^{1,0}(C_2;\mathbb{C}) \\ \omega_a & \to & (\pi_1)_*(\omega_a \cup \omega_Z) \end{array} \tag{2.69}$$

is just

$$(\rho_1^a)^*(H^{1,0}(\mathcal{T}((V_1^a)^*))).$$

At last, $H^1 = H_a^1$ for any variety and cupping with hyperplane sections pre-

[5]This statement is similar to the Lefschetz hyperplane theorem, but it is not because of the higher codimensions. We call it quasi-Lefschetz-hyperplane-theorem. It is incorrect in general. However it is true in our case because of the special construction of Z.

serves the partial algebraicity. Hence $l\cup u^q\in V_1^a$. Then we obtain that

$$\begin{array}{c} span(\bar{l}_i)_i \\ \bigcap \\ (\pi_1)*\left(H^{1,0}(X;\mathbb{C})\cup u^q\cup\omega_Z\right) \\ \bigcap \\ (\pi_1)*(F^{q+1}H_a^{2q+1}(X;\mathbb{C})\cup\omega_Z) \\ \bigcap \\ (\pi_1)*(F^{q+1}H^{2q+1}(X;\mathbb{C})\cup\omega_Z) \\ \| \\ span(\bar{l}_i)_i. \end{array} \tag{2.70}$$

Finally $E_1=E_1^a$. This completes the proof of vanishing $\sigma(C_2)$.

This verifies all assumptions in theorem 2.5. Hence APD holds on such a variety X with $H^1(X)\neq 0$.

Step 2: without the assumption $dim(H^1(X;\mathbb{Q}))\neq 0$.

APD can be easily handled in various situations once it is proved in a special case. Let's see this. Let E be an elliptic curve. Let $G=E\times X$. For any non-zero cycle $c\in H_a^{2p+1}(X)$, $[E]\otimes c\in H_a^{2p+1}(G)$ is non-zero. We can apply the step 1 to G. We obtain a $\beta\in H_a^{2q+3}(G)$ such that

$$\mathcal{B}_G([E]\otimes c,\beta)\neq 0. \tag{2.71}$$

Let's see it in the homology. There is a correspondence

$$\mathcal{I}\in CH^{q+1}(E\times X) \tag{2.72}$$

such that there is a singular cycle β_h Poincaré dual to β, contained in the support $|\mathcal{I}|$, and the projections from both $\mathcal{I}$ and β_h to X are finite-to-one. Let $\pi: G\to X$ be the projection. Then the cohomology class $\pi_*(\beta)$ lies in $\pi(|\mathcal{I}|)$ which is a codimension q algebraic set. Hence

$$\pi_*(\beta)\in H_a^{2q+1}(X;\mathbb{C}).$$

On the other hand, using the projection formula, we obtain that

$$\mathcal{B}_G([E]\otimes c,\beta)=\mathcal{B}_X(c,\pi_*(\beta))\neq 0. \tag{2.73}$$

This shows that the restricted intersection pairing is also non-degenerate on right. Then reversing the order of the intersection, we repeat the same proof to obtain that the restricted intersection pairing is also non-degenerate on left. This completes the proof of theorem 1.1.

2.4 Application–a proof of Griffiths' conjecture

Theorem 1.1 is equivalent to the Griffiths' conjecture on the incidence equivalence. Let's introduce the conjecture.

Let X be a smooth projective variety of dimension n over the complex numbers. There is the Chow group $CH^r(X)$ which is the group of all codimensional r algebraic cycles with integer coefficients modulo rational equivalence. Let $CH^r_{alg}(X)$ denote the subgroup of $CH^r(X)$ whose cycles are algebraically equivalent to zero. There is the Griffiths' intermediate Jacobian $J^r(X)$ (or $J_{n-r}(X)$) which is a complex torus defined via Hodge structure of X as follows. Let

$$H^{2r-1}(X;\mathbb{R})$$

be the cohomology group of the real manifold. Then the complexified cohomology group $H^{2r-1}(X;\mathbb{C})$ has a Hodge decomposition as a vector space

$$H^{2r-1}(X;\mathbb{C}) = \oplus_{i=0}^{2r-1} H^{i,2r-1-i}(X) \tag{2.74}$$

The decomposition can be re-grouped as

$$H^{2r-1}(X;\mathbb{C}) = \overline{F^r H^{2r-1}(X)} \oplus F^r H^{2r-1}(X), \tag{2.75}$$

where $F^r H^{2r-1}(X)$ denotes the sum of summands $H^{p,q}$ in the Hodge decomposition (2.74) with $p > q$. Then it is clear the complex conjugate $\overline{F^r H^{2r-1}(X)}$ is just the rest of other summands with $p < q$. Next we identify $H_{2n-2r+1}(X;\mathbb{Z})$ with a subgroup of $\overline{F^r H^{2r-1}(X)}$ via Poincaré duality and the projection in the vector space. Then the Griffiths' intermediate Jacobian $J^r(X)$ (or $J_r(X)$) is defined to be

$$J^r(X) = \frac{\overline{F^r H^{2r-1}(X)}}{H_{2n-2r+1}(X;\mathbb{Z})}. \tag{2.76}$$

Another expression of this is:

$$J^r(X) = \frac{(F^{n-r+1}H^{2n-2r+1}(X))^*}{H_{2n-2r+1}(X;\mathbb{Z})}. \tag{2.77}$$

where the identification is made through the cup product between

$$H^{2r-1}(X), H^{2n-2r+1}(X).$$

The Jacobian $J^r(X)$ is equipped with a natural complex structure inherited from $\overline{F^r H^{2r-1}(X)}$. There is the Abel-Jacobi map AJ from $CH^r_{alg}(X)$ to $J^r(X)$ defined as follows. For any $B \in CH^r_{alg}(X)$, there is a real singular chain Γ_B in X such that in cohomology $[\partial\Gamma_B] = [B]$. The map AJ,

$$B \to \frac{\int_{\Gamma_B}(\cdot)}{H_{2n-2r+1}(X;\mathbb{Z})} \quad \in \frac{(F^{n-r+1}H^{2n-2r+1}(X))^*}{H_{2n-2r+1}(X;\mathbb{Z})} \tag{2.78}$$

is defined to be the Abel-Jacobi map. This is a well-defined map by Hodge theory. Denote the image of $CH^r_{alg}(X)$ under the Abel-Jacobi map by $J^r_a(X)$ (or $J^a_{n-r}(X)$). The Abel-Jacobi map is a *regular* homomorphism in the sense that for any smooth projective variety T with a fixed point $t_0 \in T$ and a correspondence

$$Z \in CH^r(T \times X), \tag{2.79}$$

the map

$$\begin{array}{ccc} T & \stackrel{AJ}{\to} & J^r(X) \\ t & \to & AJ(Z(t) - Z(0)) \end{array} \tag{2.80}$$

is a complex analytic map, where $Z(t) = (Pr_X)_*(Z \cdot (\{t\} \times X))$ with projection Pr_X to X. In order to understand the kernel of AJ, Griffiths in [3] introduced another equivalence, called "incidence equivalence": let T be a smooth projective variety parametrizing $n-q-1$-algebraic cycles and

$$\Sigma \subset CH^{q+1}(T \times X)$$

be a correspondence. Now the cycle $B \in CH^{n-q}_{alg}(X)$ is called incidence equivalent to zero if for all couples (T, Σ) above, the divisor

$$\Sigma^*(B) = (Pr_T)_*\bigg(\Sigma \cdot ([T] \times B)\bigg)$$

is well-defined and rationally equivalent to zero on T. Let

$$CH^{n-q}_{inc}(X) \subset CH^{n-q}_{alg}(X)$$

be the collection of all cycles in $CH^{n-q}_{alg}(X)$ that are incidence equivalent to zero. Then he further proved that for an algebraic cycle $B \in CH^{n-q}_{alg}(X)$,

$$AJ(B) = 0 \Longrightarrow B \in CH^{n-q}_{inc}(X). \tag{2.81}$$

The converse was left as a conjecture ([3]):

Conjecture 2.1. *(Griffiths' conjecture) For $B \in CH^{n-q}_{alg}(X)$,*

$$B \in CH^{n-q}_{inc}(X) \Longrightarrow AJ(mB) = 0, for\ some\ positive\ integer\ m. \tag{2.82}$$

The conjecture has an importance in the study of algebraic cycles. In the past 46 years, there had been a number of work on the conjecture. But they only yielded partial solutions. The most noticeable one is the following theorem given by J. Murre ([8]).

THEOREM 2.7. *(Murre) For $B \in CH^2_{alg}(X)$, the conjecture 2.1 is correct.*

Using the APD on the cohomology of odd degree, we prove the full conjecture

THEOREM 2.8. *The Griffiths' conjecture 2.1 is correct.*

Why is the APD related to the Griffiths' conjecture? A straitforward reason is that the incidence relation factors through the algebraic parts of cohomology, or equivalently it factors through the algebraic parts of intermediate Jacobian. Let's be more specific on this.

Incidence structure and Archimeadean height pairing

Our result on APD only concerns Abelian sub varieties inside of the intermediate Jacobians. But Griffiths conjecture will relate the APD on the intermediate Jacobians to incidence relations. In this section, we interpret the incidence equivalence through Abel-Jacobi maps.

Let $dim(X) = n$ be any positive integer. We let p, q be any two whole numbers satisfying $p + q = n - 1$. Let $\mathcal{C}_r(X)$ denote an irreducible component of Chow-variety of X of effective algebraic cycles of dimension $r \geq 0$.

DEFINITION 2.9. *(Archimedean height pairing, [1], [2]) Assume X is equipped with a Kähler metric. Let $A \in \mathcal{C}_p(X), B \in \mathcal{C}_q(X)$. Assume*

$$|A| \cap |B| = \emptyset.$$

Define the Archimedean height pairing $\langle A, B\rangle$ by the integral

$$\int_A G_B.$$

where G_B is a normalized Green's form of B. A normalized Green's form of B is a smooth form on $X \backslash |B|$ and L^1 on X that satisfies

(1) $dd^c < G_b >= \delta_B - < \omega_B >$ where $< \cdot >$ is the notation for currents, δ_B is the current of integration over B, ω_B is the harmonic, Poincaré dual to B.

(2) Harmonic projection of the current $< G_B >$ is zero.

THEOREM 2.10. *([10]). Let $B \in Z_q(X)$. Assume that there is a cycle $A \in \mathcal{C}_p(X)$ such that $|A| \cap |B| = \emptyset$. Then B determines a rational section s_B of some metrized line bundle $\mathcal{L}_{[B]}$ such that the Archimedean height pairing, as a real function on $\mathcal{C}_p \backslash |div(s_B)|$, is*

$$\langle A, B\rangle = \frac{1}{p!} \log ||s_B(A)||^2. \tag{2.83}$$

The line bundle $\mathcal{L}_{[B]}$ is called "Mazur's incidence line bundle", and $div(s_B)$ is called incidence divisor of B, denoted by $\mathcal{D}_B$.

By pulling back Mazur's incidence line bundle, we obtain

Corollary 2.11. *Let T_p be a smooth projective variety with a regular map*

$$\phi : T_p \quad \rightarrow \quad \mathcal{C}_p(X), \tag{2.84}$$

whose image does not lie in $|div(s_B)|$. Then there is a rational section s'_B of some metrized line bundle $\mathcal{L}'_B$ on T_p such that the Archimedean height pairing, as a real function on $T_p\backslash|div(s'_B)|$, is

$$\langle A, B\rangle = \log||s'_B(A)||^2. \tag{2.85}$$

Remark . There is an easy, but non-trivial assertion: since T_p is smooth,

$$\frac{\phi^*(\mathcal{L}_B)}{p!}$$

is also a line bundle on T_p.

Definition 2.12.

(1) The divisor $div(s'_B)$ will be denoted by $\mathcal{D}_B$, called the incidence divisor of B.

(2) If $\mathcal{D}_B$ is zero in the Chow group $CH(T_p)$ for all T_p, we say B is incidence equivalent to zero.

Definition (2) coincides with Griffiths' mentioned in introduction.

Next we see the map $[B] \rightarrow \mathcal{L}_{[B]}$ factors through intermediate Jacobian.

Using the result of ([5]), we obtained that

Corollary 2.13. *(Theorem 3.15, [11]). Let $CH^{n-q}_{hom}(X)$ denote the Chow group of algebraic cycles of codimension $n-q$ homologous to zero.*

Let

$$\begin{array}{rcl} \mathcal{L} : CH^{n-q}_{hom}(X) & \rightarrow & Pic^0(\mathcal{C}_p(X)) \\ {[B]} & \rightarrow & \mathcal{L}_{[B]}. \end{array} \tag{2.86}$$

Then $\mathcal{L}$ factors as:

$$\begin{array}{ccccccc} CH^{n-q}_{hom}(X) & \stackrel{AJ}{\rightarrow} & J_q(X) & \stackrel{f}{\rightarrow} & Pic^0(J_p(X)) & & \\ & & & & \downarrow p! & & \\ & & & & Pic^0(J_p(X)) & \stackrel{AJ^*}{\rightarrow} & Pic^0(\mathcal{C}_p(X)). \end{array} \tag{2.87}$$

where f is induced from the Poincaré line bundle, AJ^* is induced from the Abel-Jacobi map AJ with a fixed base point in $\mathcal{C}_p(X)$ and $p!$ is the map $\otimes^{p!}$ on the line bundles.

The sequence (2.87) is modified to the following sequence

$$\frac{CH^{n-q}_{hom}(X)}{CH^{n-q}_{Jac}(X)} \stackrel{AJ}{\to} J^a_q(X) \stackrel{f}{\to} Pic^0(J^a_p(X)) \stackrel{AJ^*_a}{\to} Pic^0(\mathcal{C}) \tag{2.88}$$

where AJ^*_a is the pullpack map restricted to $Pic^0(J^a_p(X))$.

LEMMA 2.14. *The homomorphism AJ^*_a is injective.*

Proof. Let L be a line bundle on $J^{n-p}_a(X)$. First L is determined by a representation η of the lattice Λ of $J^{n-p}_a(X)$ in $U(1)$. Suppose L is not trivial. Then η is not a constant map 1. Let $\lambda \in \Lambda$ such that $\eta(\lambda) \neq 1$. Then there exists a smooth curve $C \subset (J^{n-p}_a(X))$ such that

$$\lambda \in i_*(H_1(C;\mathbb{Z})),$$

where

$$i : C \hookrightarrow J^{n-p}_a(X)$$

is the embedding. Also through the Abel-Jacobi map AJ_a, there is a parameter curve T parametrizing p-cycles in X such that

$$AJ_*(H_1(T;Z)) = i_*(H_1(C;\mathbb{Z})),$$

i.e. the induced map AJ_* is surjective on the fundamental groups (this is because C is smooth, and AJ is surjective to C). Then the pull-back bundle $AJ^*(L)$ is also a flat bundle represented by the representation

$$\eta \circ AJ_*, \tag{2.89}$$

which is not trivial because $\eta(\lambda) \neq 1$. Hence $AJ^*(L)$ is not trivial. This completes the proof.
□

Griffiths' conjecture

Let's prove theorem 2.8, the Griffiths' conjecture. Recall

$$\frac{CH^{n-q}_{hom}(X)}{CH^{n-q}_{Jac}(X)} \stackrel{AJ}{\to} J^a_q(X) \stackrel{f}{\to} Pic^0(J^a_p(X)) \stackrel{AJ^*_a}{\to} Pic^0(\mathcal{C}) \tag{2.90}$$

where $\mathcal{C}$ is any smooth parameter space of p-cycles. By the lemma 2.14, to prove the Griffiths' conjecture it is sufficient to prove the f has a finite kernel when it is restricted to the image of AJ, i.e. $J^{n-q}_a(X)$. By theorem 3.1 in [12], this is implied by the APD in theorem 1.1. We complete the proof of theorem 2.8 which is also the corollary 1.3.

3 Algebraic Poincaré duality on cohomology of even degrees

3.1 APD on cohomology of even degrees

Now we assume the algebraic Poincaré duality in theorem 1.1.

Proof. of theorem 1.2. Let E be an elliptic curve. Let

$$G = E \times X. \tag{3.1}$$

Let $B \subset Z^p(X)$ be an algebraic cycle in X that represents a non-zero cohomology class. let c_1 be an integral 1 dimensional singular cycle on E. Assume the class $[c_1] \in H^1(E;\mathbb{C})$ is non-zero. Hence

$$c_1 \times B$$

represents a non-zero cohomology class $[c_1] \otimes [B]$ in $H_a^{2p+1}(G;\mathbb{Z})$. By the theorem 1.1, there exists a cycle class

$$[\Sigma] \in H^a_{2p+1}(G;\mathbb{Z}) \tag{3.2}$$

such that

$$\mathcal{B}_G([c_1 \times B], [\Sigma]) \neq 0. \tag{3.3}$$

We may assume Σ is a singular cycle representing the class. We would like to dissect this cycle to extract the algebraic cycles from it.

Since Σ is partially algebraic, it is supported on irreducible, algebraic varieties

$$W_j \subset G, \tag{3.4}$$

of dimension $p+1$. Since $dim(E) = 1$, there are only two kinds of W_j:

(1) those whose projection to E is a point, denoted by W_j^0

(2) those whose projection to E is surjective, denoted by W_j^1

For the second type of variety all fibres have the dimension p. It is easy to see that all components of Σ carried by W_j^0 will have zero intersection with $c_1 \times B$. Let

$$\Sigma_1$$

be the part of Σ carried by W_j^1. Assume Σ_1 is triangulated by the sum of simplexes

$$\begin{array}{rcl} \sigma_i^1 : \Delta_i^1 & \rightarrow & \Sigma_1 \\ \sigma_i^2 : \Delta_i^2 & \rightarrow & \Sigma_1, \end{array} \tag{3.5}$$

where $\sigma_i^k, k = 1, 2$ are such that the following compositions have k-dimensional (real) simplexes as their images,

$$\theta_i^k : \Delta_i^1 \quad \rightarrow \quad \Sigma_1 \quad \rightarrow \quad E \ .$$

Then we denote $\sum_i \sigma_i^k$ by Σ_k^1. Now we see that Σ_k^1 must be closed. It suffices to consider Σ_1^1 only. Notice that for each fixed j, fibres of W_1^j over E are all homotopic in X, thus represent the same cohomology class of X. Therefore by the Künneth decomposition

$$[\Sigma_1^1] = \sum_l c_l \otimes \beta_l \tag{3.6}$$

where $\sum_l c_l = \sum_i \theta_i^1(\Delta_i^1)$, and β_l are represented by fibres of W_j^1. Hence the classes β_l are all algebraic. We'll denote $\sum_l c_l = c$. Hence

$$\mathcal{B}_{E\times X}([c_1 \times B], [\Sigma]) = \mathcal{B}_{E\times X}([c_1 \times B], [\Sigma_1^1]) = \sum_h a_h \mathcal{B}_X([B], \beta_h) \neq 0. \tag{3.7}$$

where h represents an intersection point e_h of $c \cap c_1$ in E, a_h is some integer and

$$\beta_h = [(\{e_h\} \times X) \cdot \Sigma_1^1]$$

are algebraic cycle classes of dimension p.

Then (3.7),

$$\mathcal{B}_X(B, \sum_h a_h \beta_) \neq 0. \tag{3.8}$$

shows that the intersection pairing is non-degenerate on algebraic cycles. This completes the proof of theorem 1.2. □

3.2 Standard conjectures

THEOREM 3.1. *The Grothendieck's standard conjectures over $\mathbb{C}$ are correct.*

Remark We are not going to introduce the Grothendieck's standard conjectures. We'll refer the readers to [4], [6] and [7] for the introduction.

Proof. We'll recall one of his conjectures in that paper, the "A(X)" conjecture. Let X be a smooth projective variety of dimension n over the complex numbers. Recall p, q are two non-negative integers such that $p + q = n - 1$. Assume $p \leq q$. For any $0 \leq i \leq n$, we define $A^i \subset H^{2i}(X; \mathbb{C})$ to be the sub-space spanned by algebraic cycles of codimension i. Suppose that there is a polarization of X with hyperplane section $u \in H^2(X; \mathbb{C})$, which determines Lefschetz operators

$$L : H^*(X; \mathbb{C}) \quad \rightarrow \quad H^{*+2}(X; \mathbb{C}) \tag{3.9}$$

$$\Lambda : H^*(X; \mathbb{C}) \quad \rightarrow \quad H^{*-2}(X; \mathbb{C}). \tag{3.10}$$

Conjecture 3.1. *(Grothndieck's "A(X)" conjecture). For all polarizations, the restriction map*

$$L^{q-p+1} : A^p \quad \to \quad A^{q+1} \tag{3.11}$$

is an isomorphism.

In [6], Kleimann proved the Grothendieck's standard conjectures over the complex number $\mathbb{C}$ are equivalent to the "A(X)" conjecture. Because L^{q-p+1} is injective restricted to A^p, then the isomorphism follows from the

$$dim(A^p) = dim(A^{q+1}). \tag{3.12}$$

The equation (3.12) follows from theorem 1.2. We complete the proof of theorem 3.1 which is also the corollary 1.4.

□

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