A classical probability space exists for the measurement theory based on the truth values

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Recently, a new measurement theory based on the truth values is proposed [37]. The results of measurements are either 0 or 1. The measurement theory accepts a hidden variables model for a single Pauli observable. Therefore we can introduce a classical probability space for the measurement theory. Our discussion provides new insight to formulate quantum measurement theory based on the truth values.

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I. INTRODUCTION

The projective measurement theory (cf. [1–6]) gives accurate and at times remarkably accurate numerical predictions. Much experimental data has fit to the quantum predictions for long time.

From the incompleteness argument of Einstein, Podolsky, and Rosen (EPR) [7], a hidden-variables interpretation of quantum mechanics has been an attractive topic of research [3, 4]. One is the Bell-EPR theorem [8]. The other is the no-hidden-variables theorem of Kochen and Specker (the KS theorem) [9]. Greenberger, Horne, and Zeilinger discover [10, 11] the so-called GHZ theorem for four-partite GHZ state. And, the Bell-KS theorem becomes very simple form (see also Refs. [12–16]).

Leggett-type nonlocal hidden-variable theory [17] is experimentally investigated [18–20]. The experiments report that the quantum theory does not accept Leggett-type nonlocal hidden-variable theory. These experiments are done in four-dimensional space (two parties) in order to study nonlocality of hidden-variable theories. However there are debates for the conclusions of the experiments. See Refs. [21–23].

As for the applications of the projective measurement theory, implementation of a quantum algorithm to solve Deutsch's problem [24, 25] on a nuclear magnetic resonance quantum computer is reported firstly [26]. Implementation of the Deutsch-Jozsa algorithm on an ion-trap quantum computer is also reported [27]. There are several attempts to use single-photon two-qubit states for quantum computing. Oliveira *et al.* implement Deutsch's algorithm with polarization and transverse spatial modes of the electromagnetic field as qubits [28]. Single-photon Bell states are prepared and measured [29]. Also the decoherence-free implementation of Deutsch's algorithm is reported by using such single-photon and by using two logical qubits [30]. More recently, a one-way based experimental implementation of Deutsch's algorithm is reported [31]. In 1993, the Bernstein-Vazirani algorithm was reported [32]. It can be considered as an extended Deutsch-Jozsa algorithm. In 1994, Simon's algorithm was reported [33]. Implementation of a quantum algorithm to solve the Bernstein-Vazirani parity problem without entanglement on an ensemble quantum computer is reported [34]. Fiber-optics implementation of the Deutsch-Jozsa and Bernstein-Vazirani quantum algorithms with three qubits is discussed [35]. A quantum algorithm for approximating the influences of Boolean functions and its applications is recently reported [36].

More recently, a new measurement theory based on the truth values is proposed [37]. The results of measurements are either 0 or 1. The measurement theory accepts a hidden variables model for a single Pauli observable. Therefore we can introduce a classical probability space for the measurement theory. Our discussion provides new insight to formulate quantum measurement theory based on the truth values.

II. MEASUREMENT THEORY BASED ON THE TRUTH VALUES MEETS A HIDDEN VARIABLES MODEL OF A SINGLE SPIN OBSERVABLE

We discuss the new measurement theory meets a hidden variables model of a single spin observable. Assume a spin-1/2 state ρ. Let σ_x be a single Pauli observable. We have a quantum expected value as

$$\mathrm{Tr}[\rho\sigma_x]. \tag{1}$$

We derive a necessary condition for the quantum expected value for the system in a spin-1/2 state given in (1). We have

$$0 \leq (\mathrm{Tr}[\rho\sigma_x])^2 \leq 1. \tag{2}$$

It is worth noting here that we have $(\mathrm{Tr}[\rho\sigma_x])^2 = 1$ if ρ is a pure state lying in the x-direction. Hence we derive the following proposition concerning quantum mechanics when the system is in a state lying in the x-direction

$$(\mathrm{Tr}[\rho\sigma_x])^2_{\max} = 1. \tag{3}$$

$(\mathrm{Tr}[\rho\sigma_x])^2_{\max}$ is the maximal possible value of the product. It is worth noting here that we have $(\mathrm{Tr}[\rho\sigma_x])^2 = 0$ when the system is in a pure state lying in the z-direction. Thus we have

$$(\mathrm{Tr}[\rho\sigma_x])^2_{\min} = 0. \tag{4}$$

$(\mathrm{Tr}[\rho\sigma_x])^2_{\min}$ is the minimal possible value of the product. In short, we have

$$(\mathrm{Tr}[\rho\sigma_x])^2_{\min} = 0 \text{ and } (\mathrm{Tr}[\rho\sigma_x])^2_{\max} = 1. \tag{5}$$

In what follows, we derive the above proposition (5) assuming the following form:

$$\mathrm{Tr}[\rho\sigma_x] = \int d\lambda\rho(\lambda)f(\sigma_x,\lambda). \tag{6}$$

where λ denotes some hidden variable and $f(\sigma_x,\lambda)$ is the hidden result of measurements of the Pauli observable σ_x. We assume the values of $f(\sigma_x,\lambda)$ are 1 and 0 (in $\hbar/2$ unit).

Let us assume a hidden variables theory of a single spin observable based on the new measurement theory. In this case, the quantum expected value in (1), which is the average of the hidden results of the new measurements, is given by

$$\mathrm{Tr}[\rho\sigma_x] = \int d\lambda\rho(\lambda)f(\sigma_x,\lambda). \tag{7}$$

The possible values of the hidden result $f(\sigma_x,\lambda)$ are 1 and 0 (in $\hbar/2$ unit). Same expected value is given by

$$\mathrm{Tr}[\rho\sigma_x] = \int d\lambda'\rho(\lambda')f(\sigma_x,\lambda'), \tag{8}$$

because we only change the notation as $\lambda \to \lambda'$. Of course, the possible values of the hidden result $f(\sigma_x,\lambda')$ are 1 and 0 (in $\hbar/2$ unit). By using these facts, we derive a necessary condition for the expected value for the system in a spin-1/2 state lying in the x-direction. We derive the possible values of the product $(\mathrm{Tr}[\rho\sigma_x])^2$. We have

$$\begin{aligned}
&(\mathrm{Tr}[\rho\sigma_x])^2\\
&= \int d\lambda\rho(\lambda)f(\sigma_x,\lambda) \times \int d\lambda'\rho(\lambda')f(\sigma_x,\lambda')\\
&= \int d\lambda\rho(\lambda) \cdot \int d\lambda'\rho(\lambda')f(\sigma_x,\lambda)f(\sigma_x,\lambda')\\
&\le \int d\lambda\rho(\lambda) \cdot \int d\lambda'\rho(\lambda')|f(\sigma_x,\lambda)f(\sigma_x,\lambda')|\\
&= \int d\lambda\rho(\lambda) \cdot \int d\lambda'\rho(\lambda') = 1.
\end{aligned} \tag{9}$$

Clearly, the above inequality can have the upper limit since the following case is possible:

$$\|\{\lambda|f(\sigma_x,\lambda)=1\}\| = \|\{\lambda'|f(\sigma_x,\lambda')=1\}\|, \tag{10}$$

and

$$\|\{\lambda|f(\sigma_x,\lambda)=0\}\| = \|\{\lambda'|f(\sigma_x,\lambda')=0\}\|. \tag{11}$$

Thus we derive a proposition concerning the hidden variables theory based on the new measurement theory (in a spin-1/2 system), that is, $(\mathrm{Tr}[\rho\sigma_x])^2 \le 1$. Hence we derive the following proposition concerning the hidden variables theory:

$$(\mathrm{Tr}[\rho\sigma_x])^2_{\max} = 1. \tag{12}$$

We derive another necessary condition for the expected value for the system in a pure spin-1/2 state lying in the z-direction. We have

$$\begin{aligned}
&(\mathrm{Tr}[\rho\sigma_x])^2\\
&= \int d\lambda\rho(\lambda)f(\sigma_x,\lambda) \times \int d\lambda'\rho(\lambda')f(\sigma_x,\lambda')\\
&= \int d\lambda\rho(\lambda) \cdot \int d\lambda'\rho(\lambda')f(\sigma_x,\lambda)f(\sigma_x,\lambda')\\
&\ge \int d\lambda\rho(\lambda) \cdot \int d\lambda'\rho(\lambda')(0)\\
&= (0)\left(\int d\lambda\rho(\lambda) \cdot \int d\lambda'\rho(\lambda')\right) = 0.
\end{aligned} \tag{13}$$

Clearly, the above inequality can have the lower limit since the following case is possible:

$$\|\{\lambda|f(\sigma_x,\lambda)=1\}\| = \|\{\lambda'|f(\sigma_x,\lambda')=0\}\|, \tag{14}$$

and

$$\|\{\lambda|f(\sigma_x,\lambda)=0\}\| = \|\{\lambda'|f(\sigma_x,\lambda')=1\}\|. \tag{15}$$

Thus we derive a proposition concerning the hidden variables theory based on the new measurement theory (in a spin-1/2 system), that is, $(\mathrm{Tr}[\rho\sigma_x])^2 \ge 0$. Hence we derive the following proposition concerning the hidden variables theory

$$(\mathrm{Tr}[\rho\sigma_x])^2_{\min} = 0. \tag{16}$$

Thus from (12) and (16) we have

$$(\mathrm{Tr}[\rho\sigma_x])^2_{\min} = 0 \text{ and } (\mathrm{Tr}[\rho\sigma_x])^2_{\max} = 1. \tag{17}$$

Clearly, we can assign the truth value "1" for two propositions (5) (concerning quantum mechanics) and (17) (concerning the hidden variables theory based on the new measurement theory), simultaneously. Therefore, the new measurement theory meets the existence of the hidden variables theory of a single spin observable.

III. CONCLUSIONS

In conclusions, recently, a new measurement theory based on the truth values has been proposed [37]. The results of measurements have been either 0 or 1. The measurement theory has accepted a hidden variables model for a single Pauli observable. Therefore we can have introduced a classical probability space for the measurement theory. Our discussion has provided new insight to formulate quantum measurement theory based on the truth values.

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