The Complex Form of the Law of Cosines

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This paper is prepared to show the mathematical derivation of the complex form of the law of cosines and show how it can help in the vector algebra.

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1. The derivation of the complex form of the law of cosines

For any triangle $A'B'C'$ (not necessarily a right triangle) with sides a, b and c and angles α, β and γ. Where γ is the angle between the sides a and b, α is the angle between the sides b and c and β is the angle between the sides a and c:

Law of cosines:

$$c^2 = a^2 + b^2 - 2ab\cos\gamma. \tag{1.1}$$
$$b^2 = a^2 + c^2 - 2ac\cos\beta. \tag{1.2}$$
$$a^2 = b^2 + c^2 - 2bc\cos\alpha. \tag{1.3}$$

Finding the roots of the quadratic equation (R.H.S of equation (1.1)) using the quadratic formula and solving for a:

$$\begin{aligned} a &= \frac{-(-2b\cos\gamma) \pm \sqrt{4b^2\cos^2\gamma - 4b^2}}{2}, \\ &= b\left(\cos\gamma \pm \sqrt{\cos^2\gamma - 1}\right), \\ &= b\left(\cos\gamma \pm \sqrt{-\sin^2\gamma}\right), \\ &= b\left(\cos\gamma \pm i\sin\gamma\right), \end{aligned}$$

Therefore,

$$a = be^{\pm i\gamma}. \tag{1.4}$$

and the roots are:

$$(a - be^{i\gamma})(a - be^{-i\gamma}) = \left|a - be^{i\gamma}\right|^2. \tag{1.5}$$

Hence, the law of cosines becomes

$$c^2 = \left|a - be^{i\gamma}\right|^2. \tag{1.6}$$

Which is the *complex form* of the law of cosines.

***Example*(1):** Consider the right triangle where $a = b = 1$ and $\gamma = \frac{\pi}{2}$. Substituting into equation (1.6) gives

$$c = \left|1 - (1)e^{i\frac{\pi}{2}}\right| = \sqrt{2}.$$

which is the true length of the hypotenuse as we expected.

***Example*(2):** Consider the unit equilateral triangle where $a = b = c = 1$ and $\gamma = \beta = \alpha = \frac{\pi}{3}$. Thus we have

$$c = \left|1 - (1)e^{i\frac{\pi}{3}}\right| = 1.$$

***Example*(3):** Consider an arbitrary triangle with $a = 1.5,\ b = 3.8$ and $\gamma = \frac{2\pi}{5}$. Hence we obtain

$$c = \left|1.5 - (3.8)e^{i\frac{2\pi}{5}}\right| \approx 3.62866.$$

You can check this result using equation (1.1) .

2. The law of cosines and the vector addition

Consider the sides of the triangle to be consisted of vectors and write down every side as a vector subtraction of the other remaining vectors (sides), thus we have

$$\vec{c} = \vec{a} - \vec{b}. \tag{2.1}$$
$$\vec{b} = \vec{c} - \vec{a}. \tag{2.2}$$
$$\vec{a} = \vec{b} - \vec{c}. \tag{2.3}$$

To find the length of the vector $\vec{c}$ (equation (2.1)) we take the dot product of the vector with itself, so we have

$$\begin{aligned} \vec{c} \circ \vec{c} &= (\vec{a} - \vec{b}) \circ (\vec{a} - \vec{b}), \\ &= \vec{a} \circ \vec{a} - \vec{a} \circ \vec{b} - \vec{b} \circ \vec{a} + \vec{b} \circ \vec{b}, \end{aligned}$$

Therefore, we obtain

$$c^2 = a^2 + b^2 - 2ab\cos\gamma. \tag{2.4}$$

which is exactly equation (1.1).

Similarly, from equations (2.2) and (2.3), one can obtains

$$\begin{aligned} b^2 &= c^2 + a^2 - 2ac\cos\beta. \\ a^2 &= b^2 + c^2 - 2bc\cos\alpha. \end{aligned}$$

Thus, the law of cosines can find the magnitude of vector subtraction with *phase difference* equal to γ, β or α, where in the 2-dimension Cartesian coordinate they are:

$$\left.\begin{aligned} \gamma &= \left|\arctan\left(\frac{a_y}{a_x}\right) - \arctan\left(\frac{b_y}{b_x}\right)\right|, \\ \beta &= \left|\arctan\left(\frac{a_y}{a_x}\right) - \arctan\left(\frac{c_y}{c_x}\right)\right|, \\ \alpha &= \left|\arctan\left(\frac{b_y}{b_x}\right) - \arctan\left(\frac{c_y}{c_x}\right)\right|. \end{aligned}\right\} \quad \text{Phase difference} \tag{2.5}$$

where $\vec{a} = a_x\hat{\imath} + a_y\hat{\jmath}$, $\vec{b} = b_x\hat{\imath} + b_y\hat{\jmath}$ and $\vec{c} = c_x\hat{\imath} + c_y\hat{\jmath}$.

Now consider the parallel transportation of the same vector, which led to equation (1.6), from the tail of the other vector to its tip. Thus the vector subtraction becomes a vector addition and the phase difference angle (the angle between the two vectors) becomes the supplement angle of this angle. Substituting this angle (the supplement angle) into equation (1.6), we find

$$\begin{aligned} c^2 &= \left|a - be^{i(\pi-\gamma)}\right|^2, \\ &= \left|a - be^{i\pi}e^{-i\gamma}\right|^2, \\ &= \left|a + be^{-i\gamma}\right|^2, \end{aligned}$$

Since $\pm\gamma$ makes no difference under the modulus, we can write

$$c^2 = \left|a + be^{i\gamma}\right|^2. \tag{2.6}$$

Therefore, equation (2.6) gives the magnitude of vector addition. To verify equation (2.6) we consider the following sum of vector:

$$\vec{c} = \vec{a} + \vec{b},$$

The length of this vector:

$$\begin{aligned} \vec{c} \circ \vec{c} &= (\vec{a} + \vec{b}) \circ (\vec{a} + \vec{b}), \\ &= \vec{a} \circ \vec{a} + \vec{a} \circ \vec{b} + \vec{b} \circ \vec{a} + \vec{b} \circ \vec{b}, \end{aligned}$$

Therefore, we obtain

$$c^2 = a^2 + b^2 + 2ab\cos\gamma. \tag{2.7}$$

Using the quadratic formula will yield equation (2.6).

Finally, equation (1.4) helps finding solution of the vectors or triangle sides in the complex plane, then equations (1.6) and (2.6) will yield the same results as with the real values. (see the following example).

***Example*(4):** Consider the triangle $a = 5,\ b = 3$ and $\gamma = \frac{2\pi}{3}$. Therefore,

$$\begin{aligned} c_- &= \left|5 - (3)e^{i\frac{2\pi}{3}}\right| = 7. \\ c_+ &= \left|5 + (3)e^{i\frac{2\pi}{3}}\right| = \sqrt{19}. \end{aligned}$$

Finding one of the solutions of a and b in the complex plane:

$$\begin{aligned} a &= (5)e^{i\frac{2\pi}{3}} = -\frac{5}{2} + \frac{5\sqrt{3}}{2}i. \\ b &= (3)e^{i\frac{2\pi}{3}} = -\frac{3}{2} + \frac{3\sqrt{3}}{2}i. \end{aligned}$$

Hence, equations (1.6) and (2.6) will give

$$c_- = \left| \left(-\frac{5}{2} + \frac{5\sqrt{3}}{2} i \right) - \left(-\frac{3}{2} + \frac{3\sqrt{3}}{2} i \right) e^{i\frac{2\pi}{3}} \right| = 7.$$
$$c_+ = \left| \left(-\frac{5}{2} + \frac{5\sqrt{3}}{2} i \right) + \left(-\frac{3}{2} + \frac{3\sqrt{3}}{2} i \right) e^{i\frac{2\pi}{3}} \right| = \sqrt{19}.$$

That is, the same solutions as when we substituted the real values.