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Apollonius's Circle of Second Rank

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This article highlights some properties of **Apollonius's circle of second rank** in connection with the **adjoint circles** and the **second Brocard's triangle**.

1st Definition.

It is called Apollonius's circle of second rank relative to the vertex A of the triangle ABC the circle constructed on the segment determined on the simedians' feet from A on BC as diameter.

1st Theorem.

The Apollonius's circle of second rank relative to the vertex A of the triangle ABC intersect the circumscribed circle of the triangle ABC in two points belonging respectively to the cevian of third rank (antibisector's isogonal) and to its external cevian.

The theorem's proof follows from the theorem relative to the Apollonius's circle of k^{th} rank (see [1]).

1^{st} Proposition.

The Apollonius's circle of second rank relative to the vertex A of the triangle ABC intersects the circumscribed circle in two points Q and P (Q on the same side of BC as A). Then, $(QS$ is a bisector in the triangle QBC, S is the simedian's foot from A of the triangle ABC.

Proof.

Q belongs to the Apollonius's circle of second rank, therefore:

$$\frac{QB}{QC} = \left(\frac{AB}{AC}\right)^2. \quad (1)$$

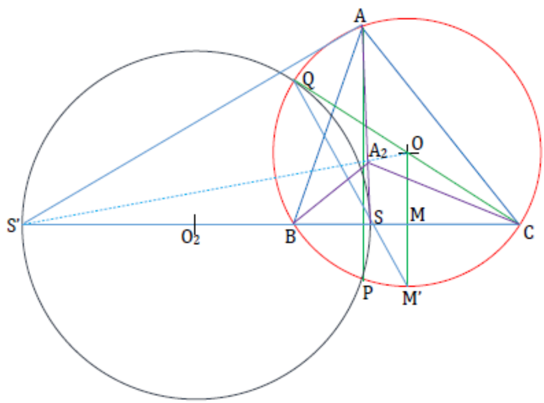

Figure 1.

On the other hand, S being the simedian's foot, we have:

$\frac{SB}{SC} = \left(\frac{AB}{AC}\right)^2$. (2)

From relations (1) and (2), we note that

$\frac{QB}{QC} = \frac{SB}{SC}$,

a relation showing that QS is bisector in the triangle QBC.

Remarks.

1. The Apollonius's circle of second relative to the vertex A of the triangle ABC (see *Figure 1*) is an Apollonius's circle for the triangle QBC. Indeed, we proved that QS is an internal bisector in the triangle QBC, and since S', the external simedian's foot of the triangle ABC, belongs to the Apollonius's Circle of second rank, we have $m(\sphericalangle S'QS) = 90^0$, therefore QS' is an external bisector in the triangle QBC.
2. QP is a simedian in QBC. Indeed, the Apollonius's circle of second rank, being an Apollonius's circle for QBC, intersects the circle circum-scribed to QBC after QP, which is simedian in this triangle.

2nd Definition.

It is called adjoint circle of a triangle the circle that passes through two vertices of the triangle and in one of them is tangent to the triangle's side. We denote $(B\bar{A})$ the adjoint circle that passes through B and A, and is tangent to the side AC in A.

About the circles $(B\bar{A})$ and $(C\bar{A})$, we say that they are adjoint to the vertex A of the triangle ABC.

3rd Definition.

It is called the second Brocard's triangle the triangle $A_2B_2C_2$ whose vertices are the projections of the center of the circle circumscribed to the triangle ABC on triangle's simedians.

2nd Proposition.

The Apollonius's circle of second rank relative to the vertex A of triangle ABC and the adjoint circles relative to the same vertex A intersect in vertex A_2 of the second Brocard's triangle.

Proof.

It is known that the adjoint circles $(B\bar{A})$ and $(C\bar{A})$ intersect in a point belonging to the simedian AS; we denote this point A_2 (see [3]).

We have:

$\sphericalangle BA_2S = \sphericalangle A_2BA + \sphericalangle A_2AB,$

but:

$\sphericalangle A_2BA \equiv \sphericalangle BA_2S = \sphericalangle A_2AB + A_2AC = \sphericalangle A.$

Analogously, $\sphericalangle CA_2S = \sphericalangle A$, therefore $(A_2S$ is the bisector of the angle BA_2C. The bisector's theorem in this triangle leads to:

$$\frac{SB}{SC} = \frac{BA_2}{CA_2},$$

but:

$$\frac{SB}{SC} = \left(\frac{AB}{AC}\right)^2,$$

consequently:

$$\frac{BA_2}{CA_2} = \left(\frac{AB}{AC}\right)^2,$$

so A_2 is a point that belongs to the Apollonius's circle of second rank.

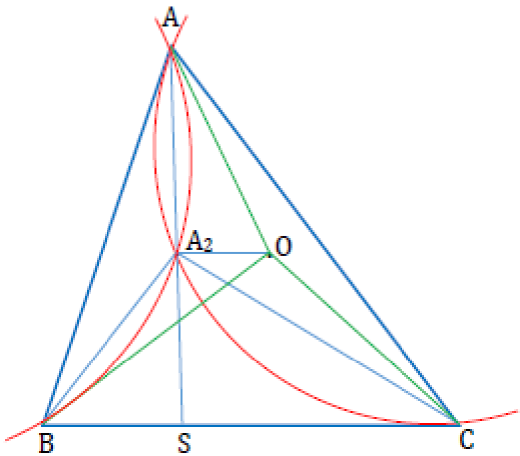

Figure 2.

We prove that A_2 is a vertex in the second Brocard's triangle, i.e. $OA_2 \perp AS$, O the center of the circle circumscribed to the triangle ABC.

We pointed (see *Figure 2*) that $m(\widehat{BA_2C}) = 2A$, if $\sphericalangle A$ is an acute angle, then also $m(\widehat{BOC}) = 2A$, therefore the quadrilateral OCA_2B is inscriptible.

Because $m(\widehat{OCB}) = 90^0 - m(\hat{A})$, it follows that $m(\widehat{BA_2O}) = 90^0 + m(\hat{A})$.

On the other hand, $m(\widehat{AA_2B}) = 180^0 - m(\hat{A})$, so $m(\widehat{BA_2O}) + m(\widehat{AA_2B}) = 270^0$ and, consequently, $OA_2 \perp AS$.

Remarks.

1. If $m(\hat{A}) < 90^0$, then four remarkable circles pass through A_2: the two circles adjoint to the vertex A of the triangle ABC, the circle circumscribed to the triangle BOC (where O is the center of the circumscribed circle) and the Apollonius's circle of second rank corresponding to the vertex A.
2. The vertex A_2 of the second Brocard's triangle is the middle of the chord of the circle circumscribed to the triangle ABC containing the simedian AS.
3. The points O, A_2 and S' (the foot of the external simedian to ABC) are collinear. Indeed, we proved that $OA_2 \perp AS$; on the other hand, we proved that $(A_2S$ is an internal bisector in the triangle BA_2C, and

since $S'A_2 \perp AS$, the outlined collinearity follows from the uniqueness of the perpendicular in A_2 on AS.

Open Problem.

The Apollonius's circle of second rank relative to the vertex A of the triangle ABC intersects the circle circumscribed to the triangle ABC in two points P and Q (P and A apart of BC).

We denote by X the second point of intersection between the line AP and the Apollonius's circle of second rank.

What can we say about X?

Is X a remarkable point in triangle's geometry?

References.

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[3] R. Johnson: *Advanced Euclidean Geometry*. New York: Dover Publications, Inc. Mineola, 2007.

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