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Anti-Geometry

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ANTI-GEOMETRY

It is possible to de-formalize entirely Hilbert's groups of axioms of the Euclidean Geometry, and to construct a model such that none of his fixed axiom holds.

Let's consider the following things:

- a set of <points>: $A, B, C, \ldots$

- a set of <lines>: $h, k, l, \ldots$

-a set of <planes>: $\alpha, \beta, \gamma, \ldots$

and

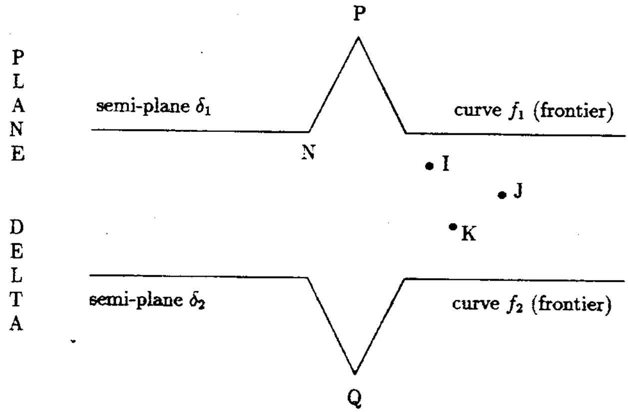
- a set of relationship among these elements: "are situated", "between", "parallel", "congruent", "continous", etc.

Then, we can deny all Hilbert's twenty axioms [see David Hilbert, "Foundation of Geometry", translated by E.J.Towsend, 1950; and Roberto Bonola, "Non-Euclidean Geometry", 1938]. There exist casses, whithin a geometric model, when the same axiom is verifyed by certain points/lines/planes and denied by others.

GROUP I. ANTI-AXIOMS OF CONNECTION

I.1. Two distinct points A and B do not always completely determine a line.

Let's consider the following model MD: get an ordinary plane δ, but with an infinite hole in of the following shape:



Plane delta is a reunion of two disjoint planar semi-planes; f_1 lies in MD, but f_2 does not; P, Q are two extreme points on f that belong to MD.

One defines a LINE l as a geodesic curve: if two points A, B that belong to MD lie in l, then the shortest curve lied in MD between A and B lies in l also. If a line passes two times throught the same point, then it is called double point ($KNOT$).

One defines a PLANE α as a surface such that for any two points A, B that lie in α and belong to MD there is a geodesic which passes trought A, B and lies in α also.

Now, let's have two strings of the same length: one ties P and Q with the first string s_1 such that the curve s_1 is folded in two or more different planes and s_1 is under the plane δ; next, do the same with string s_2, tie Q with P, but over the plane δ and such that s_2 has a different form from s_1; and a third string s_3, from P to Q, much longer than s_1. s_1, s_2, s_3 belongs to MD.

Let I, J, K be three isolated points - as some islands, i.e. not joined with any other poits of MD, exterior to the plane δ.

This model has measure, because the (pseudo-) line is the shortesr way (length) to go from a point to another (when possible).

Question 37:

Of course, the model is not perfect, and is far from the best. Readers are asked to improve it, or to make up a new one that is better.

(Let A, B be two distinct points in $\delta_1 - f_1$. P and Q are two points on s_1, but they do not completely determine a line, referring to the first axiom of Hilbert, because $A - P - s_1 - Q$ are different from $B - P - s_1 - Q$.)

I.2. There is at least a line l and at least two different points A and B of l, such that A and B do not completely determine the line l.
(Line $A - P - s_1 - Q$ are not completely determine by P and Q in the previous construction, because $B - P - s_1 - Q$ is another line passing through P and Q too.)

I.3. Three points A, B, C not situated in the same line do not always completely determine a plane α.
(Let A, B be two distinct points in $\delta_1 - f_1$, such that A, B, P are not co-linear. There are many planes containing these three points: δ_1 extended with any surface s containing s_1, but not cutting s_2 in between P and Q, for example.)

I.4. There is at least a plane, α, and at least three points A, B, C in it not lying in the same line, such that A, B, C do not completely determine the plane α.
(See the previous example.)

I.5. If two points A, B of line l lie in a plane α, it doesn't mean that every point of l lies in α.
(Let A be a point in $\delta_1 - f_1$, and B another point on s_1 in between P and Q. Let α be the following plane: δ_1 extended with a surface s containing s_1, but not cutting s_2 in between P and Q, and tangent to δ_2 on a line QC, where C is a point in $\delta_2 - f_2$. Let D be point in $\delta_2 - f_2$, not lying on the line QC. Now, A, B, D are lying on the same line $A - P - s_1 - Q - D, A, B$ are in the plane α, but D does not.)

I.6. If two planes α, β have a point A in common, it doesn't mean they have at least a second point in common.
(Construct the following plane α): a closed surface containing s_1 and s_2, and intersecting δ_1 in one point only, P. Then α and δ_1 have a single point in common.)

I.7. There exist lines where only one point lies, or planes where only two points lie, or space where only three points lie.

(Hilbert's I.7 axiom may be contradicted if the model has discontinuities. Let's consider the isolated points area.

The point I may be regarded as a line, because it's not possible to add any new point to I to form a line.

One constructs a surface that intersects the model only in the points I and J.)

GROUP II. ANTI-AXIOMS OF ORDER

II.1. If A, B, C are points of line and B lies between A and C, it doesn't mean that always B lies also between C and A.

[Let T lie in s_1, and V lie in s_2, both of them closer to Q, but different from it. Then:

P, T, V are points on the line $P - s_1 - Q - s_2 - P$ (i.e. the closed curve that starts from the point P) and lies in s_1 and passes through the point Q and lies back to s_2 and ends in P), and T lies between P and V

- because PT and TV are both geodesics, but T doesn't lie between V and P
- because from V the line goes to P and then to T, therefore P lies between V and T.]

[By defenition: a segment AB is a system of points lying upon a line between A and B (the extremes are included.)

Warning: AB may be different from BA; for example:]

the segment PQ formed by the system of points starting with P, ending with Q, and lying in s_1, is different from the segment PQ formed by the system of points starting with P, ending with Q, but belong to s_2.]

II.2. If A and C are two points of a line, then: there does not always exist a point B lying between A and C, or there does not always exist a point D such that C lies between A and D.

[For example:

let F be a point on f_1, F different from P, and G a point in δ_1, G doesn't belong to f_1; draw the line l which passes through G and F; then: there exists a point B

lying between G and F - because GF is an obvious segment, but there is no point D such that F lies between G and D - because GF is right bounded in F (GF may not be extended to the other side of F, because otherwise the line will not remain a geodesic anymore).]

II.3. There exist at least three points situated on a line such that:

one point lies between the other two, and another point lies also between the other two.

[For example:

let R, T be two distinct points, different from P and Q, situated on the line $P - s_1 - Q - s_2 - P$, such that the lenghts PR, RT, TP are all equal; then:

R lies between P and T, and T lies between R and P; also P lies between T and R].

II.4. Four points A, B, C, D of a line can not always be arranged: Such that B lies between A and C and also between A and D, and such that C lies between A and D and also between B and D.

[For example:

let R, T be two distinct points, different from P and Q, situated on the line $P - s_1 - Q - s_2 - P$ such that the lenghts PR, RQ, QT, TP are all equal, therefore R belongs to s_1, and T belongs to s_2; then P, Q, R, T are situated on the same line: such that R lies between P and Q, but not between P and T - because the geodesic PT does not pass through R, and such that Q does not lie between P and T, because the geodesic PT does not pass through Q, but lies between R and T; let A, B be two points in $\delta_2 - f_2$ such that A, Q, B are colinear, and C, D two points on s_1, s_2 respectively, all of the four points being different from P and Q; then A, B, C, D are points situated on the same line $A - Q - s_1 - P - s_2 - Q - B$, which qis the same with line $A - Q - s_2 - P - s_1 - Q - B$, therefore we may have two different orders of these four points in the same time: A, C, D, B and A, D, C, B.]

II.5. Let A, B, C be three points not lying in the same line, and l a line lying in the same plane ABC and not passsing through any of the points A, B, C. Then, if the line l passes through a point of the segment AB, it doesn't mean that always the line l will pass through either a point of the segment BC or a point of the segment AC.

[For example:

let AB be a segment passing through P in the semi-plane δ_1, and C point lying in δ_1 too on the left side of the line AB; thus A, B, C do not lie on the same line; now, consider the line $Q - s_2 - P - s_1 - Q - D$, where D is a point lying in the semi-plane δ_2 not on f_2: therefore this line passes through the point P of the segment AB, but does not pass through any point od the segment BC, nor through any point of the segment AC.]

GROUP III. ANTI-AXIOMS OF PARALLELS

In a plane α there can be drawn through a point A, lying outside of a line l, either no line, or only one line, or a finite number of lines which do not intersect the line l. (At least two of these situations should occur.) The line(s) is (are) called the parallel(s) to l through the given point A.

[For examples:

- let l_0 be the line $N - P - s_1 - Q - R$, where N is a point lying in δ_1 not on f_1, and R is a similar point lying in δ_2 not on f_2, and let A be a point lying on s_2, then: no parallel to l_0 can be drawn through A (because any line passing through A, hence through s_2, will intersect s_1, hence l_0, in P and Q);
- if the line l_1 lies in δ_1 such that l_1 does not intersect the frontier f_1, then: through any point lying on the left side of l_1 one and only one parellel will pass;
- let B be a point lying in f_1, different from P, and another point C lying in δ_1, not on f_1; let A be a point lying in δ_1 outside of BC; then: an infinite number of parallels to the line BC can be drawn through the point A.]

Theorem. *There are at least two lines l_1, l_2 of a plane, which do not meet a third line l_3 of the same plane, but they meet each other, (i.e. if l_1 is parallel to l_3, and l_2 is parallel to l_3, and all of them are in the same plane, it's not necessary that l_1 is parallel to l_2).*

[For example:

consider three points A, B, C lying in f_1, and different from P, and D a point in δ_1 not on f_1; draw the lines AD, BE and CE such that E is a point in δ_1 not on f_1 and both

BE and CE do not intersect AD; then: BE is parallel to AD, CE is also parallel to AD, but BE is not parallel to CE because the point E belong to both of them.]

GROUP IV. ANTI-AXIOMS OF CONGRUENCE

IV.1. If A, B are two points on a line l, and A' is a point upon the same or another line l', then: upon a given side of A' on the line l', we can not always find only one point B' so that the segment AB is congruent to the segment $A'B'$.

[For exemples:

- let AB be segment lying in δ_1 and having no point in common with f_1, and construct the line $C-P-s_1-Q-s_2-P$ (noted by l') which is the same with $C-P-s_2-Q-s_1-P$, where C is a point lying in δ_1 not on f_1 nor on AB; take the point A' on l', in between C and P, such that $A'P$ is smaller than AB; now, there exist two distinct points B'_1 on s_1 and B'_2 on s_2, such that $A'B'_1$ is congruent to AB and $A'B'_2$ is congruent to AB, with $A'B'_1$ different from $A'B'_2$;

- but if we consider a line l' lying in δ_1 and limited by the frontier f_1 on the right side (the limit point being noted by M), and take a point A' on l', close to M, such that $A'M$ is less than $A'B'$, then: there is no point B' on the right side of l' so that $A'B'$ is congruent to AB.]

A segment may not be congruent to itself!

[For example:

- let A be a point on s_1, closer to P, and B a point on s_2, closer to P also; A and B are lying on the same line $A-Q-B-P-A$ which is the same with line $A-P-B-Q-A$, but AB meseared on the first repersentation of the line is strictly greater than AB meseared on the second representation of their line.]

IV.2. If a segment AB is congruent to the segment $A'B'$ and also to the segment $A''B''$, then not always the segment $A'B'$ is congruent to the segment $A''B''$.

[For example:

- let AB be a segment lying in $\delta_1 - f_1$, and consider the line $C-P-s_1-Q-s_2-P-D$, where C, D are two distinct points in $\delta_1 - f_1$ such that C, P, D are colinear. Suppose tat the segment AB is congruent to the segment CD (i.e. $C-P-s_1-Q-s_2-P-D$). Get also an obvious segment $A'B'$ in $\delta_1 - f_1$, different from the preceding ones, but congruent to AB.

Then the segment $A'B'$ is not congruent to the segment CD (considered as $C-P-D$, i.e. not passing through Q.)

IV.3. If AB, BC are two segments of the same line l which have no points in common aside from the point B, and $A'B', B'C'$ are two segments of the same line or of another line l' having no point other than B' in common, such that AB is congruent to $A'B'$ and BC is congruent to $B'C'$, then not always the segment AC is congruent to $A'C'$.

[For example:

let l be a line lying in δ_1, not on f_1, and A, B, C three distinct points on l, such that AC is greater than s_1; let l' be the following line: $A'-P-s_1-Q-s_2-P$ where A' lies in δ_1, not on f_1, and get B' on s_1 such that $A'B'$ is congruent to AB, get C' on s_2 such that BC is congruent to $B'C'$ (the points A, B, C are thus chosen); then: the segment $A'C'$ which is first seen as $A'-P-B'-Q-C'$ is not congruent to AC, because $A'C'$ is the geodesic $A'-P-C'$ (the shortest way from A' to C' does not pass through B') which is strictly less than AC.]

Definitions. *Let h, k be two lines having a point O in common. Then the system (h, O, k) is called the angle of the lines h and k in the point O.*

(Because some of our lines are curves, we take the angle of the tangents to the curves in their common point.)

The angle formed by the lines h and k situated in the same plane, noted by $< h, k >$, is equal to the arithmetic mean of the angles formed by h and k in all their common points.

IV.4. Let an angle (h, k) be given in the plane α, and let a line h be given in the plane β. Suppose that in the plane β a definite side of the line h' is assigned, and a point O'. Then in the plane β there are one, or more , or even no half-line(s) k' emanating from the point O' such that the angle (h, k) is congruent to the angle (h', k'), and at the same time the interior points of the angle (h', k') lie upon one or both sides of h'.

[Examples:

- Let A be a point in $\delta_1 - f_1$, and B, C two distinct points in $\delta_2 - f_2$; let h be the line $A-P-s_1-Q-B$, and k be the line $A-P-s_2-Q-C$; because h and k intersect in an infinite number of points (the segment AP), where they normally coincide - i.e. in each such point their angle is congruent to zero, the angle (h, k) is congruent to zero.

Now, let A' be a point in $\delta_1 - f_1$, different from A, and B' a point in $\delta_2 - f_2$, different from B, and draw the line h' as $A' - P - s_1 - Q - B'$; there exist an infinite number of lines k', of the form $A' - P - s_2 - Q - C'$ (where C' is any point in $\delta_2 - f_2$, not on the line QB'), such that the angle (h,k) is congruent to (h',k'), because (h',k') is al- so congruent to zero, and the line $A' - P - s_2 - Q - C'$ is different from the line $A' - P - s_2 - Q - D'$ id D' is not on the line QC'

- If h, k and h' are three lines in $\delta_1 - P$, which intersect the frontier f_1 in at most one point, then there exists only one line k' on a given part of h' such that the angle (h,k) is congruent to the angle (h',k').

- *Is there any case when, with these hypotheses, no k' exists?

- Not every angle is congruent to itself; for example: $(< s_1, s_2 >)$ is not congruent to $(< s_1, s_2 >)$ [because one can construct two distinct lines: $P - s_1 - Q - A$ and $P - s_2 - Q - A$, where A is point in $\delta_2 - f_2$, for the first angle, which becomes equal to zero; and $P - s_1 - Q - A$ and $P - s_2 - Q - B$, where B is another point in $\delta_2 - f_2$, B different from A, for the second angle, which becomes strictly greater than zero!].

IV.5. If the angle (h,k) is congruent to the angle (h',k') and to the angle (h'',k''), then the angle (h',k') is not always congruent to the angle (h'',k'').
(A similar construction to the previous one.)

IV.6. Let ABC and $A'B'C'$ be two triangles such that AB is congruent to $A'B'$, AC is congruent to $A'C'$, $< BAC$ is congruent to $< B'A'C'$. Then not always $< ABC$ is congruent to $< A'B'C'$ and $< ACB$ is congruent to $< A'C'B'$.
[For example:
Let M, N be two distinct points in $\delta_2 - f_2$, thus obtaining the triangle PMN; now take three points R, M', N' in $\delta_1 - f_1$, such that RM' is congruent to PM, RN' is congruent to RN, and the angle (RM', RN') is congruent to the angle (PM, PN). $RM'N'$ is an obvious triangle. Of course, the two triangle are not congruent, because for example PM and PN cut each other twice - in p and Q - while RM' and RN' only once - in R. (These are geodesical triangles.)]

Definitions. *Two angles are called supplementary if they have the same vertex, one side in common, and the other sides not common form a line.*

A right angle is an angle congruent to its supplementary angle.

Two triangles are congruent if their angles are congruent two by two, and its sides are *congruent two by two.*

Propositions:

A right angle is not always congruent to another right angle.

For example:

Let $A - P - s_1 - Q$ be a line, with A lying in $\delta_1 - f_1$, and $B - P - s_1 - Q$ another line, with B lying in $\delta_1 - f_1$ and B not lying in the line AP; we consider the tangent t at s_1 in P, and B chosen in a way that $< (AP, t)$ is not congruent to $< (BP, t)$; let A', B' be other points lyng in $\delta_1 - f_1$ such that $< APA'$ is congruent to $< A'P - s_1 - Q$, and $< BPB'$ is congruent to $< B'P - s_1 - Q$. Then:

- the angle APA' is right, because it is congruent to its supplementary (by construction);

- the BPB' is also right, because it is congruent to its supplementary (by construction);

- but $< APA'$ is not congruent to $< BPB'$, because the first one is half of the angle $A - P - s_1 - Q$, i.e. half of $< (AP, t)$, while the second one is half of the $B - P - s_1 - Q$, i.e. half of $< (BP, t)$.

The theorems of congruence for triangles [side, side, and angle in between; angle, angle, and common side; side, side, side] may not hold either in the Critical Zone (s_1, s_2, f_1, f_2) of the Model.

Property:

The sum of the angles of a triangle can be:

- 180 degrees, if all its vertexes A, B, C are lying, for example, in $\delta_1 - f_1$;

-strictly less than 180 degrees [any value in the interval $(0, 180)$], for example:

let R, T be two points in $\delta_2 - f_2$ such that Q does not lie in RT, and S another point on s_2; then the triangle SRT has $< (SR, ST)$ congruent to O because SR and ST have an infinite number of common points (the segment SQ), and $< QTR+ < TRQ$ congruent to $180- < TQR$ [by construction we may very $< TQR$ in the interval $(0, 180)$];

-even O degree!

let A be a point in $\delta_1 - f_1, B$ a point in $\delta_2 - f_2$, and C a point on s_3, very close to P; then ABC is a non-degenerated triangle (because its vertexes are non-colinear), but $< (A - P - s_1 - Q - B, A - P - s_3 - C) =< (B - Q - s_1 - P - A, B - Q - s_1 - P - s_3 - C) = < (C - s_3 - P - A, C - s_3 - P - s_1 - Q - B) = 0$ (one considers the lenght $C - s_3 - P - s_1 - Q - B$ strictly less than $C - s_3 - B$); the area of this triangle is also 0!

- more than 180 degrees, for example:

let A, B be two points in $\delta_1 - f_1$, such that $< PAB+ < PBA+ < (s_1, s_2;$ in $Q)$ is strictly greater than 180 degrees; then triangle ABQ, formed by the intersection of the lines $A - P - s_2 - Q, Q - s_1 - P - B, AB$ will have the sum of its angles strictly greater than 180 degrees.

Defenition. *A circle of center M is a totality of all points A for which the segments MA are congruent to one another.*

For example, if the center is Q, and the length of the segments MA is chosen greater than the length of s_1, then the circle is formed by the arc of circle centered in Q, of radius MA, and lying in δ_2, plus another arc of circle centered in P, of radius MA - length of s_1, lying in δ_1.

GROUP V. ANTI-AXIOMS OF CONTINUITY (ANTI-ARCHIMEDEAN AXIOM)

Let A, B be two points. Take the point $A_1, A_2, A_3, A_4, \ldots$ so that A_1 lies between A and A_2, A_2 lies between A_1 and A_3, A_3 lies between A_2 and A_4, etc. and the segments AA_1, A_1A_2, $A_2A_3, A_3A_4, \ldots$ are congruent to one another.

Then, among this series of points, not always there exists a certain point A_n such that B lies between A and A_n.

For example:

let A be a point in $\delta_1 - f_1$, and B a point on f_1, B different from P; on the line AB consider the points $A_1, A_2, A_3, A_4, \ldots$ in between A and B, such that $AA_1, A_1A_2, A_2A_3, A_3A_4$, etc. are congruent to one another; then we finde that there is no point behind B (considering the direction from A to B), because B is a limit point (the line AB ends in B).

The Bolzano's (intermediate value) theorem may not hold in the Critical Zone of the Model.

Question 38:

It's very intresting to find out if this system of axiom is complete and consistent (!) The apparent unsientific or wrong geometry, which looks more like an amalgam, is somehow supported by its attached model.

Question 39:

How will the differential equations look like in this field?

Question 40:

How will the (so called by us:) "PARADOXIST" TRIGONOMETRY look like in this field?

Question 41:

First, one can generalize this using more bridges (conections/strings between δ_1 and δ_2) of many lengths, and many gates (points like P and Q on f_1 and f_2, respectively) - from a finite to an infinite number of such bridges and gates.

If one put all bridges in the δ plane, one gates a dimension-2 model; otherwise, the dimension is ≥ 3.

Some bridges may be replaced with (round or not necessaryly) bodies, tangent (or not necessaryly) to the frontiers f_1 and f_2.

Question 42:

Should it be indicated to remove the discontinuities?

But what about DISCONTINUOUS MODELS (on spaces not everywhere continouus - like our MD)? generating in this way DISCONTINUOUS GEOMETRIES.

Question 43:

The model MD can also be generalized to n-dimensional space as a hypersurfece, considering the group of all projective transformations of an $(n+1)$-dimensional real projective space that leave MD invariant.

Questions 44-47:

Find geometric models for each of the following four cases:

- No point/line/plane in the model space verifies any of Hilbert's twenty axioms; (in our MD, some points/linse/planes did verify, and some others did not);

- The Hilbert's groups of axioms I, II, IV, V are denied for any point/line/plane in the model space, but the III-th one (axiom of parallels) is verified; this is an Opposite-(Lobachevski+ Reimann) Geomatry:

neither hyperbolic, nor eliptic ... and yet Non-Euclidean!

- The groups og anti-axioms I, II, IV, V are all verified, but the III-th one (anti-axiom of parallels) is denied;

- Some of the groups of anti-axioms I, II, III, IV, V are verified, while the others atre not - except the previous case; (there are particular cases already known).

Question 48:

What connections may be found among this Paradoxist Model, and the Cayley, Klein, Poincare, Beltrami (differential geometric) models?

Questions 49-120: (combining by twos, each new geometry - out of 4 - with an old geometry - out of 18 - all mentionned below):

What connections among these Paradoxist Geometry, Non-Geometry, Counter-Projective Geometry, Anti-Geometry and the other ones: Conformal (Mobius) Geometry, Pseudo-Conformal Geometry, Laguerre Geometry, Spectral Geometry, Spherical Geometry, Hiper-Sphere Geometry, wave Geometry (Y. Mimura), Non-Holonomic Geometry (G. Vranceanu), Cartan's Geometry of Connection, Integral Geometry (W. Blaschke), Continuous Geometry (von Neumann), Affine Geometry, Generalized Geometries (of H. Weyl, O. Veblen, J.A. Schoutten), etc.

CONCLUSION

The above 120 OPEN QUESTIONS are not impossible at all. "The world is moving so fast nowadays that the person, who says <it can't be done>, is often interrupted by someone doing it"! [<Leadirship> journal, Editor Arthur F. Lenehan, October 24, 1995, p. 16, Fairfield, NJ].

The author encourages readers to send not only comments, but also new (solved or unsolved) questions arising from them.

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