BENCZE MIHÁLY
FLORIN POPOVICI
FLORENTIN SMARANDACHE

A Generalization of the Leibniz Theorem

In Florentin Smarandache: "Collected Papers", vol. III. Oradea (Romania): Abaddaba, 2000.

A Generalization Of The Leibniz Theorem[1]

In this paper we show a generalization of Leibniz's theorem and an application of this.

Leibniz's theorem. Let M be an arbitrary point in the plane of the ABC triangle, then $MA^2+MB^2+MC^2 = 1/3(a^2+b^2+c^2)+3MG^2$, where G is the centroid ofthe triarigle. We generalize this theorem:

Theorem. Let $A_1, A_2, \ldots, A_n$ be arbitrary points in the space and G the centroid of this points system; then for an arbitrary M point of the space is valid the following equation

$$\sum_{i=1}^{n} MA_i^2 = \frac{1}{n} \sum_{1 \le i < j \le n} A_i A_j^2 + n \cdot MG^2 .$$

Proof. First, we interpret the centroid of the n points system in a recurrent way. If n =2, then is the midpoint of the segment. If n = 3, then it is the centroid of the triangle. Suppose that we found the centroid of the n-1 points created system. Now we join each of the n points with the centroid of the n-1 points created system; so we obtain n bisectors of the sides. It is easy to show that these n medians are concurrent segments. In this way we obtain the centroid of the n points created system. Denote G_i the centroid of the $A_k, k = 1,2,\ldots,i-1,i+1,\ldots,n$ points created system. It can be showed that $(n-1)A_iG=GG_i$. Now by induction we prove the theorem. If n=2 the $MA_1^2 + MA_2^2 = \frac{1}{2} A_1 A_2^2 + 2MG^2$ or $MG^2 == \frac{1}{4}\left(2\left(MA_1^2 + MA_2^2\right)\right)$, where G is the midpoint of the A_1A_2 segment. The above formula is the side bisector's formula in the MA_1A_2 triangle. The proof can be done by Stewart's theorem, cosines theorem, generalized theorem of Pythagoras or can be done vectorially. Suppose that the assertion of the theorem is true for n = k. If $A_1, A_2, \ldots, A_k$ are arbitrary points in the space, G_0 is the centroid of this points system, then we have the following relation $\sum_{i=1}^{k} MA_i^2 = \frac{1}{k} \sum_{1 \le i < j \le k} A_i A_j^2 + k \cdot MG_o^2$.

Now we prove for n=k+1. Let $A_{k+1} \notin \{A_1, A_2, \ldots, A_k, G_o\}$ an arbitrary point in the space and let G be the centroid of the $A_1, A_2, \ldots, A_k, A_{k+1}$, points system. Taking into account that G is on the $A_{k+1}G_o$ segment and

[1] Together with Mihály Bencze and Florin Popovici

$k\cdot A_{k+1}G=GG_o$, we apply Stewart's theorem to the M,G_o,G,A_{k+1} points, from where

$$MA_{k+1}^2\cdot GG_o+MG_o^2\cdot GA_{k+1}-MG^2\cdot A_{k+1}G_o=GG_o\cdot GA_{k+1}\cdot A_{k+1}G_o.$$

According to the previous observation $A_{k+1}G=\frac{k}{k+1}A_{k+1}G_o$ *and* $GG_o=\frac{1}{k+1}A_{k+1}G_o$.

Using these, the above relation becomes

$$MA_{k+1}^2+k\cdot MG_o^2=\frac{k}{k+1}A_{k+1}G_o^2+(k+1)MG^2.$$

From here

$$k\cdot MG_o^2=\sum_{i=1}^{k}MA_i^2-\frac{1}{k}\sum_{1\le i<j\le k}A_iA_j^2.$$

From the supposition of the induction with $M\equiv A_{k+1}$ as substitution we get $\sum_{i=1}^{k}A_iA_j^2=\frac{1}{k}\sum_{1<i<j\le k}A_iA_j^2+k\cdot A_{k+1}G_o^2$ and thus

$$\frac{k}{k+1}A_{k+1}G_0^2=\frac{1}{k+1}\sum_{i=1}^{k}A_iA_{k+1}^2-\frac{1}{k(k+1)}\sum_{1\le i<j\le k}A_iA_j^2.$$

Substituting this in the above relation we obtain that

$$\sum_{i=1}^{k+1}MA_i^2=\left(\frac{1}{k}-\frac{1}{k(k+1)}\right)\sum_{1\le i<j\le k}A_iA_j^2+\frac{1}{k+1}\sum_{i=1}^{k}A_iA_{k+1}^2$$

$$+(k+1)MG^2=\frac{1}{k+1}\sum_{1\le i<j\le k+1}A_iA_j^2+(k+1)MG^2.$$

With this we proved that our assertion is true for $n = k+1$. According to the induction it is true for every $n \ge 2$ natural numbers.

1. Application. If the $A_1, A_2, ..., A_n$ points are on the sphere with the center O and radius R, then using in the theorem the substitution $M\equiv O$ we get the identity $OG^2=R^2-\frac{1}{n^2}\sum_{1\le i<j\le n}A_iA_j^2$.

In case of a triangle $OG^2=R^2-\frac{1}{9}\left(a^2+b^2+c^2\right)$.

In case of a tetrahedron $OG^2=R^2-\frac{1}{16}\left(a^2+b^2+c^2+d^2+e^2+f^2\right)$.

2. Application. If the $A_1, A_2, ..., A_n$ points are on the sphere with the

center O and radius R, then $\sum_{1\le i<j\le n} A_iA_j^2 \le n^2R^2$.

Equality holds if and only if G≡O. In case of a triangle $a^2+b^2+c^2 \le 9R^2$, in case of a tetrahedron $a^2+b^2+c^2+d^2+e^2+f^2 \le 16R^2$.

3. Application. Using the arithmetic and harmonic mean inequality, from the previous application results the following inequality results: $\sum_{1\le i<j\le n} \frac{1}{A_iA_j^2} \ge \frac{(n-1)^2}{4R^2}$.

In case of a triangle $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} \ge \frac{1}{R^2}$, in case of a tetrahedron

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}+\frac{1}{e^2}+\frac{1}{f^2} \ge \frac{9}{4R^2}.$$

4. Application. Considering the Cauchy-Buniakowski-Schwarz inequality from the application 2 we obtain the following inequality

$$\sum_{1\le i<j\le n} A_iA_j \le nR\sqrt{\frac{n(n-1)}{2}}.$$

In case of a triangle $a+b+c\le 3\sqrt{3}R$, in case of a tetrahedron $a+b+c+d+e+f \le 4\sqrt{6}R$.

5. Application. Using the arithmetic and harmonic mean inequality, from the previous application we get the following inequality

$$\sum_{1\le i<j\le n} \frac{1}{A_iA_j} \ge \frac{(n-1)\sqrt{n(n-1)}}{2R\sqrt{2}}.$$

In case of a triangle $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \ge \frac{\sqrt{3}}{R}$, in case of a tetrahedron

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e}+\frac{1}{f} \ge \frac{3}{R}\sqrt{\frac{3}{2}}.$$

6. Application. Considering application 3, we obtain the following inequality: $\frac{n^2(n-1)^2}{4} \le \left(\sum_{1\le i<j\le n} A_iA_j^k\right)\left(\sum_{1\le i<j\le n} \frac{1}{A_iA_j^k}\right) \le$

$$\le \begin{cases} \frac{(M+m)^2 n^2(n-1)^2}{16M\cdot m} \text{ if } \frac{n(n-1)}{2} \text{ is even}, \\ \frac{(M+m)^2 n^2(n-1)^2 - 4(M-m)^2}{16M\cdot m} \text{ if } \frac{n(n-1)}{2} \text{ is odd}, \end{cases}$$

where $m = \min\{A_iA_j^k\}$ and $M = \max\{A_iA_j^k\}$ In case of a triangle

$$9 \le (a^k+b^k+c^k)(a^{-k}+b^{-k}+c^{-k}) \le \frac{2M^2+5M\cdot m+2m^2}{M\cdot m},$$

in case of a tetrahedron

$$36 \le \left(a^k + b^k + c^k + d^k + e^k + f^k\right)\left(a^{-k} + b^{-k} + c^{-k} + d^{-k} + e^{-k} + f^{-k}\right) \le \frac{9(M+m)^2}{M \cdot m}.$$

7. Application. Let $A_1, A_2, \ldots, A_n$ be the vertexes of the polygon inscribed in the sphere with the center O and radius R. First we interpret the orthocenters of the $A_1A_2 \ldots A_n$ inscribable polygon. For three arbitrary vertexes, one orthocenter corresponds. Now we take four vertexes. In the obtained four orthocenters of the triangles we construct the circles with radius R, which has one common point. This will be the orthocenter of the inscribable quadrilateral. We continue in the same way. The circles with radius R that we construct in the orthocenters of the n-1 sides inscribable polygons have one common point. This will be the orthocenter of the n sides, inscribeable polygon. It can be shown that O, H, G are collinears and n·OG=OH. From the first application

$$OH^2 = n^2R^2 - \sum_{1 \le i < j \le n} A_iA_j^2 \quad \text{and} \quad GH^2 = (n-1)R^2 - \left(1 - \frac{1}{n}\right)^2 \sum_{1 \le i < j \le n} A_iA_j^2.$$

In case of a triangle $OH^2 = 9R^2 - (a^2+b^2+c^2)$ and $GH^2 = 4R^2 - \frac{4}{9}\left(a^2+b^2+c^2\right)$.

8. Application. In the case of an $A_1A_2 \ldots A_n$ inscribable polygon $\sum_{1 \le i < j \le n} A_iA_j^2 = n^2R^2$ if and only if O≡H≡G. In case of a triangle this is equivalent with an equilateral triangle.

9. Application. Now we compute the length of the midpoints created by the $A_1, A_2, \ldots, A_n$ space points system. Let S={1,2,...,i-1, i+ 1,...,n} and G_0 be the centroid of the A_k, $k \in S$, points system. By substituting $M \equiv A_i$ in the theorem, for the length of the midpoints we obtain the following relation $A_iG_0^2 = \frac{1}{n-1}\sum_{k \in S} A_iA_k^2 - \frac{1}{(n-1)^2}\sum_{u,v \in S,\, u \ne v} A_uA_v^2$.

10. Application. In case of a triangle $m_a^2 = \frac{2\left(b^2+c^2\right) - a^2}{4}$ and its permutations. From here

$$m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}\left(a^2+b^2+c^2\right),$$

$$m_a^2 + m_b^2 + m_c^2 \le \frac{27}{4}R^2, \quad m_a + m_b + m_c \le \frac{9}{2}R.$$

11. Application. In case of a tetrahedron $m_a^2 = \frac{1}{9}\left(3\left(a^2+b^2+c^2\right) - \left(d^2+e^2+f^2\right)\right)$ and its permutations. From here $\sum m_a^2 = \frac{4}{9}\left(\sum a^2\right)$, $\sum m_a^2 \le \frac{64}{9}R^2$, $\sum m_a \le \frac{16}{3}R$.

12. Application. Denote $m_{a,f}$ the length of the segments which join

midpoint of the a and f skew sides of the tetrahedron (bimedian). In the interpretation of the application

$$9m_{a,f}^2 = \frac{1}{4}\left(b^2 + c^2 + d^2 + e^2 - a^2 - f^2\right)$$

and its permutations. From here

$$m_{a,f}^2 + m_{b,d}^2 + m_{c,e}^2 = \frac{1}{4}\left(\sum a^2\right),$$

$$m_{a,f}^2 + m_{b,d}^2 + m_{c,e}^2 \le 4R^2, \qquad m_{a,f} + m_{b,d} + m_{c,e} \le 2R\sqrt{3}.$$

References:

1. Hajós Gy., Bevezetés a geometriába, Tankönyvkiadó, Bp. 1966.
2. Kazarinoff N.D., Geometriai egyenlötlenségek, Gondolat, 1980.
3. Stoica Gh., Egy ismert maximum feladatról, Matematikai Lapok, Kolozsvár, 9/1987, pp. 330-332.
4. Caius Jacob, Lagrange egyik képletéröl és ennek kiterjesztéséröl, Matematikai Lapok, Kolozsvár, 2/1987, pp. 50-56.
5. Sándor József, Geometriai egyenlötlenségek, Kolozsvár, 1988.

["Octogon", Vol.6, No.1, 67-70, 1998.]