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About the characteristic function of the set

In Florentin Smarandache: “Collected Papers”,
vol. III. Oradea (Romania): Abaddaba, 2000.

About the characteristic function of the set [1]

In our paper we give a method, based on characteristic function of the set, of resolving some difficult problem of set theory found in high school study.

Definition: Let be $A \subset E \neq \theta$ (a universal set), then the $f_A : E \to \{0, 1\}$, where the function
$$f_A(x) = \begin{cases} 1, & \text{if } x \in A; \\ 0, & \text{if } x \notin A, \end{cases}$$
is named the characteristic function of the set A.

Theorem 1. Let $A, B \subset E$. In this case $f_A = f_B$ if and only if A=B.

Proof.
$$f_A(x) = \begin{cases} 1, & \text{if } x \in A = B \\ 0, & \text{if } x \notin A = B \end{cases} = f_B(x)$$

Reciprocally: In case of any $x \in A$, $f_A(x) = 1$, but $f_A = f_B$ and for that $f_B(x) = 1$, namely $x \in B$ from where $A \subset B$. The same way we prove that $B \subset A$, namely $A = B$.

Theorem 2. $f_{\tilde{A}} = 1 - f_A$, where $\tilde{A} = C_E A$.

Proof.
$$f_{\tilde{A}}(x) = \begin{cases} 1, & \text{if } x \in \tilde{A} \\ 0, & \text{if } x \notin \tilde{A} \end{cases} = \begin{cases} 1, & \text{if } x \notin A \\ 0, & \text{if } x \in A \end{cases}$$
$$= \begin{cases} 1 - 0, & \text{if } x \notin A \\ 1 - 1, & \text{if } x \in A \end{cases} = 1 - \begin{cases} 0, & \text{if } x \notin A \\ & \text{if, } x \notin A \end{cases} = 1 - f_A(x).$$

Theorem 3. $f_{A \cap B} = f_A * f_B$

Proof.
$$f_{A \cap B}(x) = \begin{cases} 1, & \text{if } x \in A \cap B \\ 0, & \text{if } x \notin A \cap B \end{cases} = \begin{cases} 1, & \text{if } x \in A \text{ and } x \in B \\ 0, & \text{if } x \notin A \text{ or } x \notin B \end{cases}$$
$$= \begin{cases} 1, & \text{if } x \in A, x \in B \\ 0, & \text{if } x \in A, x \notin B \\ 0, & \text{if } x \notin A, x \in B \\ 0, & \text{if } x \notin A, x \notin B \end{cases} = \left(\begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} \right) \cdot \left(\begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \notin B \end{cases} \right)$$
$$= f_A(x) f_B(x)$$

The theorem can be generalized by induction:

Theorem 4. $$f_{\bigcap_{k=1}^{n} A_k} = \prod_{k=1}^{n} f_{A_k}$$

[1] Together with Mihály Bencze

Consequence. For any $n \in N^*$ $f_M^n = f_M$

Proof. In the previous theorem we write $A_1 = A_2 = \ldots = A_n = M$.

Theorem 5.

$f_{A \cup B} = f_A + f_B - f_A f_B$.

Proof. $f_{A \cup B} = f_{\overline{\overline{A \cup B}}} = f_{\overline{\bar{A} \cap \bar{B}}} = 1 - f_{\bar{A} \cap \bar{B}} = 1 - f_{\bar{A}} f_{\bar{B}} =$
$= 1 - (1 - f_A)(1 - f_B) = f_A + f_B - f_A f_B$.

Can be generalized by induction:

Theorem 6. $f_{\bigcup_{k=1}^{n} A_k} = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \le i_1 < \ldots < i_k \le n} f_{A_{i_1}} f_{A_{i_2}} \ldots f_{A_{i_k}}$

Theorem 7. $f_{A-B} = f_A(1 - f_B)$

Proof. $f_{A-B} = f_{A \cap \bar{B}} = f_A f_{\bar{B}} = f_A(1 - f_B)$.

Can be generalized by induction :

Theorem 8. $f_{A_1 - A_2 - \ldots - A_n} = \sum_{k=1}^{n} (-1)^{k-1} f_{A_{i_1}} f_{A_{i_2}} \ldots f_{A_{i_k}}$.

Theorem 9. $f_{A \Delta B} = f_A + f_B - 2 f_A f_B$

Proof. $f_{A \Delta B} = f_{A \cup B - A \cap B} = f_{A \cup B}(1 - f_{A \cap B}) =$
$= (f_A + f_B - f_A f_B)(1 - f_A f_B) = f_A + f_B - 2 f_A f_B$.

Can be generalized by induction:

Theorem 10.

$F_{\Delta_{k=1}^{n} A_k} = \sum_{k=1}^{n} (-2)^{k-1} \sum_{1 \le i_1 < \ldots < i_k \le n} f_{A_{i_1} A_{i_2} \ldots A_{i_k}}$.

Theorem 11. $f_{A \times B}(x, y) = f_A(x) f_B(y)$

Proof. If $(x,y) \in AXB$, then $f_{AXB}(x,y) = 1$ and $x \in A$, namely $f_A(x) = 1$ and $y \in B$, namely $f_B(y) = 1$, so $f_A(x) f_B(y) = 1$. If $(x,y) \notin AXB$, then $f_{AXB}(x,y) = 0$ and $x \notin A$, namely $f_A(x) = 0$ or $y \notin B$, namely $f_y(B) = 0$ so $f_A(x) f_B(y) = 0$. Can be generalized by induction.

Theorem 12.

$$f_{\times_{k=1}^{n} A_k}(x_1, x_2, \ldots, x_n) = \prod_{k=1}^{n} f_{A_k}(x_k).$$

Theorem 13. (De Morgan) $\overline{\bigcup_{k=1}^{n} A_k} = \bigcap_{k=1}^{n} \bar{A}_k$.

Proof. $f_{\overline{\bigcup_{k=1}^{n} A_k}} = 1 - f_{\bigcup_{k=1}^{n} A_k} =$

$1 - \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \le i_1 < \ldots < i_k \le n} f_{A_{i_1}} f_{A_{i_2}} \ldots f_{A_{i_k}} =$

$\prod_{k=1}^{n} (1 - f_{A_k}) = \prod_{k=1}^{n} f_{\bar{A}_k} = f_{\bigcap_{k=1}^{n} \bar{A}_k}$.

We prove in the same way the following theorem:

Theorem 14. (De Morgan) $\overline{\bigcap_{k=1}^{n} A_k} = \bigcup_{k=1}^{n} \overline{A}_k$.

Theorem 15.

$$\left(\bigcup_{k=1}^{n} A_k\right) \cap M = \bigcup_{k=1}^{n} \left(A_k \cap M\right) .$$

Proof. $f_{\left(\bigcup_{k=1}^{n} A_k\right) \cap M} = f_{\bigcup_{k=1}^{n} A_k} \, f_M =$

$$\sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \le i_1 < \ldots < i_k \le n} f_{A_{i_1}} f_{A_{i_2}} \ldots f_{A_{i_k}} f_M =$$

$$\sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \le i_1 < \ldots < i_k \le n} f_{A_{i_1}} f_{A_{i_2}} \ldots f_{A_{i_k}} f^k_M =$$

$$\sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \le i_1 < \ldots < i_k \le n} f_{A_{i_1} \cap M} \, f_{A_{i_2} \cap M} \ldots f_{A_{i_k} \cap M} = f_{\bigcup_{k=1}^{n} (A_k \cap M)} .$$

In the same way we prove that:

Theorem 16. $\left(\bigcap_{k=1}^{n} A_k\right) \cup M = \bigcap_{k=1}^{n} \left(A_k \cup M\right)$.

Theorem 17. $\left(\Delta_{k=1}^{n} A_k\right) \cap M = \Delta_{k=1}^{n} \left(A_k \cap M\right)$.

Application.

$\left(\Delta_{k=1}^{n} A_k\right) \cup M = \Delta_{k=1}^{n} \left(A_k \cup M\right)$ if and only if $M = \phi$.

Theorem 18.

$$MX\left(\bigcup_{k=1}^{n} A_k\right) = \bigcup_{k=1}^{n} \left(MXA_k\right) .$$

Proof. $f_{MX\left(\bigcup_{k=1}^{n} A_k\right)}(x,y) = f_M(y) \, f_{\bigcup_{k=1}^{n} A_k}(x) =$

$$\sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \le i_1 < \ldots < i_k \le n} f_{A_{i_1}}(x) \, f_{A_{i_2}}(x) \ldots f_{A_{i_k}}(x) \, f_M(y) =$$

$$\sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \le i_1 < \ldots < i_k \le n} f_{A_{i_1}}(x) \, f_{A_{i_2}}(x) \ldots f_{A_{i_k}}(x) \, f^k_M(y) =$$

$$\sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \le i_1 < \ldots < i_k \le n} f_{A_{i_1} X M}(x,y) \ldots f_{A_{i_k} X M}(x,y) = f_{\bigcup_{k=1}^{n} (MXA_k)}$$

In the same way we prove that:

Theorem 19. $MX\left(\bigcap_{k=1}^{n} A_k\right) = \bigcap_{k=1}^{n}\left(MXA_k\right)$.

Theorem 20.

$MX(A_1 - A_2 - \ldots - A_n) = (MXA_1) - (MXA_2) - \ldots - (MXA_n)$.

Theorem 21. $(A_1 - A_2) \cup (A_2 - A_3) \cup \ldots \cup (A_{n-1} - A_n) \cup (A_n - A_1) = \bigcup_{k=1}^{n} A_k - \bigcap_{k=1}^{n} A_k$.

Proof 1. $f_{(A_1-A_2)\cup\ldots\cup(A_n-A_1)} =$

$$\sum_{k=1}^{n}(-1)^{k-1} \sum_{1\le i_1<\ldots<i_k\le n} f_{A_{i_1}-A_{i_2}} \ldots f_{A_{i_k}-A_{i_1}} =$$

$$\sum_{k=1}^{n}(-1)^{k-1} \sum_{1\le i_1<\ldots<i_k\le n} (f_{A_{i_1}} f_{A_{i_2}} - f_{A_{i_1}} f_{A_{i_2}}) \ldots (f_{A_{i_k}} f_{A_{i_1}} f_{A_{i_k}} f_{A_{i_1}}) =$$

$$\sum_{k=1}^{n}(-1)^{k-1} \sum_{1\le i_1<\ldots<i_k\le n} f_{A_{i_1}} \ldots f_{A_{i_k}}\left(1 - \prod_{p=1}^{n} f_{A_p}\right) =$$

$$f_{\bigcup_{k=1}^{n} A_k}\left(1 - f_{\bigcap_{k=1}^{n} A_k}\right) = f_{\bigcup_{k=1}^{n} A_k - \bigcap_{k=1}^{n} A_k}.$$

Proof 2. Let $x \in \bigcup_{i=1}^{n}(A_i - A_{i+1})$, (where $A_{n+1} = A_1$), then there exists k such that $x \in (A_k - A_{k+1})$, namely $x \notin (A_k \cap A_{k+1}) \subset A_1 \cap A_2 \cap \ldots \cap A_n$, namely $x \notin A_1 \cap \ldots \cap A_n$ and

$$x \in \bigcup_{k=1}^{n} A_k - \bigcap_{k=1}^{n} A_k.$$

Now we prove the inverse statement:

Let $x \in \bigcup_{k=1}^{n} A_k - \bigcap_{k=1}^{n} A_k$, we show that there exists k such that $x \in A_k$ and $x \notin A_{k+1}$. On the contrary it would result that for any $k \in \{1,2,\ldots,n)$, $x \in A_k$ and $x \in A_{k+1}$ namely $x \in \bigcup_{k=1}^{n} A_k$, it results that there exists p such that $x \in A_p$, but from the previous reasoning it result that $x \in A_{p+1}$, and using this we consequently obtain that $x \in A_k$ for $k = \overline{p,n}$. But from $x \in A_n$ we get that $x \in A_1$ using consequently, it results that $x \in A_k$, $k = \overline{1,p}$, from where $x \in A_k$, $k = \overline{1,n}$, namely $x \in A_1 \cap \ldots \cap A_n$, that is a contradiction. Thus there exists r such that $x \in A_r$

and $x \notin A_{r-1}$, namely $x \in (A_r - A_{r-1})$ and so $x \in \bigcup_{k=1}^{n} (A_k - A_{k-1})$.

In the same way we prove the following theorem:

Theorem 22. $(A_1 \Delta A_2) \cup (A_2 \Delta A_3) \cup \ldots \cup (A_{n-1} \Delta A_n) = \bigcup_{k=1}^{n} A_k - \bigcap_{k=1}^{n} A_k$.

Theorem 23. $(A_1 X A_2 X \ldots X A_k) \cap (A_{k+1} X A_{k+2} X \ldots X A_{2k}) \cap (A_n X A_1 X \ldots X A_{k-1}) = (A_1 \cap A_2 \cap \ldots \cap A_n)^k$.

Proof. $f_{(A_1 x \ldots x A_k) \cap \ldots \cap (A_n x A_1 x \ldots x A_{k-1})}(x_1, \ldots, x_n) =$
$f_{A_1 x \ldots x A_k}(x_1, \ldots, x_n) \ldots f_{A_n x \ldots x A_{k-1}}(x_1, \ldots, x_n) =$
$(f_{A_1}(x_1) \ldots f_{A_k}(x_k)) \ldots (f_{A_n}(x_n) \ldots f_{A_{k-1}}(x_{k-1}) =$
$f^k_{A_1}(x_1) \ldots f^k_{A_n}(x_n) = f^k_{A_1 \cap \ldots \cap A_n}(x_1, \ldots, x_n) =$
$f_{(A_1 \cap \ldots \cap A_n)^k}(x_1, \ldots, x_n)$.

Theorem 24. $(P(E), \cup)$ is a commutative monoid.

Proof. For any $A, B \in P(E)$; $A \cup B \in P(E)$, namely the intern operation. Because $(A \cup B) \cup C = A \cup (B \cup C)$ is associative, $A \cup B = B \cup A$ commutative, and because $A \cup \phi = A$ then ϕ is the neutral element.

Theorem 25. $(P(E), \cap)$ is a commutative monoid.

Proof. For any $A, B \in P(E)$; $A \cap B \in P(E)$ namely intern operation. $(A \cap B) \cap C = A \cap (B \cap C)$ associative, $A \cap B = B \cap A$, commutative $A \cap E = A$, E is the neutral element.

Theorem 26. $(P(E), \Delta)$ is an abellan group.

Proof. For any $A, B \in P(E)$; $A \Delta B \in P(E)$, namely the intern operation. $A \Delta B = B \Delta A$ commutative. The proof of associativity is in the XII class manual as a problem. We prove it, using the characteristic function of the set.

$f_{(A \Delta B) \Delta C} = 4 f_A f_B f_C - 2 f_A f_B + f_B f_C + f_C f_A + f_A + f_B + f_C = f_{A \Delta (B \Delta C)}$

Because $A \Delta \phi = A$, ϕ is the neutral element and because $A \Delta A = \phi$; A is the symmetric element itself.

Theorem 27. $(P(E), \Delta, \cap)$ is a commutative Boole ring with divisor of zero.

Proof. Because of the previous theorem it satisfies the commutative ring axioms. Now we prove that it has a divisor of zero. If $A \neq \phi$ and $B \neq \phi$ are two disjoint sets, then $A \cap B = \phi$, thus it has divisor of zero. From Theorem 17 we get that it is distributive for n = 2. Because for any $A \in P(E)$; $A \cap A = A$ and $A \Delta A = \phi$ it also satisfies the Boole-type axioms.

Theorem 28. Let be $H = \{ f \mid f : E \to \{0, 1\} \}$, then $(H, \oplus)$ is an Abelian group, where $f_A \oplus f_B = f_A + f_B - 2 f_A f_B$ and $(P(E), \Delta) \cong (H, \oplus)$.

Proof. Let F : P(E) → H, where F(A) = f_A , then from the previous theorem we get that it is bijective and because

F(AΔB) = $f_{A\Delta B}$ = F(A) ⊕ F(B) it is compatible.

Theorem 29. $card(A_1 \Delta A_n) \leq card(A_1 \Delta A_2) +$
$+ card(A_2 \Delta A_3) + \cdots + card(A_{n-1} \Delta A_n)$

Proof. By induction. If n = 2, then it is true, we show that for n = 3 it is also true. Because $(A_1 \cap A_2) \cup (A_2 \cap A_3) \subseteq A_2 \cup (A_1 \cap A_3)$;

$card((A_1\, capA_2) \cup (A_2 \cap A_3)) \leq card(A_2 \cup (A_1 \cap A_3))$ but

$card(M \cup N) = cardM + cardN - card(M \cap N)$ and thus

$cardA_2 + card(A_1 \cap A_3) - card(A_1 \cap A_2) - card(A_2 \cap A_3) \geq 0$ can be writen as $cardA_1 + cardA_3 - 2card(A_1 \cap A_3) \leq$

$(cardA_1 + cardA_2 - 2card(A_1 \cap A_2)) + (cardA_2 + cardA_3 - 2card(A_2 \cap A_3))$.

But because of $(M \Delta N) = cardM + cardN - 2card(M \cap N)$ then $card(A_1 \Delta A_3) \leq card(A_1 \Delta A_2) + card(A_2 \Delta A_3)$. The proof of this step of the induction relies on the above method.

Theorem 30. $(P^2(E), card(A\Delta B))$ is a metric space.

Proof. Let d(A,B) = card(AΔB) : P(E)xP(E) → R.

1. d(A, B) = 0 ⇔ card(AΔB) = 0 ⇔ card((A - B) ∪ (B - A)) = 0 but because (A - B) ∩ (B - A) = ϕ we get (A - B) + card(B - A) = 0 and because (A - B) = 0 and card(B - A) = O, then A - B = ϕ, B - A = ϕ and A = B.

2. d(A, B) = d(B,A) results from AΔB = BΔA.

3. In consequence of the previous theorem

d(A, C) ≤ d(A, B) + d(B, C).

As result of the above three properties it is a metric space.

PROBLEMS

Problem 1.

Let A = B ∪ C and f : P(A) → P(A)XP(A), where

f(x) = (X ∪ B, X ∪ C). Prove that f is injective if and only if B∩C=ϕ.

Solution 1. If f is injective. Then

f(ϕ) = (ϕ∪B,ϕ∪C) = (B,C) = ((B∩C)∪B,(B∩C)∪C) = f(B∩C) from where B ∩ C = ϕ. Now reciprocally: Let B ∩ C = ϕ, then f(x) = f(Y), it result, that X ∪ B = Y ∪ B and X ∪ C = Y ∪ C or X = X∪ϕ = X∪(B∩C) = (X∪B)∩(X∪C)=(Y∪B)∩(Y∪C) = Y ∪ (B ∩ C) = Y ∪ ϕ = Y namely it is injective.

Solution 2. Let $B \cap C = \phi$ passing over the set function $f(x) = f(Y)$ if and only if $X \cup B = Y \cup B$ and $X \cup C = Y \cup C$, namely $f_{X\cup B} = f_{Y\cup B}$ and

$f_{X\cup C} = f_{Y\cup C}$ or $f_X + f_B - f_X f_B = f_Y + f_B - f_Y f_B$ and

$f_X + f_C - f_X f_C = f_Y + f_C - f_Y f_C$ from where

$(f_X - f_Y)(f_B - f_C) = 0$. Because $A = B \cup C$ and $B \cap C = \phi$ therefore

$$(f_B - f_C)(u) = \begin{cases} 1, & \text{if } u \in B \\ -1, & \text{if } u \in C \end{cases} \neq 0$$

therefore $f_X - f_Y = 0$, namely $X = Y$ and thus it is injective.

Generalization. Let $M = \bigcup_{k=1}^{n} A_k$ and $f : P(A) \to P^n(A)$, where

$f(X) = (X \cup A_1, X \cup A_2, \dots, X \cup A_n)$. Prove that f is injective if and only if $A_1 \cap A_2 \cap \dots \cap A_n = \phi$.

Problem 2. Let $E \neq \phi$ and $A \in P(E)$ and

$f : P(E) \to P(E)xP(E)$, where $f(X) = (X \cap A, X \cup A)$.

a. Prove that f is injective

b. Prove that $\{f(x), x \in P(E)\} = \{(M,N) \mid M \subset A \subset N \subset E\} = K$.

c. Let $g : P(E) \to K$, where $g(X) = f(X)$. Prove that g is bijective and compute its inverse.

Solution.

a. $f(X) = f(Y)$, namely $(X \cap A, X \cup A) = (Y \cap A, Y \cup A)$ and so

$X \cap A = Y \cap A$, $X \cup A = Y \cup A$, from where $X \Delta A = Y \Delta A$ or

$(X\Delta A)\Delta A = (Y\Delta A)\Delta A$, $X\Delta(A\Delta A) = Y\Delta(A\Delta A)$, $X\Delta\phi = Y\Delta\phi$ and thus $X = Y$, namely f is injective.

b. $\{f(X), X \in P(E)\} = f(P(E))$. We show that $f(P(E)) \subset K$. For any $(M,N) \in f(P(E))$, $\exists X \in P(E) : f(X) = (M,N)$;

$(X \cap A, X \cup A) = (M, N)$. From here $X \cap A = M$, $X \cup A = N$, namely $M \subset A$ and $A \subset N$ thus $M \subset A \subset N$ and so $(M, N) \in X$. Now we show that $K \subset f(P(E))$, for any $(M, N) \in K$, $\exists X \in P(E)$ so that $f(X) = (M, N)$.$f(X) = (M, N)$, namely $(X \cap A, X \cup A) = (M, N)$ from where $X \cap A = M$ and $X \cup A = N$, namely

$X\Delta A = N - M$, $(X\Delta A)\Delta A = (N - M)\Delta A$, $X\Delta\phi = (N - M)\Delta A$,

$X = (N-M)\Delta A$, $X = (N\cap\bar{M})\Delta A$, $X = ((N\cap\bar{M}) - A) \cup (A - (N\cap\bar{M})) =$

$((N\cap\bar{M})\cap\bar{A}) \cup (A\cap(\overline{N\cap\bar{M}})) = (N\cap(\bar{M}\cap\bar{A})) \cup (A\cap(\bar{N}\cup M)) =$

$(N\cap\bar{A}) \cup ((A\cap\bar{N}) \cup (A\cap M)) = (N\cap\bar{A}) \cup (\phi \cup M) = (N-A) \cup M$.

From here we get the unic solution:

$X = (N-A) \cup M$.

We test $f((N-A)\cup M)=(((N-A)\cup M)\cap A,((N-A)\cup M)\cup A)$ but
$((N-A)\cup M)\cap A=((N\cap\bar{A})\cup M)\cap A=((N\cap\bar{A})\cap A)\cup(M\cap A)=$
$(N\cap(\bar{A}\cap A))\cup M=(N\cap\phi)\cup M=\phi\cup M=M$ and
$((N-A)\cup M)\cup A=(N-A)\cup(M\cup A)=(N-A)\cup A=$
$(N\cap\bar{A})\cup A=(N\cup A)\cap(\bar{A}\cup A)=N\cap E=N$, $f((N-A)\cup M)=(M,N)$. Thus $f(P(E))=K$.

c. From point a. we get g is injective, from point b. we get g is surjective, thus g is bijective.The inverse function is :

$g^{-1}(M,N)=(N-A)\cup M$.

Problem 3. Let $E\neq\phi, A, B\in P(E)$ and

$f: P(E)\to P(E)XP(E)$, where $f(X)=(X\cap A, X\cap B)$.

a. Give the necessary and sufficient condition such that f is injective.

b. Give the necessary and suffcient condition such that f is surjective.

c. Supposing that f is bijective, compute its inverse.

Solution.

a. Suppose f is injective. Then: $f(A\cup B)=$
$((A\cup B)\cap A,(A\cup B)\cap B)=(A,B)=(E\cap A,E\cap B)=f(E)$, from where $A\cup B=E$, Now we suppose that $A\cup B=E$, it results that
$X=X\cap E=X\cap(A\cup B)=(X\cap A)\cup(X\cap B)=(Y\cap A)\cup(Y\cap B)=Y\cap(A\cup B)=Y\cap E=Y$, namely from $f(X)=f(Y)$ we get that
$X=Y$, namely f is injective.

b. Suppose f is surjective, for any $M,N\in P(A)XP(B)$, there exists $X\in P(E)$, $f(X)=(M,N)$,$(X\cap A,X\cap B)=(M,N)$,$X\cap A=M$,$X\cap B=N$. In special cases $(M,N)=(A,\phi)$, there exists $X\in P(E)$, from $X\supset A$, $\phi=X\cap B\supset A\cap B$,$A\cap B=\phi$. Now we suppose that$A\cap B=\phi$ and show that it is surjective. Let $(M,N)\in P(A)XP(B)$ then $M\subset A$, $N\subset B$ and $M\cap B\subset A\cap B=\phi$ and$N\cap A\subset B\cap A=\phi$ namely $M\cap B=\phi$, $N\cap A=\phi$ and $f(M\cup N)=((M\cup N)\cap A, (M\cup N)\cap B)=((M\cap A)\cup(N\cap A), (M\cap B)\cup(N\cap B))=(M\cup\phi, \phi\cup N)=(M,N)$, for any (M, N) there exists $X=M\cup N$ such that $f(X)=(M,N)$, namely f is surjective.

c. We show that $f^{-1}((M,N))=M\cup N$.

Observation. In the previous two problems we can use the characteristic function of the set as in the first problem. This method we leave to the readers.

Application. Let $E\neq\phi, A_k\in P(E)(k=1,...,n)$ and

$f: P(E) \to P^n(E)$, where $f(X) = (X \cap A_1, X \cap A_2, \dots, X \cap A_n)$.
Prove that f is injective if and only if $\bigcup_{k=1}^{n} A_k = E$.

Application. Let $E \neq \phi, A_k \in P(E)(k = 1,\dots,n)$ and
$f: P(E) \to P^n(E)$, where $f(X) = (X \cap A_1, X \cap A_2, \dots, X \cap A_n)$.
Prove that f is surjective if and only if $\cap_k^n \bar{A}_k = \phi$.

Problem 4. We name the set M convex if for any $x,y \in M$
$tx + (1 - t)y \in M$, for any $t \in [0, 1]$.

Prove that if $A_k(k= 1,\dots,n)$ are convex sets, then $\bigcap_{k=1}^{n} A_k$ is also convex.

Problem 5. If $A_k(k = 1, \dots, n)$ are convex sets, then $\bigcap_{k=1}^{n} A_k$ is also convex .

Problem 6. Give the necessary and sufficient condition such that if A, B are convex /concave sets then $A \cup B$ is also convex /concave. Generalization for n set.

Problem 7. Give the necessary and sufficient condition such that if A, B are convex /concave sets then $A \Delta B$ is also convex /concave. Generalization for n set.

Problem 8. Let $f,g : P(E) \to P(E)$, where $f(X) = A\text{-}X$ and $g(X) = A \Delta X$, $A \in P(E)$. Prove that f, g are bijective and compute their inverse functions.

Problem 9. Let

$A \circ B = \{(x,y) \in R \times R \mid \exists z \in R : (x,z) \in A \text{ and } (z,y) \in B\}$. In a particular case let $A = \{(x, \{x\}) \mid x \in R\}$ and $B = \{(\{y\},y) \mid y \in R\}$.
Represent the A o A, B o A, B o B cases.

Problem 10.
i. If $A \cup B \cup C = D$, $A \cup B \cup D = C$, $A \cup C \cup D = B$, $B \cup C \cup D = A$, then $A = B = C = D$.

ii. Are there different A, B, C, D sets such that
$A \cup B \cup C = A \cup B \cup D = A \cup C \cup D = B \cup C \cup D$?

Problem 11. Prove that $A \Delta B = A \cup B$ if and only if $A \cap B = \phi$.

Problem 12. Prove the following identity.

$$\bigcap_{i,j=1,i<j}^{n} A_k \cup A_j = \bigcup_{i=1}^{n} \left(\bigcap_{j=1,\, j\neq i}^{n} A_j \right).$$

Problem 13. Prove the following identity.
$(A \cup B)\text{-}(B \cap C) = [A\text{-}(B \cap C)) \cup (B\text{-}C) = (A - B) \cup (A - C) \cup (B - C)$ and

$A - [(A \cap C) - (A \cap B)] = (A - \bar{B}) \cup (A - C)$.

Problem 14. Prove that $A \cup (B \cap C) = (A \cup B) \cap C = (A \cup C) \cap B$ if and only if $A \subset B$ and $A \subset C$.

Problem 15. Prove the following identities:

$(A-B)-C = (A-B)-(C-B)$,

$(A \cup B)-(A \cup C) = B-(A \cap C)$,

$(A \cap B)-(A \cap C)=(A \cap B)-C$.

Problem 16. Solve the following system of equations:

$$\begin{cases} A \cup X \cup Y = (A \cup X) \cap (A \cup Y) \\ A \cap X \cap Y = (A \cap X) \cup (A \cap Y). \end{cases}$$

Problem 17. Solve the following system of equations:

$$\begin{cases} A \Delta X \Delta B = A \\ A \Delta Y \Delta B = B. \end{cases}$$

Problem 18. Let $X, Y, Z \subseteq A$.

Prove that: $Z = (X \cap \bar{Z}) \cup (Y \cap \bar{Z}) \cup (\bar{X} \cap Z \cap \bar{Y})$ if and only if $X=Y=\phi$.

Problem 19. Prove the following identity:

$$\bigcup_{k=1}^{n} [A_k \cup (B_k - C)] = \left(\bigcup_{k=1}^{n} A_k\right) \cup \left[\left(\bigcup_{k=1}^{n} A_k\right) - C\right].$$

Problem 20. Prove that: $A \cup B = (A - B) \cup (B - A) \cup (A \cap B)$.

Problem 21. Prove that:

$(A \Delta B) \Delta C = (A \cap \bar{B} \cap \bar{C}) \cup (\bar{A} \cap B \cap \bar{C}) \cup (\bar{A} \cap \bar{B} \cap C) \cup (A \cap B \cap C)$.

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["Octogon", Vol. 6, No. 2, 86-96, 1998.]