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Theorems with Parallels Taken through a Triangle's Vertices and Constructions Performed only with the Ruler

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In this article, we solve problems of **geometric constructions only with the ruler**, using known theorems.

1^{st} Problem.

Being given a triangle ABC, its circumscribed circle (its center known) and a point M fixed on the circle, construct, using only the ruler, a transversal line A_1, B_1, C_1, with $A_1 \in BC, B_1 \in CA, C_1 \in AB$, such that $\sphericalangle MA_1C \equiv \sphericalangle MB_1C \equiv \sphericalangle MC_1A$ (the lines taken though M to generate congruent angles with the sides BC, CA and AB, respectively).

2^{nd} Problem.

Being given a triangle ABC, its circumscribed circle (its center known) and A_1, B_1, C_1, such that $A_1 \in$

$BC, B_1 \in CA, C_1 \in AB$ and A_1, B_1, C_1 collinear, construct, using only the ruler, a point M on the circle circumscribing the triangle, such that the lines MA_1, MB_1, MC_1 to generate congruent angles with BC, CA and AB, respectively.

3rd Problem.

Being given a triangle ABC inscribed in a circle of given center and AA' a given cevian, A' a point on the circle, construct, using only the ruler, the isogonal cevian AA_1 to the cevian AA'.

To solve these problems and to prove the theorems for problems solving, we need the following *Lemma*:

1st Lemma.

(Generalized Simpson's Line)

If M is a point on the circle circumscribed to the triangle ABC and we take the lines MA_1, MB_1, MC_1 which generate congruent angles ($A_1 \in BC, B_1 \in CA, C_1 \in AB$) with BC, CA and AB respectively, then the points A_1, B_1, C_1 are collinear.

Proof.

Let M on the circle circumscribed to the triangle ABC (see *Figure 1*), such that:

$\sphericalangle MA_1C \equiv \sphericalangle MB_1C \equiv \sphericalangle MC_1A = \varphi$. (1)

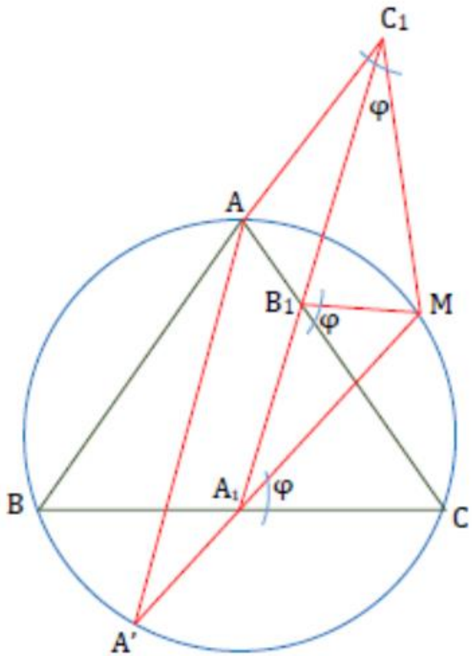

Figure 1.

From the relation (1), we obtain that the quadrilateral MB_1A_1C is inscriptible and, therefore:

$\sphericalangle A_1BC \equiv \sphericalangle A_1MC$. (2).

Also from (1), we have that MB_1AC_1 is inscriptible, and so

$\sphericalangle AB_1C_1 \equiv \sphericalangle AMC_1$. (3)

The quadrilateral MABC is inscribed, hence:

$\sphericalangle MAC_1 \equiv \sphericalangle BCM.$ (4)

On the other hand,

$\sphericalangle A_1MC = 180^0 - \left(\widehat{BCM} + \varphi\right),$

$\sphericalangle AMC_1 = 180^0 - (\widehat{MAC_1} + \varphi).$

The relation (4) drives us, together with the above relations, to:

$\sphericalangle A_1MC \equiv \sphericalangle AMC_1.$ (5)

Finally, using the relations (5), (2) and (3), we conclude that: $\sphericalangle A_1B_1C \equiv AB_1C_1$, which justifies the collinearity of the points A_1, B_1, C_1.

Remark.

The Simson's Line is obtained in the case when $\varphi = 90^0$.

2nd Lemma.

If M is a point on the circle circumscribed to the triangle ABC and A_1, B_1, C_1 are points on BC, CA and AB, respectively, such that $\sphericalangle MA_1C = \sphericalangle MB_1C = \sphericalangle MC_1A = \varphi$, and MA_1 intersects the circle a second time in A', then $AA' \parallel A_1B_1$.

Proof.

The quadrilateral MB_1A_1C is inscriptible (see *Figure 1*); it follows that:

$\sphericalangle CMA' \equiv \sphericalangle A_1B_1C.$ (6)

On the other hand, the quadrilateral $MAA'C$ is also inscriptible, hence:

$\sphericalangle CMA' \equiv \sphericalangle A'AC.$ (7)

The relations (6) and (7) imply: $\sphericalangle A'MC \equiv \sphericalangle A'AC$, which gives $AA' \parallel A_1B_1$.

3rd Lemma.

(The construction of a parallel with a given diameter using a ruler)

In a circle of given center, construct, using only the ruler, a parallel taken through a point of the circle at a given diameter.

Solution.

In the given circle $\mathcal{C}(O,R)$, let be a diameter $(AB)]$ and let $M \in \mathcal{C}(O,R)$. We construct the line BM (see *Figure 2*). We consider on this line the point D (M between D and B). We join D with O, A with M and denote $DO \cap AM = \{P\}$.

We take BP and let $\{N\} = DA \cap BP$. The line MN is parallel to AB.

Construction's Proof.

In the triangle DAB, the cevians DO, AM and BN are concurrent.

Ceva's Theorem provides:

$$\frac{OA}{OB} \cdot \frac{MB}{MD} \cdot \frac{ND}{NA} = 1. \quad (8)$$

But DO is a median, $DO = BO = R$.

From (8), we get $\frac{MB}{MD} = \frac{NA}{ND}$, which, by Thales reciprocal, gives $MN \parallel AB$.

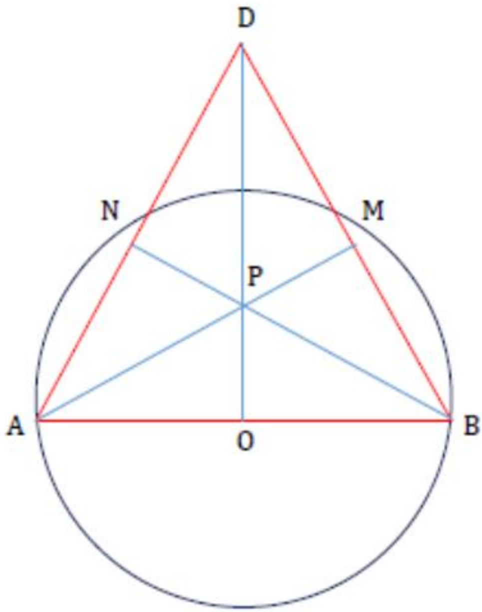

Figure 2.

Remark.

If we have a circle with given center and a certain line d, we can construct though a given point M a parallel to that line in such way: we take two diameters $[RS]$ and $[UV]$ through the center of the given circle (see *Figure 3*).

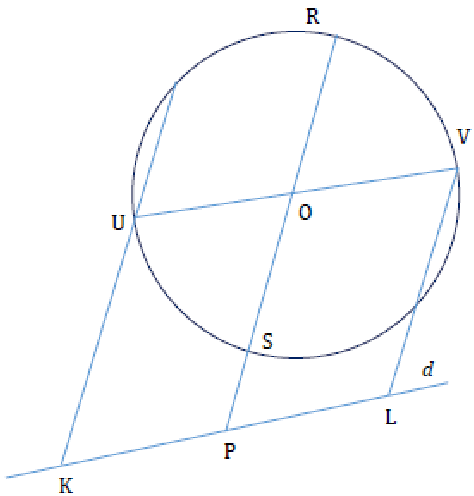

Figure 3.

We denote $RS \cap d = \{P\}$; because $[RO] \equiv [SO]$, we can construct, applying the 3^{rd} *Lemma*, the parallels through U and V to RS which intersect d in K and L, respectively. Since we have on the line d the points K, P, L, such that $[KP] \equiv [PL]$, we can construct the parallel through M to d based on the construction from 3^{rd} *Lemma*.

1st Theorem.

(P. Aubert – 1899)

If, through the vertices of the triangle ABC, we take three lines parallel to each other, which intersect the circumscribed circle in A', B' and C', and M is a point on the circumscribed circle, as well $MA' \cap BC = \{A_1\}$, $MB' \cap CA = \{B_1\}$, $MC' \cap AB = \{C_1\}$, then A_1, B_1, C_1 are collinear and their line is parallel to AA'.

Proof.

The point of the proof is to show that MA_1, MB_1, MC_1 generate congruent angles with BC, CA and AB, respectively.

$$m(\widehat{MA_1C}) = \frac{1}{2}\left[m(\breve{MC}) + m(\breve{BA'})\right] \qquad (9)$$

$$m(\widehat{MB_1C}) = \frac{1}{2}\left[m(\breve{MC}) + m(\breve{AB'})\right] \qquad (10)$$

But $AA' \parallel BB'$ implies $m(\breve{BA'}) = m(\breve{AB'})$, hence, from (9) and (10), it follows that:

$$\sphericalangle MA_1C \equiv \sphericalangle MB_1C, \qquad (11)$$

$$m(\widehat{MC_1A}) = \frac{1}{2}\left[m(\breve{BM}) - m(\breve{AC'})\right]. \qquad (12)$$

But $AA' \parallel CC'$ implies that $m(\breve{AC'}) = m(\breve{A'C})$; by returning to (12), we have that:

$$m(\widehat{MC_1A}) = \frac{1}{2}\left[m(\breve{BM}) - m(\breve{AC'})\right] =$$

$$= \frac{1}{2}\left[m(\breve{BA'}) + m(\breve{MC})\right]. \qquad (13)$$

The relations (9) and (13) show that:

$$\sphericalangle MA_1C \equiv \sphericalangle MC_1A. \qquad (14)$$

From (11) and (14), we obtain: $\sphericalangle MA_1C \equiv \sphericalangle MB_1C \equiv \sphericalangle MC_1A$, which, by *1st Lemma*, verifies the collinearity of points A_1, B_1, C_1. Now, applying the *2nd Lemma*, we obtain the parallelism of lines AA' and A_1B_1.

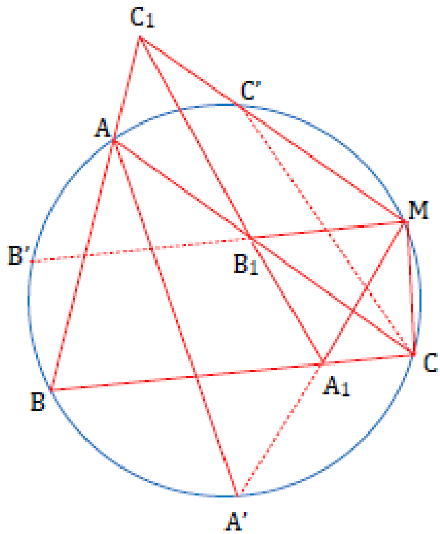

Figure 4.

2nd Theorem.

(M'Kensie – 1887)

If $A_1B_1C_1$ is a transversal line in the triangle ABC ($A_1 \in BC, B_1 \in CA, C_1 \in AB$), and through the triangle's vertices we take the chords AA', BB', CC' of a circle circumscribed to the triangle, parallels with the transversal line, then the lines AA' , BB', CC' are concurrent on the circumscribed circle.

Proof.

We denote by M the intersection of the line A_1A' with the circumscribed circle (see *Figure 5*) and with B_1', respectively C_1' the intersection of the line MB' with AC and of the line MC' with AB.

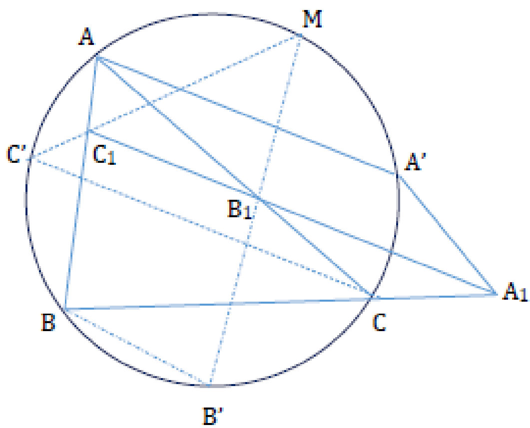

Figure 5.

According to the P. Aubert's theorem, we have that the points A_1, B_1', C_1' are collinear and that the line A_1B_1' is parallel to AA'.

From hypothesis, we have that $A_1B_1 \parallel AA'$; from the uniqueness of the parallel taken through A_1 to AA', it follows that $A_1B_1 \equiv A_1B_1'$, therefore $B_1' = B_1$, and analogously $C_1' = C_1$.

Remark.

We have that: MA_1, MB_1, MC_1 generate congruent angles with BC, CA and AB, respectively.

3rd Theorem.

(Beltrami – 1862)

If three parallels are taken through the three vertices of a given triangle, then their isogonals intersect each other on the circle circumscribed to the triangle, and vice versa.

Proof.

Let AA', BB', CC' the three parallel lines with a certain direction (see *Figure 6*).

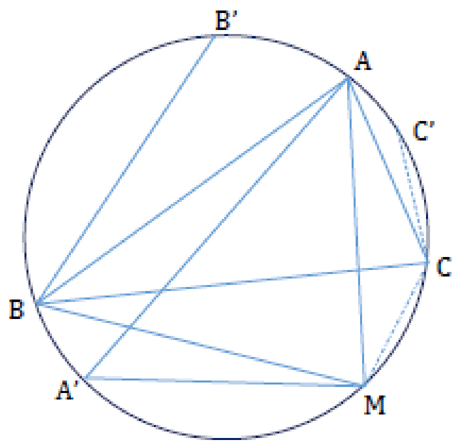

Figure 6.

To construct the isogonal of the cevian AA', we take $A'M \parallel BC$, M belonging to the circle circumscribed to the triangle, having $\overset{\frown}{BA'} \equiv \overset{\frown}{CM}$, it follows that AM will be the isogonal of the cevian AA'. (Indeed, from $\overset{\frown}{BA'} \equiv \overset{\frown}{CM}$ it follows that $\sphericalangle BAA' \equiv \sphericalangle CAM$.)

On the other hand, $BB' \parallel AA'$ implies $\overset{\frown}{BA'} \equiv \overset{\frown}{AB'}$, and since $\overset{\frown}{BA'} \equiv \overset{\frown}{CM}$ we have that $\overset{\frown}{AB'} \equiv \overset{\frown}{CM}$, which shows that the isogonal of the parallel BB' is BM. From $CC' \parallel AA'$, it follows that $A'C \equiv AC'$, having $\sphericalangle B'CM \equiv \sphericalangle ACC'$, therefore the isogonal of the parallel CC' is CM'.

Reciprocally.

If AM, BM, CM are concurrent cevians in M, the point on the circle circumscribed to the triangle ABC, let us prove that their isogonals are parallel lines. To construct an isogonal of AM, we take $MA' \parallel BC$, A' belonging to the circumscribed circle. We have $\overset{\frown}{MC} \equiv \overset{\frown}{BA'}$. Constructing the isogonal BB' of BM, with B' on the circumscribed circle, we will have $\overset{\frown}{CM} \equiv \overset{\frown}{AB'}$, it follows that $\overset{\frown}{BA'} \equiv \overset{\frown}{AB'}$ and, consequently, $\sphericalangle ABB' \equiv \sphericalangle BAA'$, which shows that $AA' \parallel BB'$. Analogously, we show that $CC' \parallel AA'$.

We are now able to solve the proposed problems.

Solution to the 1st problem.

Using the *3rd Lemma*, we construct the parallels AA', BB', CC' with a certain directions of a diameter of the circle circumscribed to the given triangle.

We join M with A' , B', C' and denote the intersection between MA' and BC, A_1; $MB' \cap CA = \{B_1\}$ and $MA' \cap AV = \{C_1\}$.

According to the Aubert's Theorem, the points A_1, B_1, C_1 will be collinear, and MA', MB', MC' generate congruent angles with BC, CA and AB, respectively.

Solution to the 2nd problem.

Using the *3rd Lemma* and the remark that follows it, we construct through A, B, C the parallels to A_1B_1; we denote by A', B', C' their intersections with the circle circumscribed to the triangle ABC. (It is enough to build a single parallel to the transversal line $A_1B_1C_1$, for example AA').

We join A' with A_1 and denote by M the intersection with the circle. The point M will be the point we searched for. The construction's proof follows from the M'Kensie Theorem.

Solution to the 3rd problem.

We suppose that A' belongs to the little arc determined by the chord $\overset{\frown}{BC}$ in the circle circumscribed to the triangle ABC.

In this case, in order to find the isogonal AA_1, we construct (by help of the *3rd Lemma* and of the remark that follows it) the parallel $A'A_1$ to BC, A_1 being on the circumscribed circle, it is obvious that AA' and AA_1 will be isogonal cevians.

We suppose that A' belongs to the high arc determined by the chord $\overset{\frown}{BC}$; we consider $A' \in \overset{\frown}{AB}$ (the arc $\overset{\frown}{AB}$ does not contain the point C). In this situation, we firstly construct the parallel BP to AA', P belongs to the circumscribed circle, and then through P we construct the parallel PA_1 to AC, A_1 belongs to the circumscribed circle. The isogonal of the line AA' will be AA_1. The construction's proof follows from *3rd Lemma* and from the proof of Beltrami's Theorem.

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Apollonius's Circles of k^{th} Rank

The purpose of this article is to introduce the notion of **Apollonius's circle of k^{th} rank**.

1st Definition.

It is called an internal cevian of k^{th} rank the line AA_k where $A_k \in (BC)$, such that $\frac{BA}{A_kC} = \left(\frac{AB}{AC}\right)^k$ $(k \in \mathbb{R})$.

If A_k' is the harmonic conjugate of the point A_k in relation to B and C, we call the line AA_k' an external cevian of k^{th} rank.

2nd Definition.

We call Apollonius's circle of k^{th} rank with respect to the side BC of ABC triangle the circle which has as diameter the segment line A_kA_k'.

1st Theorem.

Apollonius's circle of k^{th} rank is the locus of points M from ABC triangle's plan, satisfying the relation: $\frac{MB}{MC} = \left(\frac{AB}{AC}\right)^k$.

Proof.

Let O_{A_k} the center of the Apollonius's circle of k^{th} rank relative to the side BC of ABC triangle (see *Figure 1*) and U, V the points of intersection of this circle with the circle circumscribed to the triangle ABC. We denote by D the middle of arc BC, and we extend DA_k to intersect the circle circumscribed in U'.

In $BU'C$ triangle, $U'D$ is bisector; it follows that $\frac{BA_k}{A_kC} = \frac{U'B}{U'C} = \left(\frac{AB}{AC}\right)^k$, so U' belongs to the locus.

The perpendicular in U' on $U'A_k$ intersects BC on A_k'', which is the foot of the BUC triangle's outer bisector, so the harmonic conjugate of A_k in relation to B and C, thus $A_k'' = A_k'$.

Therefore, U' is on the Apollonius's circle of rank k relative to the side BC, hence $U' = U$.

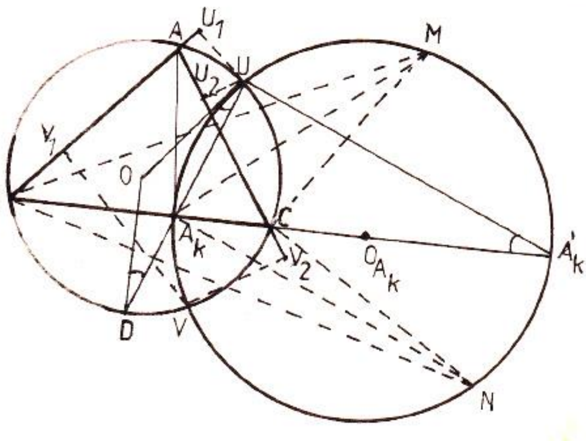

Figure 3

Let M a point that satisfies the relation from the statement; thus $\frac{MB}{MC} = \frac{BA_k}{A_kC}$; it follows – by using the reciprocal of bisector's theorem – that MA_k is the internal bisector of angle BMC. Now let us proceed as before, taking the external bisector; it follows that M belongs to the Apollonius's circle of center O_{A_k}. We consider now a point M on this circle, and we construct C' such that $\sphericalangle BNA_k \equiv \sphericalangle A_kNC'$ (thus (NA_k is the internal bisector of the angle $\widehat{BNC'}$). Because $A'_kN \perp NA_k$, it follows that A_k and A'_k are harmonically conjugated with respect to B and C'. On the other hand, the same points are harmonically conjugated with respect to B and C; from here, it follows that $C' = C$, and we have $\frac{NB}{NC} = \frac{BA_k}{A_kC} = \left(\frac{AB}{AC}\right)^k$.

3rd Definition.

It is called a complete quadrilateral the geometric figure obtained from a convex quadrilateral by extending the opposite sides until they intersect. A complete quadrilateral has 6 vertices, 4 sides and 3 diagonals.

2nd Theorem.

In a complete quadrilateral, the three diagonals' middles are collinear (Gauss - 1810).

Proof.

Let $ABCDEF$ a given complete quadrilateral (see *Figure 2*). We denote by H_1, H_2, H_3, H_4 respectively the orthocenters of ABF, ADE, CBE, CDF triangles, and let A_1, B_1, F_1 the feet of the heights of ABF triangle.

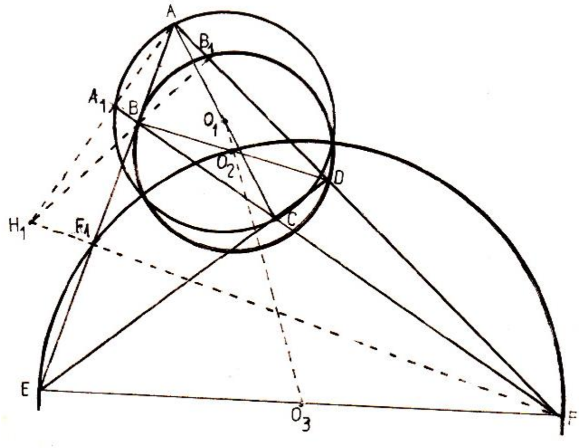

Figure 4

As previously shown, the following relations occur: $H_1A.H_1A_1 - H_1B.H_1B_1 = H_1F.H_1F_1$; they express that the point H_1 has equal powers to the circles of diameters AC, BD, EF, because those circles contain respectively the points A_1, B_1, F_1, and H_1 is an internal point.

It is shown analogously that the points H_2, H_3, H_4 have equal powers to the same circles, so those points are situated on the radical axis (common to the circles), therefore the circles are part of a fascicle, as such their centers – which are the middles of the complete quadrilateral's diagonals – are collinear.

The line formed by the middles of a complete quadrilateral's diagonals is called Gauss's line or Gauss-Newton's line.

3rd Theorem.

The Apollonius's circle of k^{th} rank of a triangle are part of a fascicle.

Proof.

Let AA_k , BB_k, CC_k be concurrent cevians of k^{th} rank and AA'_k, BB'_k, CC'_k be the external cevians of k^{th} rank (see *Figure 3*). The figure $B'_kC_kB_kC'_kA_kA'_k$ is a complete quadrilateral and 2nd theorem is applied.

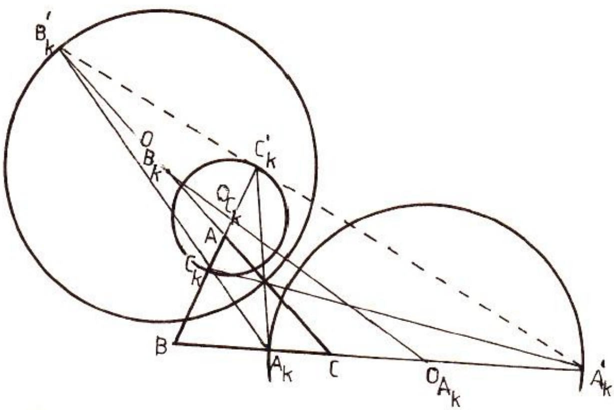

Figure 5

4th Theorem.

The Apollonius's circle of k^{th} rank of a triangle are the orthogonals of the circle circumscribed to the triangle.

Proof.

We unite O to D and U (see *Figure 1*), $OD \perp BC$ and $m\left(\widehat{A_k U A'_k}\right) = 90^0$, it follows that $\widehat{UA'_kA_k} = \widehat{ODA_k} = \widehat{OUA_k}$.

The congruence $\widehat{UA'_kA_k} \equiv \widehat{OUA_k}$ shows that OU is tangent to the Apollonius's circle of center O_{A_k}.

Analogously, it can be demonstrated for the other Apollonius's Circle.

1st Remark.

The previous theorem indicates that the radical axis of Apollonius's circle of k^{th} rank is the perpendicular taken from O to the line $O_{A_k}O_{B_k}$.

5th Theorem.

The centers of Apollonius's Circle of k^{th} rank of a triangle are situated on the trilinear polar associated to the intersection point of the cevians of $2k^{th}$ rank.

Proof.

From the previous theorem, it results that $OU \perp UO_{A_k}$, so UO_{A_k} is an external cevian of rank 2 for BCU triangle, thus an external symmedian. Henceforth, $\frac{O_{A_k}B}{O_{A_k}C} = \left(\frac{BU}{CU}\right)^2 = \left(\frac{AB}{AC}\right)^{2k}$ (the last equality occurs because U belong to the Apollonius's circle of rank k associated to the vertex A).

6th Theorem.

The Apollonius's circle of k^{th} rank of a triangle intersects the circle circumscribed to the triangle in two points that belong to the internal and external cevians of k+1th rank.

Proof.

Let U and V points of intersection of the Apollonius's circle of center O_{A_k} with the circle circumscribed to the ABC (see *Figure 1*). We take from U and V the perpendiculars UU_1, UU_2 and VV_1, VV_2 on AB and AC respectively. The quadrilaterals $ABVC$, $ABCU$ are inscribed, it follows the similarity of triangles BVV_1, CVV_2 and BUU_1, CUU_2, from where we get the relations:

$$\frac{BV}{CV} = \frac{VV_1}{VV_2}, \qquad \frac{UB}{UC} = \frac{UU_1}{UU_2}.$$

But $\frac{BV}{CV}=\left(\frac{AB}{AC}\right)^k$, $\frac{UB}{UC}=\left(\frac{AB}{AC}\right)^k$, $\frac{VV_1}{VV_2}=\left(\frac{AB}{AC}\right)^k$ and $\frac{UU_1}{UU_2}=\left(\frac{AB}{AC}\right)^k$, relations that show that V and U belong respectively to the internal cevian and the external cevian of rank $k+1$.

4th Definition.

If the Apollonius's circle of k^{th} rank associated with a triangle has two common points, then we call these points isodynamic points of k^{th} rank (and we denote them W_k, W_k').

1st Property.

If W_k, W_k' are isodynamic centers of k^{th} rank, then:

$W_kA.BC^k = W_kB.AC^k = W_kC.AB^k$;

$W_k'A.BC^k = W_k'B.AC^k = W_k'C.AB^k$.

The proof of this property follows immediately from *1st Theorem*.

2nd Remark.

The Apollonius's circle of 1st rank is the investigated Apollonius's circle (the bisectors are cevians of 1st rank). If $k=2$, the internal cevians of 2nd rank are the symmedians, and the external cevians of 2nd rank are the external symmedians, i.e. the tangents in triangle's vertices to the circumscribed circle. In this case, for the Apollonius's circle of 2nd rank, the *3rd Theorem* becomes:

7th Theorem.

The Apollonius's circle of 2nd rank intersects the circumscribed circle to the triangle in two points belonging respectively to the antibisector's isogonal and to the cevian outside of it.

Proof.

It follows from the proof of the 6th theorem. We mention that the antibisector is isotomic to the bisector, and a cevian of 3rd rank is isogonic to the antibisector.

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