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Solving Problems by Using a Function in The Number Theory

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SOME LINEAR EQUATIONS INVOLVING A FUNCTION IN THE NUMBER THEORY

We have constructed a function η which associates to each non-null integer m the smallest positive n such that $n!$ is a multiple of m.

(a) Solve the equation $\eta(x) = n$, where $n \in N$.

*(b) Solve the equation $\eta(mx) = x$, where $m \in Z$.

Discussion.

(c) Let $\eta^{(i)}$ denote $\eta \circ \eta \circ \ldots \circ \eta$ of i times. Prove that there is a k for which

$$\eta^{(k)}(m) = \eta^{(k+1)}(m) = n_m, \text{ for all } m \in Z^* \setminus \{1\}.$$

**Find n_m and the smallest k with this property.

Solution

(a) The cases $n = 0, 1$ are trivial.

We note the increasing sequence of primes less or equal than n by $P_1, P_2, \ldots, P_k$, and

$$\beta_t = \sum_{h \geq 1} [n/p_t^h], t = 1, 2, \ldots, k;$$

where $[y]$ is greatest integer less or equal than y.

Let $n = p_{i_1}^{\alpha_{i_1}} \ldots p_{i_s}^{\alpha_{i_s}}$, where all p_{i_j} are distinct primes and all α_{i_j} are from N.

Of course we have $n \leq x \leq n!$

Thus $x = p_1^{\sigma_1} \ldots p_k^{\sigma_k}$ where $0 \leq \sigma_t \leq \beta_t$ for all $t = 1, 2, \ldots, k$ and there exists at least a $j \in \{1, 2, \ldots, s\}$ for which

$$\sigma_{i_j} \in \beta_{ij}, \ \{\beta_{i_j}^{-1}, \ldots, \beta_{i_j} - \alpha_{i_j} + 1\}.$$

Clearly $n!$ is a multiple of x, and is the smallest one.

(b) See [1] too. We consider $m \in N^*$.

Lemma 1. *$\eta(m) \leq m$, and $\eta(m) = m$ if and only if $m = 4$ or m is a prime.*

Of course $m!$ is a multiple of m.

If $m \neq 4$ and m is not a prime, the Lemma is equivalent to there are m_1, m_2 such that $m = m_1 \cdot m_2$ with $1 < m_1 \leq m_2$ and ($2m_2 < m$ or $2m_1 < m$). Whence $\eta(m) \leq 2m_2 < m$, respectively $\eta(m) \leq \max\{m_2, 2m\} < m$.

Lemma 2. *Let p be a prime ≤ 5. Then $= \eta(px) = x$ if and only if x is a prime $> p$, or $x = 2p$.*

Proof: $\eta(p) = p$. Hence $x > p$.

Analogously: x is not a prime and $x \neq 2p \Leftrightarrow x = x_1x_2, 1 < x_1 \leq x_2$ and $(2x_2 < x_1, x_2 \neq p_1$, and $2x_1 < x) \Leftrightarrow \eta(px) \leq \max\{p, 2x_2\} < x$ respectively $\eta(px) \leq \max\{p, 2x_1, x_2\} < x$.

Observations

$$\eta(2x) = x \Leftarrow x = 4 \text{ or } x \text{ is an odd prime.}$$

$$\eta(3x) = x \Leftrightarrow x = 4, 6, 9 \text{ or } x \text{ is a prime} > 3.$$

Lemma 3. *If $(m, x) = 1$ then x is a prime $> \eta(m)$.*

Of course, $\eta(mx) = \max\{\eta(m), \eta(x)\} = \eta(x) = x$. And $x \neq \eta(m)$, because if $x = \eta(m)$ then $m \cdot \eta(m)$ divides $\eta(m)!$ that is m divides $(\eta(m)-1)!$ whence $\eta(m) \leq \eta(m) - 1$.

Lemma 4. *If x is not a prime then $\eta(m) < x \leq 2\eta(m)$ and $x = 2\eta(m)$ if and only if $\eta(m)$ is a prime.*

Proof: If $x > 2\eta(m)$ there are x_1, x_2 with $1 < x_1 \leq x_2, x = x_1x_2$. For $x_1 < \eta(m)$ we have $(x-1)!$ is a multiple of mx. Same proof for other cases.

Let $x = 2\eta(m)$; if $\eta(m)$ is nopt a prime, then $x = 2ab, 1 < a \leq b$, but the product $(\eta(m)+1)(\eta(m)+2)\dots(2\eta(m)-1)$ is divided by x.

If $\eta(m)$ is a prime, $\eta(m)$ divides m, whence $m \cdot 2\eta(m)$ is divided by $\eta(m)^2$, it results in $\eta(m \cdot 2\eta(m)) \geq 2 \cdot \eta(m)$, but $(\eta(m)+1)(\eta(m)+2)\dots(2\eta(m))$ is a multiple of $2\eta(m)$, that is $\eta(m \cdot 2\eta(m)) = 2\eta(m)$.

Conclusion.

All x, prime number $> \eta(m)$, are solutions.

If $\eta(m)$ is prime, then $x = 2\eta(m)$ is a solution.

*If x is not a prime, $\eta(m) < x < 2\eta(m)$, and x does not divide $(x-1)!/m$ then x is a solution (semi-open question). If $m = 3$ it adds $x = 9$ too. (No other solution exists yet.)

(c)

Lemma 5. $\eta(ab) \leq \eta(a) + \eta(b)$.

Of course, $\eta(a) = a'$ and $\eta(b) = b'$ involves $(a'+b')! = b'!(b'+1\dots(b'+a')$. Let $a' \leq b'$. Then $\eta(ab) \leq a' + b'$, because the product of a' consecutive positive integers is a multiple of $a'!$

Clearly, if m is a prime then $k = 1$ and $n_m = m$.

If m is not a prime then $\eta(m) < m$, whence there is a k for which $\eta^{(k)}(m) = \eta^{(k+1)}(m)$.

If $m \neq 1$ then $2 \leq n_m \leq m$.

Lemma 6. *$n_m = 4$ or n_m is a prime.*

If $n_m = n_1 n_2, 1 < n_1 \leq n_2$, then $\eta(n_m) < n_m$. Absurd. $n_m \neq 4$.

(**) This question remains open.

References

[1] F.Smarandache, A Function in the Number Theory, An. Univ. Timisoara, seria st. mat., Vol. XVIII, fasc. 1, pp.79-88, 1980; Mathematical Reviews: 83c:10008.

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