Superattractive Fixed-points of the Hardy Z Function and the Riemann Hypothesis

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Abstract

An assertion about the superaattractiveness of the fixed-points of the Newton map of the Hardy Z function is shown to be equivalent to the Riemann Hypothesis.

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1 Derivations

1.1 Preliminaries

Let $\zeta(t)$ be the Riemann zeta function

$$\begin{aligned} \zeta(t) &= \sum_{n=1}^{\infty} n^{-s} && \forall \mathrm{Re}(s) > 1 \\ &= (1 - 2^{1-s}) \sum_{n=1}^{\infty} n^{-s} (-1)^{n-1} && \forall \mathrm{Re}(s) > 0 \end{aligned} \tag{1}$$

and $\vartheta(t)$ be Riemann-Siegel vartheta function $\vartheta(t)$

$$\vartheta(t) = -\frac{i}{2}\left(\ln\Gamma\left(\frac{1}{4} + \frac{i\,t}{2}\right) - \ln\Gamma\left(\frac{1}{4} - \frac{i\,t}{2}\right)\right) - \frac{\ln(\pi)\,t}{2} \tag{2}$$

The Hardy Z function[Ivi13] can then be written as

$$Z(t) = e^{i\vartheta(t)}\,\zeta\left(\frac{1}{2} + i\,t\right) \tag{3}$$

which can be mapped isometrically back to the ζ function

$$\zeta(t) = e^{-i\vartheta\left(\frac{i}{2} - i\,t\right)} Z\left(\frac{i}{2} - i\,t\right) \tag{4}$$

due to the isometry

$$t = \frac{1}{2} + i\left(\frac{i}{2} - i\,t\right) \tag{5}$$

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of the Mobius transforms[2] $f(t)=\frac{at+b}{ct+d}$ with

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -i & \frac{i}{2} \\ 0 & 1 \end{pmatrix} \text{and its inverse} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} i & \frac{1}{2} \\ 0 & 1 \end{pmatrix} \tag{6}$$

making possible the Riemann-Siegel-Hardy correspondence. The Bäcklund counting formula gives the exact number of zeros on the critical strip up to level t, not just on the criticial line $\mathrm{Re}(t)=\frac{1}{2}$, [Bor08, 3.2]

$$N(t) = \#\{\zeta(x+iy)=0\colon 0\leqslant x\leqslant 1, 0\leqslant y\leqslant t\} = \mathrm{Im}(R(t)) = \langle N(t)\rangle + S(t) \tag{7}$$

where $\langle N(t)\rangle$ is the smooth part of the counting function

$$\langle N(t)\rangle = \pi^{-1}\vartheta(t)+1 \tag{8}$$

and $S(t)$ is normalised phase of ζ at the point t

$$\begin{aligned} S(t) &= \pi^{-1}\arg\left(\zeta\left(\frac{1}{2}+it\right)\right) \\ &= -\frac{i}{2\pi}\left(\ln\zeta\left(\frac{1}{2}+it\right) - \ln\zeta\left(\frac{1}{2}-it\right)\right) \\ &= \frac{1}{\pi}\lim_{\varepsilon\to 0}\mathrm{Im}\left(\ln\zeta\left(\frac{1}{2}+it+\varepsilon\right)\right) \end{aligned} \tag{9}$$

[FL14] The relationship between the functions $N(t)$, $S(t)$, and $Z(t)$ is demonstrated by

$$\ln\zeta\left(\frac{1}{2}+it\right) = \ln|Z(t)| + i\pi S(t) \tag{10}$$

These formulas are true independent of the Riemann hypothesis which posits that all complex zeros of $\zeta(s+it)$ have real part $s=\frac{1}{2}$. [Ivi13, Corrollary 1.8 p.13]

1.2 Complex Holomorphic Dynamics and the Riemann Hypothesis

Theorem 1. *(Kawahira's Theorem) In [Kaw08][Kaw16] it is proved that the Riemann hypothesis is equivalent to the statement that the Newton map N_ξ of the Riemann ξ function has no attractive fixed points*

1.3 The Newton Map $N_Z(t)$ of $Z(t)$

The Newton map [Ber93, 6.1][Bro04] of $Z(t)$ is defined by

$$\begin{aligned} N_Z(t) &= t - \frac{Z(t)}{\dot{Z}(t)} \\ &= t + \frac{e^{i\vartheta(t)}\,\zeta\left(\frac{1}{2}+it\right)}{\dot{\zeta}\left(\frac{1}{2}+it\right)\dot{\vartheta}(t)e^{i\vartheta(t)}} \end{aligned} \tag{11}$$

Proposition 2. *If the Newton map $N_Z(t)$ from Definiton (?) is iterated from the initial position given by the Franca-Leclaire formula for the approximate location of the n-th Riemann zero, $\tilde{y}_n$ then its iterates always converge to the n-th Riemann zero at y_n, rather than diverging or converging to any other zero $\{y_m\colon m\neq n\}$. That is,*

$$\lim_{m\to\infty} N_G^{(m)}(\tilde{y}_n) = y_n \tag{12}$$

2. Thanks to Matti Pitkänen for pointing out this is a Mobius transform pair, among other things

In other words, $\tilde{y}_n$ *lies within the Newton map* N_G*'s immediate basin of attraction around the* n*-th Riemann zero at the point* y_n*. TODO: make this statement more precise. [MS06]*

Proposition 3. *The Newton map* $N_Z(t)$ *of* $Z(t)$ *is maximally flat in a neighborhood of its superattractive fixed points which are separated by poles which repel trajectories away from the points of maximum curvature between the fixed points and towards those ultimately leading back to some point of minimal curvature.*

Proposition 4. *An alternative version of Theorem 1 is that the all of the fixed points of* $N_Z(t)$ *are superattractive. That is, they have multiplier 0.*

Definition 5. *The multiplier of* f *at the fixed-point* α *is the derivative of the Newton map* $N_f(t)$ *evaluated at* α, $N_f(t)|_{t=\alpha}$

Definition 6. *A point* $z_0 \in \mathbb{C}$ *is called a periodic point of* f *if* $f^n(z_0) = z_0$ *for some* $n \in \mathbb{N}$*.[Ber93, 3.1]*

Definition 7. *A family* (f_k) *of holomorphic maps* $U \to \bar{\mathbb{C}}$ *where* $U \subset \mathbb{C}$ *is a domain is called a normal family if every sequence* (f_k) *contains a subsequence that converges locally uniformly to a holomorphic limit function* $f: U \to \bar{\mathbb{C}}$*. [SSC+10, Definition A.1]*

Definition 8. *The Fatou set is defined by*

$$F = \{z \in \hat{\mathbb{C}}: \{f^n(z): n \in \mathbb{N}\} \text{ is defined and is } a \text{ normal family in some neighborhood of } z\} \tag{13}$$

Definition 9. *The Julia set is defined as the complement of the Fatou set*

$$J = \hat{\mathbb{C}} \backslash F \tag{14}$$

Definition 10. *The approximate location of the* n*-th Riemann zero is given by the Franca-Leclair formula*

$$\tilde{y}_n = \frac{2\pi\left(n - \frac{11}{8}\right)}{W\left(\frac{n - \frac{11}{8}}{e}\right)} \tag{15}$$

where $W(x)$ is the Lambert W function defined as the solution to the equation $W(x)e^{W(x)} = x$.[FL15]

Corollary 11. *The Julia set of* N_G *contains the zeros of* $H(t) = G'(t)$ *interlacing the zeros along the critical strip.*

Corollary 12. *The (stable) Fatou set of* N_G *contains the Riemann zeros to which trajectories are stably attracted.*

[BFJK14][BFJK15]

Definition 13. *If* f *is a complex function defined in* Ω *and the derivative of* f *at* z_0 *defined by*

$$\dot{f}(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \forall z_0 \in \Omega \tag{16}$$

exists then f *is said to be holomorphic or analytic in* Ω*. The class of all holomorphic functions is denoted* $H(\Omega)$*. A function* f *is known as an entire function when* Ω *is the entire complex plane* $\mathbb{C}$*. [Rud06, Definition 10.2]*

Definition 14. *The open (Fatou) set* F_f *of a meromorphic function* f *is said to be completely invariant, that is,* $z \in F_f$ *if and only if* $f(z) \in F_f$*. [BKY91][Dom98]*

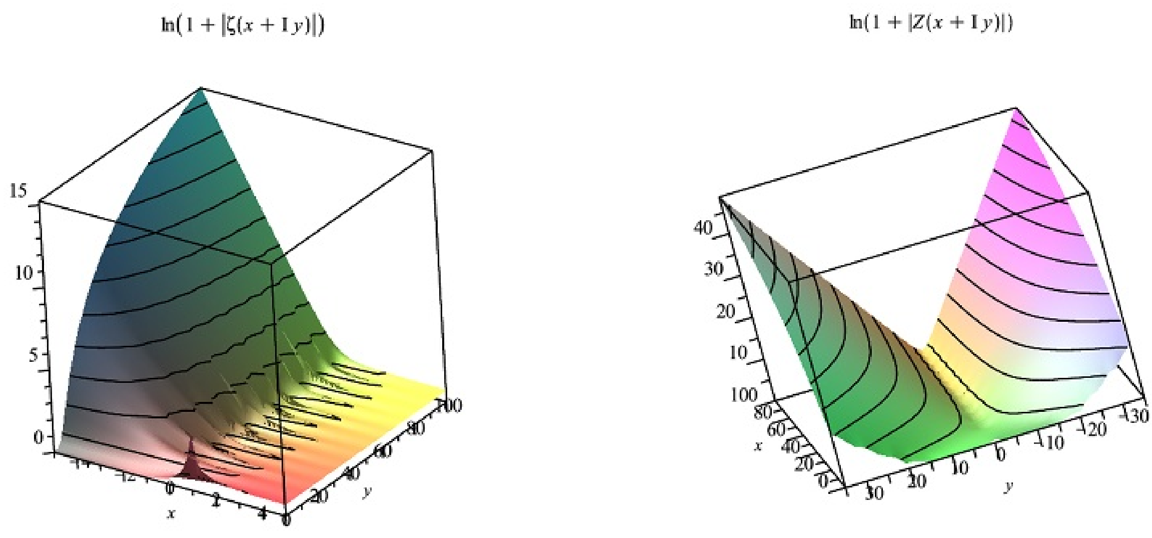

Figure 1. $\ln(1+|\zeta(x+iy)|)$ and $\ln(1+|Z(x+iy)|)$

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