# Superattractive Fixed-points of the Hardy Z Function and the Riemann Hypothesis

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An assertion about the superaattractiveness of the fixed-points of the Newton map of the Hardy Z function and their immediate basins is shown to be equivalent to the Riemann Hypothesis.
[1]

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# 1 Derivations

## 1.1 Preliminaries

Let $\zeta(t)$ be the Riemann zeta function

$$\begin{aligned} \zeta(t) &= \sum_{n=1}^{\infty} n^{-s} && \forall \mathrm{Re}(s) > 1 \\ &= (1 - 2^{1-s}) \sum_{n=1}^{\infty} n^{-s} (-1)^{n-1} && \forall \mathrm{Re}(s) > 0 \end{aligned} \tag{1}$$

and $\vartheta(t)$ be Riemann-Siegel vartheta function $\vartheta(t)$

$$\vartheta(t) = -\frac{i}{2}\left(\ln\Gamma\left(\frac{1}{4} + \frac{i t}{2}\right) - \ln\Gamma\left(\frac{1}{4} - \frac{i t}{2}\right)\right) - \frac{\ln(\pi)\, t}{2} \tag{2}$$

The exact equation for the $n$-th Riemann zero is given by [FL15a, Equation 20]

$$\vartheta(t) + S(t) = \left(n - \frac{3}{2}\right)\pi \tag{3}$$

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The Hardy $Z$ function[Ivi13] can then be written as

$$Z(t) \quad = e^{i\vartheta(t)}\zeta\left(\frac{1}{2}+it\right) \tag{4}$$

which can be mapped isometrically back to the $\zeta$ function

$$\zeta(t) = e^{-i\vartheta\left(\frac{i}{2}-it\right)}Z\left(\frac{i}{2}-it\right) \tag{5}$$

due to the isometry

$$t = \frac{1}{2}+i\left(\frac{i}{2}-it\right) \tag{6}$$

of the Mobius transforms $f(t) = \frac{at+b}{ct+d}$ with

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -i & \frac{i}{2} \\ 0 & 1 \end{pmatrix} \text{and its inverse} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} i & \frac{1}{2} \\ 0 & 1 \end{pmatrix} \tag{7}$$

making possible the Riemann-Siegel-Hardy correspondence. The Bäcklund counting formula gives the exact number of zeros on the critical strip up to level $t$, not just on the criticial line $\mathrm{Re}(t) = \frac{1}{2}$, [Bor08, 3.2]

$$N(t) = \quad \#\{\zeta(x+iy) = 0 \colon 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant t\} = \langle N(t)\rangle + S(t) \tag{8}$$

where $\langle N(t)\rangle$ is the smooth part of the counting function

$$\langle N(t)\rangle = \pi^{-1}\vartheta(t)+1 \tag{9}$$

and $S(t)$ is normalised phase of $\zeta$ at the point $t$

$$\begin{aligned} S(t) \quad &= \pi^{-1}\arg\left(\zeta\left(\frac{1}{2}+it\right)\right) \\ &= -\frac{i}{2\pi}\left(\ln\zeta\left(\frac{1}{2}+it\right) - \ln\zeta\left(\frac{1}{2}-it\right)\right) \\ &= \frac{1}{\pi}\lim_{\varepsilon\to 0}\mathrm{Im}\left(\ln\zeta\left(\frac{1}{2}+it+\varepsilon\right)\right) \end{aligned} \tag{10}$$

[FL14] The relationship between the functions $N(t)$, $S(t)$, and $Z(t)$ is demonstrated by

$$\ln\zeta\left(\frac{1}{2}+it\right) = \ln|Z(t)| + i\pi S(t) \tag{11}$$

These formulas are true independent of the Riemann hypothesis which posits that all complex zeros of $\zeta(s+it)$ have real part $s = \frac{1}{2}$. [Ivi13, Corrollary 1.8 p.13]

## 1.2 The Newton Map $N_Z(t)$ of $Z(t)$

The Hardy Z function is a meromorphic function with a pole at $t = -\frac{i}{2}$ which corresponds to the simple pole at $\zeta(1)$. Let the new Newton map [93, 6.1][Bro04] of $Z(t)$ be defined by

$$\begin{aligned} N_Z(t) \quad &= t - \frac{Z(t)}{\dot{Z}(t)} \\ &= t + \frac{e^{i\vartheta(t)}\zeta\left(\frac{1}{2}+it\right)}{\dot{\zeta}\left(\frac{1}{2}+it\right)\dot{\vartheta}(t)e^{i\vartheta(t)}} \end{aligned} \tag{12}$$

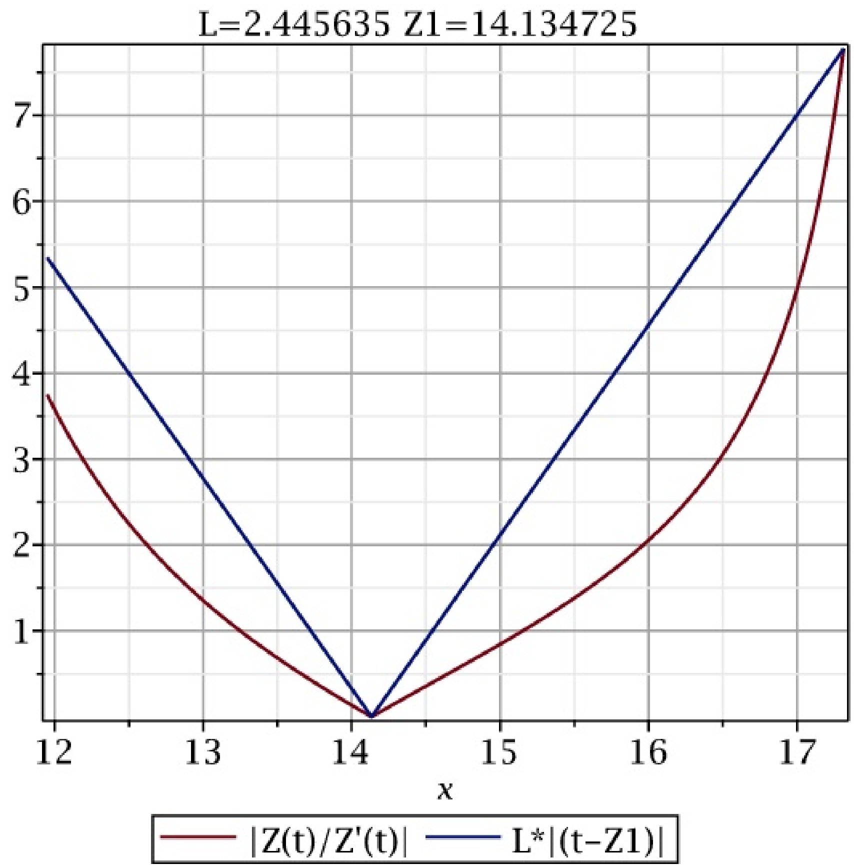


**Figure 1.** $Z1 = y1 = 14.134725$...is the first root of $Z$ on the real-line

**Definition 1.** *If $f$ is a complex function defined in $\Omega$ and the derivative of $f$ at $z_0$ defined by*

$$\dot{f}(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \forall z_0 \in \Omega \tag{13}$$

*exists then $f$ is said to be holomorphic or analytic in $\Omega$. The class of all holomorphic functions is denoted $H(\Omega)$. A function $f$ is known as an entire function when $\Omega$ is the entire complex plane $\mathbb{C}$. [Rud06, Definition 10.2]*

**Proposition 2.** *The Newton map $N_Z(t)$ of $Z(t)$ is maximally flat in a neighborhood of its superattractive fixed points which are separated by poles which repel trajectories away from the points of maximum curvature between the fixed points and towards those ultimately leading back to some point of minimal curvature.*

**Definition 3.** *The* `multiplier` *of $F$ at the fixed-point $\alpha$ is the derivative of the Newton map $N_F(t)$ evaluated at $\alpha$, $N_f(t)|_{t=\alpha}$ which can be written*

$$\dot{N}_F(t) = \frac{F(t)\ddot{F}(t)}{\dot{F}(t)^2} \tag{14}$$

**Definition 4.** *A point $z_0 \in \mathbb{C}$ is called a periodic point of $f$ if $f^n(z_0) = z_0$ for some $n \in \mathbb{N}$. A fixed-point is a 1-periodic point. [93, 3.1]*

**Definition 5.** *A family $(f_k)$ of holomorphic maps $U \to \bar{\mathbb{C}}$ where $U \subset \mathbb{C}$ is a domain is called a normal family if every sequence $(f_k)$ contains a subsequence that converges locally uniformly to a holomorphic limit function $f: U \to \bar{\mathbb{C}}$. [Mil06, 1.3 p30]*

**Definition 6.** *The (stable) Fatou set $F(f(z))$ of a function $f:\mathbb{C}\to\mathbb{C}$ is defined by*

$$F(f(z))=\{z\in\hat{\mathbb{C}}:\{f^{\circ n}(z):n\in\mathbb{N}\}\text{ is defined and constitutes }a\text{ normal family in some neighborhood of }z\} \tag{15}$$

*which is the set of points for which $\lim_{n\to\infty}f^{\circ n}(z)$ converges to a fixed-point where $f^{\circ n}(z)$ is composition of $f(x)$ with itself $n$ times, e.g. $f^{\circ 3}(z)=f(f(f(z)))$. A Fatou set $F_f$ of a meromorphic function $f$ is said to be completely invariant, that is, $z\in F_f$ if and only if $f(z)\in F_f$. [BKY][Dom98]*

**Definition 7.** *The (chaotic) Julia set $J(f(z))$ of a function $f:\mathbb{C}\to\mathbb{C}$ is defined as the complement of its Fatou set*

$$J(f(z))=\hat{\mathbb{C}}\backslash F(f(z)) \tag{16}$$

*which is the set of points for which iterated self-maps do not converge.*

### 1.2.1 The LeClaire-Guilherme Formula and Stirling's Approximation of lnΓ

**Definition 8.** *The countably infinite set of exact solutions of the approximation equations*

$$\frac{\tilde{y}_n}{2}\ln\left(\frac{\tilde{y}_n}{2\pi e}\right)-\frac{\pi}{8}+O\left(\frac{1}{t}+|S(t)|\right)=\left(n-\frac{3}{2}\right)\pi \tag{17}$$

*where the phase of $S(t)$ has been replaced with known bounds*

$$O\left(\frac{1}{t}+|S(t)|\right)=O(t^{-1}+\min(1.998+0.17\ln(t),2.45+0.111\ln(t)+0.275\ln(\ln(t)))) \tag{18}$$

*from [Tru11][Tru14] is given by the LeClaire-Guilherme formula which expressed the solutions in closed-form*

$$\tilde{y}_n=\frac{2\pi\left(n-\frac{11}{8}\right)}{W\left(\frac{n-\frac{11}{8}}{e}\right)} \tag{19}$$

*where $W(x)$ is the Lambert W function defined as the solution to the equation $W(x)e^{W(x)}=x$.[FL15b] The formula is derived by expressing the inverse of Stirling's expansion of $\ln\Gamma$ as a Lambert W function*

$$\tilde{\vartheta}(t)=\frac{t}{2}\ln\left(\frac{t}{2\pi e}\right)-\frac{\pi}{8}+O(t^{-1}) \tag{20}$$

*and dropping the $S(t)$ term in the exact equation*

$$\vartheta(y_n)+S(t)=\left(n-\frac{3}{2}\right)\pi \tag{21}$$

*and replacing $\ln\Gamma$ with its Stirling series to get*

$$\tilde{\vartheta}(y_n)=\left(n-\frac{3}{2}\right)\pi \tag{22}$$

*whose solution gives the $n$-th approximation zero*

$$\tilde{y}_n=\frac{2\pi\left(n-\frac{11}{8}\right)}{W\left(\frac{n-\frac{11}{8}}{e}\right)} \tag{23}$$

*The formula can be made more accurate by including the $S(t)$ term where the solution is given by*

$$\tilde{y}_n(p)=\frac{8\pi n-11\pi-8p}{4W\left(\frac{8\pi n-11\pi-8p}{8\pi e}\right)} \tag{24}$$

*with $p=S(y_n)$*

*however, that defeats the initial purpose of the formula which is to make a tractable approximation and care must be taken when calculating $S(y_n)$ since the phase $S(t)$ is discontinuous precisely at the roots. TODO: try replacing the phase $S(t)$ with its 1st or 2nd order approximation*

**Proposition 9.** *If the Newton map $N_Z(t)$ from Definiton (12) is iterated from the initial point given by the Franca-Leclaire formula for the approximate location of the $n$-th Riemann zero, $\tilde{y}_n$ and if its iterates form a normal family convergeing to the $n$-th Riemann zero at $y_n$, rather than diverging (showing that $\tilde{y}_n$ is contained in the Julia set for some $n$) or converging to any other zero $\{y_m : m \neq n\}$, then is is true that all non-trivial roots of $Z(t)$ are real and thus all non-trivial roots of $\zeta(s)$ have real-part $\frac{1}{2}$, e.g., the Riemann Hypothes would be true. That is, if*

$$\lim_{m\to\infty} N_Z^{(m)}(\tilde{y}_n) = y_n \forall \mathbb{N} \ni n \geqslant 1 \tag{25}$$

*In other words, $\tilde{y}_n$ lies within the Newton map $N_Z$'s immediate basin of attraction around the $n$-th Riemann zero at the point $y_n$. [MS06]*

#### 1.2.2 Convergence of Newton's Method, Lipschitz Functions, and The Mean Value Theorem

[Wan00][V.02][Pol06][Han79][CB80, Ch3p.72] See [Gra46, VIII.2] on the topic of solutions which are implicitly defined near an initial solution.

**Theorem 10.** `Theorem of the Mean`. *Suppose $f$ is continuous on the finite closed interval $[a, b]$ and has a derivative either finite or infinite at each point of the open interval $(a, b)$. Then there exists a point $c$ in the open interval $(a, b)$ such that*

$$f(b) = f(a) + f'(c)(b - a) \tag{26}$$

*[Gra46, V.I Theorem 4]*

**Definition 11.** *A function between normed spaces with the property that the distance between function values is bounded by a constant multiple of the distance between the arguments is known as a Lipschitz function. If the function satisfies the Lipschitz condition that*

$$\|f(x) - f(y)\| \leqslant L\|x - y\| \forall x, y \in A \tag{27}$$

*where possibly $A$ is a single point $A = \{x_0\}$ then $f(x)$ is said to be $k$-Lipschitz on $A$ or at $x_0$. When $k = 1$ the function is a non-expansive mapping, and is a contraction mapping when $k < 1$. [Map]*

A consequence of Theorem 10 is that the best Lipschitz constant for a derivable function on an interval is equal to the uniform norm of its derivative. As in [AH12], define the sequence of best possible Lipschitz constants over a line across the real part of the $n$-th Fatou domain

$$L_n = \frac{\eta_n}{\max\left(|y_n^{\leftarrow} - t|, |y_n^{\rightarrow} - t|\right)} \tag{28}$$

where

$$\eta_n = \sup_{t \in [y_n^{\leftarrow}, y_n^{\rightarrow}]} \left| \frac{Z(t)}{\dot{Z}(t)} \right| \tag{29}$$

where

$$y_n^{\leftarrow} = \inf_{t < y_n} \lim_{m\to\infty} N_Z^{(m)}(t) = y_n \tag{30}$$

and

$$y_n^{\rightarrow} = \sup_{t > y_n} \lim_{m\to\infty} N_Z^{(m)}(t) = y_n \tag{31}$$

Intervals where iterations of $N_Z(t)$ converge are given in the following table

| $n$ | $y_n^{\leftarrow}$ | $\tilde{y}_n$ | $y_n$ | $\tilde{y}_n - y_n$ | $y_n^{\rightarrow}$ | $\eta$ | $L_n$ |
|---|---|---|---|---|---|---|---|
| 1 | 11.949525141 | 14.52134696 | 14.13472514 | 0.38662182 | 17.313725141 | 7.774674466 | 2.44563525 |

(32)

**Table 1.** Parameters associated with each root

Proving that $\dot{N}_Z(y_n) = 0$ where $y_n$ is enumerated by the fixed-point recursion initialized by the $n$-th approximation zero would prove that all of these roots are simple since by definition the Tayor expansion about the root has the form

$$Z_n(t) = L_n(t - y_n)^{m_n} \tag{33}$$

where $m_n$ is the multiplicity of the $n$-th root

$$m_n = 1 - \frac{1}{1 - \dot{N}(y_n)} \tag{34}$$

[Mil06, Problem 4-g]

TODO: define bounds of the Leclare-approximation formula and compare with bounds for immediate basins of stable (super)attractors.

[MS06, 2.3 Theorem 2.8 p.330]

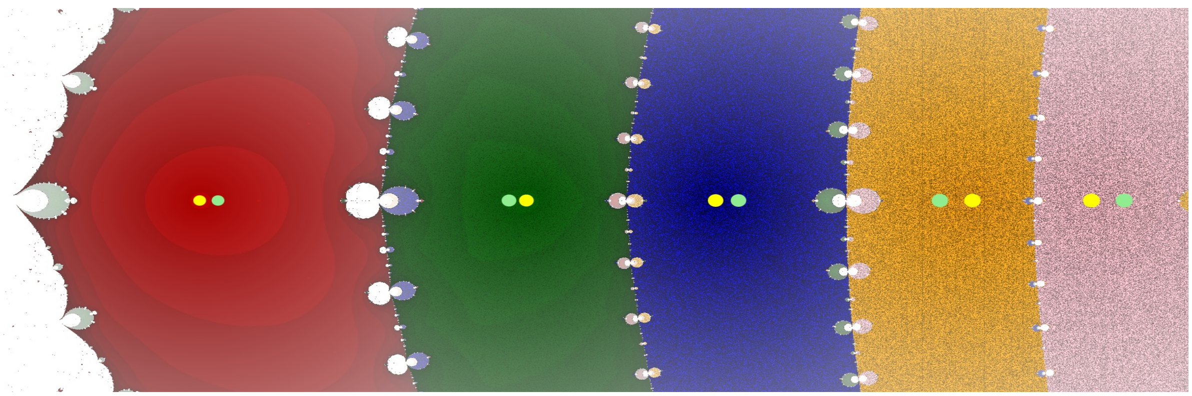

**Figure 2.** The colors represent the index of the root the points are attracted to under iteration of $N_Z(t)$. The graininess is an artifact of the floating point precision used in the implementation of the computations https://bitbucket.org/stephenc214/fastmath/src. White points are in the Julia set. The intensity of the color indicates the number of iterates required to converge. Yellow circles are drawn around the fixed point whose radius is Equation (?), green points are drawn with the same radius but centered around the Lambert W function approximation of the $n$-th zero.

### 1.2.3 Gram Points are in the Julia Set of $N_Z$

**Proposition 12.** *The approximate Gram points*

$$\tilde{g}_n = \frac{(8n-7)\pi}{4W\left(\frac{8n-7}{8e}\right)} + O\left(\frac{1}{\tilde{g}_n}\right) \tag{35}$$

*are solutions to*

$$\tilde{\vartheta}(\tilde{g}_n) = (n-1)\pi \tag{36}$$

*which are approximations of the exact equations*

$$\vartheta(g_n) = (n-1)\pi \tag{37}$$

*whose solutions $g_n$ lie in the Julia sets which seperate the neighboring basins of attraction.*

### 1.2.4 Lehmer's Phenomena

D. H. Lehmer discovered cases where the Riemann zeta function has zeros that are "only just" on the critical line: two zeros of the zeta function are so close together that it is unusually difficult to find a sign change between them. This is called "Lehmer's phenomenon", and first occurs at the zeros with imaginary parts 7005.063 and 7005.101, which differ by only .04 while the average gap between other zeros near this point is about 1. The discovery of non-simple roots, or the discovery of a local positive minimum, or a local negative maximum, implies a violation of the Riemann hypothesis. [Edw74, 8.3] TODO: plot something like Figure 2 in the region 7004 to 7006 and also compute

### 1.2.5 An Entire Conformal Transform of the Hardy Z Function

The Hardy Z functions is meromorphic, with a pole at $-\frac{i}{2}$, since the theory of iterated entire holomorphic functions is more developed than that of meromorphic functions, simply define the auxilary function which has the simple pole at $Z\left(-\frac{i}{2}\right)$ coming from $\zeta\left(\frac{1}{2}+i\left(-\frac{i}{2}\right)=1\right)=\infty$ by

$$Y(t)=Z(t)\left(t+\frac{i}{2}\right) \tag{38}$$

which makes the singularity removable by taking the limit at the point $-\frac{i}{2}$

$$\lim_{t\to-\frac{i}{2}} Y(t)=0 \tag{39}$$

so that it is transformed to an entire holomorphic function. [05][Sch10] It might be possible to work with $Z(t)$ directly but the pole at $t=-\frac{i}{2}$ makes the analysis a bit more complicated. [93, 6.1] The Newton map of the entire function $Y(t)$ is then given by

$$N_Y(t)=t-\frac{Y(t)}{\dot{Y}(t)}=t-\frac{Z(t)\left(t+\frac{i}{2}\right)}{Z(t)+\dot{Z}(t)\left(t+\frac{i}{2}\right)} \tag{40}$$

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