# Superattractive Fixed-points of a Newton-like Iterative Mappping of the Hardy Z Function and the Riemann Hypothesis

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There is shown to exist a unique solution to the LeClaire-França exact equation for the $n$-th Riemann zero via the construction of Cauchy sequences whose accumulation points are the zeros of the Hardy Z function thus proving the Riemann hypothesis.
[1]

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# 1 Derivations

## 1.1 Preliminaries

Let $\zeta(t)$ be the Riemann zeta function

$$\begin{aligned} \zeta(t) &= \sum_{n=1}^{\infty} n^{-s} && \forall \mathrm{Re}(s) > 1 \\ &= (1-2^{1-s}) \sum_{n=1}^{\infty} n^{-s} (-1)^{n-1} && \forall \mathrm{Re}(s) > 0 \end{aligned} \tag{1}$$

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and $\vartheta(t)$ be Riemann-Siegel vartheta function $\vartheta(t)$

$$\vartheta(t) = -\frac{i}{2}\left(\ln\Gamma\left(\frac{1}{4}+\frac{i t}{2}\right) - \ln\Gamma\left(\frac{1}{4}-\frac{i t}{2}\right)\right) - \frac{\ln(\pi)\, t}{2} \tag{2}$$

The exact equation for the $n$-th Riemann zero is given by [FL15, Equation 20]

$$\vartheta(t_n) + S(t_n) = \left(n - \frac{3}{2}\right)\pi \tag{3}$$

therefore the normalized argument of $\zeta$ is given by

$$\begin{aligned} S(t) &= \pi^{-1}\arg\left(\zeta\left(\frac{1}{2}+i t\right)\right) \\ &= -\frac{i}{2\pi}\left(\ln\zeta\left(\frac{1}{2}+i t\right) - \ln\zeta\left(\frac{1}{2}-i t\right)\right) \\ &= \frac{1}{\pi}\lim_{\varepsilon\to 0}\operatorname{Im}\left(\ln\zeta\left(\frac{1}{2}+i t+\varepsilon\right)\right) \end{aligned} \tag{4}$$

which is discontinuous at the the $n$-th zero where it is equal to

$$S_n = S(y_n) = \left(n - \frac{3}{2}\right)\pi - \vartheta(y_n) \tag{5}$$

which is approximated by

$$\tilde{S}_n = \tilde{S}(t_n) = \left(n - \frac{3}{2}\right)\pi - \tilde{\vartheta}(\tilde{y}_n) = S(t_n) + O(t_n^{-1}) \tag{6}$$

The Hardy $Z$ function[Ivi13] can then be written as

$$Z(t) = e^{i\vartheta(t)}\,\zeta\left(\frac{1}{2}+i t\right) \tag{7}$$

which can be mapped isometrically back to the $\zeta$ function

$$\zeta(t) = e^{-i\vartheta\left(\frac{i}{2}-i t\right)} Z\left(\frac{i}{2}-i t\right) \tag{8}$$

due to the isometry

$$t = \frac{1}{2} + i\left(\frac{i}{2} - i t\right) \tag{9}$$

of the Mobius transforms $f(t) = \frac{a t + b}{c t + d}$ with

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -i & \frac{i}{2} \\ 0 & 1 \end{pmatrix} \text{and its inverse} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} i & \frac{1}{2} \\ 0 & 1 \end{pmatrix} \tag{10}$$

making possible the Riemann-Siegel-Hardy correspondence. The Bäcklund counting formula gives the exact number of zeros on the critical strip up to level $t$, not just on the criticial line $\operatorname{Re}(t) = \frac{1}{2}$, [Bor08, 3.2]

$$N(t) = \#\{\zeta(x+i y) = 0 : 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant t\} = \langle N(t)\rangle + S(t) \tag{11}$$

where $\langle N(t)\rangle$ is the smooth part of the counting function

$$\langle N(t)\rangle = \pi^{-1}\vartheta(t) + 1 \tag{12}$$

[FL14] The relationship between the functions $N(t)$, $S(t)$, and $Z(t)$ is demonstrated by

$$\ln\zeta\left(\frac{1}{2}+it\right)=\ln|Z(t)|+i\pi S(t) \tag{13}$$

These formulas are true independent of the Riemann hypothesis which posits that all complex zeros of $\zeta(s+it)$ have real part $s=\frac{1}{2}$. [Ivi13, Corrollary 1.8 p.13]

## 1.2 The Newton Map $N_Z(t)$ of $Z(t)$

Let the new Newton map [93, 6.1][Bro04] of $Z(t)$ be defined by the meromorphic function

$$\begin{aligned} N_Z(t) &= t-\frac{Z(t)}{\dot{Z}(t)} \\ &= t+\frac{e^{i\vartheta(t)}\,\zeta\left(\frac{1}{2}+it\right)}{\dot{\zeta}\left(\frac{1}{2}+it\right)\dot{\vartheta}(t)e^{i\vartheta(t)}} \end{aligned} \tag{14}$$

which has poles at the zeros of $\dot{Z}(s)$.

**Proposition 1.** *The Newton map $N_Z(t)$ of $Z(t)$ is maximally flat in a neighborhood of its superattractive fixed points which are separated by poles which repel trajectories away from the points of maximum curvature between the fixed points and towards those ultimately leading back to some point of minimal curvature.*

### 1.2.1 The LeClair-França Formula and Stirling's Approximation of lnΓ

**Definition 2.** *The exact equation for the Riemann Zeros is*

$$\vartheta(y_n)+S(y_n)=\left(n-\frac{3}{2}\right)\pi \tag{15}$$

**Definition 3.** *The approximate equation for the n-th Riemann zero is*

$$\tilde{\vartheta}(\tilde{y}_n)=\left(n-\frac{3}{2}\right)\pi \tag{16}$$

*which is solved by*

$$\tilde{y}_n=t_n^{**}=\frac{2\pi\left(n-\frac{11}{8}\right)}{W\left(\frac{n-\frac{11}{8}}{e}\right)} \tag{17}$$

*where $W(x)$ is the Lambert W function defined as the solution to the equation $W(x)e^{W(x)}=x$.[FL15] The notation $t_n^{**}$ is the notation used in [MBM09] where this exact same function appeared. The approximate Gram points given by the exact solution of the first-order Stirling expansion of $\vartheta(t)$ in terms of the* LambertW *function (which is itself-defined in terms of a certain fixed-point iteration) take values that are very close to the true Gram points. In fact, this exact solution of the approximate equation gives an accuracy of $2.2\times10^{-3}$ for $n=0$ decreasing to $3.5\times10^{-4}$ by $n=10$. Let the approximate Riemann-Siegel vartheta function be defined*

$$\tilde{\vartheta}(t)=\frac{t}{2}\ln\left(\frac{t}{2\pi e}\right)-\frac{\pi}{8} \tag{18}$$

*which by definition*

$$\vartheta(t)=\tilde{\vartheta}(t)+\Delta(t) \tag{19}$$

*where*

$$\Delta(t)=\frac{t}{4}\log\left(1+\frac{1}{4t^2}\right)+\frac{1}{4}\arctan\left(\frac{1}{2t}\right)+\frac{t}{2}\int_0^\infty\frac{\lfloor u\rfloor-\frac{1}{2}}{(u+\frac{1}{4})^2+(\frac{t}{2})^2}\mathrm{d}u \tag{20}$$

*[Ivi13, Lemma 5.1] is conjectured to be $O(1)$ due to compelling evidence offered in [LeC16] where it is also conjectured that the suitably normalized distribution of $\delta y_n$ is a Gaussian distribution with mean 0 and variance $\sqrt{\frac{\pi}{32}}$.*

***Corollary* 4.** *The infinite integral in Equation (20) can be re-expressed as an infinite sum*

$$\begin{aligned}\int_0^\infty \frac{\lfloor u\rfloor-\frac{1}{2}}{(u+\frac{1}{4})^2+(\frac{t}{2})^2}\mathrm{d}u &= \sum_{n=0}^{\infty}\int_n^{n+1}\frac{\lfloor u\rfloor-\frac{1}{2}}{(u+\frac{1}{4})^2+(\frac{t}{2})^2}\mathrm{d}u\\ &= \sum_{n=0}^{\infty}-\frac{(2n-1)\left(\arctan(\frac{4n+1}{2t})-\arctan(\frac{4n+5}{2t})\right)}{t}\end{aligned} \tag{21}$$

so that

$$\begin{aligned}\Delta(t) &= \frac{t}{4}\log\left(1+\frac{1}{4t^2}\right)+\frac{1}{4}\arctan\left(\frac{1}{2t}\right)+\frac{t}{2}\int_0^\infty\frac{\lfloor u\rfloor-\frac{1}{2}}{(u+\frac{1}{4})^2+(\frac{t}{2})^2}\mathrm{d}u\\ &= \frac{t}{4}\log\left(1+\frac{1}{4t^2}\right)+\frac{1}{4}\arctan\left(\frac{1}{2t}\right)+\frac{t}{2}\sum_{n=0}^{\infty}-\frac{(2n-1)\left(\arctan(\frac{4n+1}{2t})-\arctan(\frac{4n+5}{2t})\right)}{t}\end{aligned} \tag{22}$$

since

$$\int_n^{n+1}\frac{\lfloor u\rfloor-\frac{1}{2}}{(u+\frac{1}{4})^2+(\frac{t}{2})^2}\mathrm{d}u = -\frac{(2n-1)\left(\arctan(\frac{4n+1}{2t})-\arctan(\frac{4n+5}{2t})\right)}{t} \tag{23}$$

which has a removable singularity at $t=0$ given by

$$\lim_{t\to 0}\int_n^{n+1}\frac{\lfloor u\rfloor-\frac{1}{2}}{(u+\frac{1}{4})^2+(\frac{t}{2})^2}\mathrm{d}u = \frac{16n-8}{16n^2+24n+5} = \frac{8(2n-1)}{16n^2+24n+5} \tag{24}$$

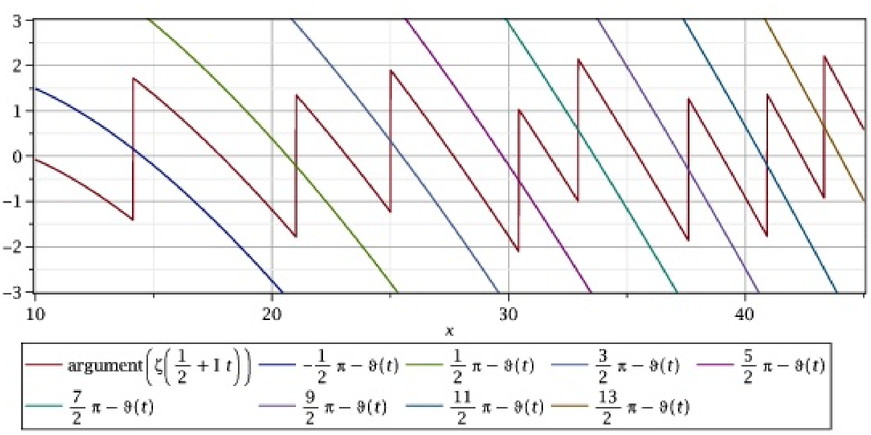


**Figure 1.** $S(t)$ and $\left\{\left(n-\frac{3}{2}\right)\pi-\vartheta(t)\colon n=1...8\right\}$

**Remark 5.** Even though convergence of the unrelaxed Newton's iteration is assured, it is not necessarily to the nearest zero $y_n$ corresponding to the initial approximation zero $\tilde{y}_n$ and instead in many instances converges to a zero several indices away from the matching zero $y_n$. A slight modification to $N_Z$ is introduced in the next section which has a much wider Fatou domains on which the Newton map iterates form normal families

#### 1.2.2 A Newton-like Map $M_Z(t)$ Which Is Holomorphic In The Critical Strip

**Definition 6.** *A function between normed spaces with the property that the distance between function values is bounded by a constant multiple of the distance between the arguments is known as a* Lipschitz function. *If the function satisfies the Lipschitz condition that*

$$\|f(x) - f(y)\| \leqslant L\|x - y\| \forall x, y \in A \tag{25}$$

*where possibly $A$ is a single point $A = \{x_0\}$ then $f(x)$ is said to be $k$-Lipschitz on $A$ or at $x_0$. When $k = 1$ the function is a* `non-expansive mapping`*, and is a* ***contraction mapping*** *when $k < 1$. [Map]*

**Definition 7.** *Let the modified Newton map [93, 6.1][Bro04] of $Z(t)$ be defined by the meromorphic function*

$$\begin{aligned} M_Z(t)_h &= t - h \cdot \tanh\left(\frac{Z(t)}{\dot{Z}(t)}\right) \\ &= t + h \cdot \tanh\left(\frac{e^{i\vartheta(t)}\,\zeta\left(\frac{1}{2} + it\right)}{\dot{\zeta}\left(\frac{1}{2} + it\right)\dot{\vartheta}(t) e^{i\vartheta(t)}}\right) \end{aligned} \tag{26}$$

*where the hypebolic tangent function $\tanh(z) = \frac{e^z e^z - 1}{e^z e^z + 1}$ maps $(-\infty, \infty) \to (-1, 1)$ and the parameter $h$ is the relaxation parameter. The $\tanh$ function has an infinite number of simple poles with residue 1 at $\frac{i}{2}\pi(2n + 1)$ for all integers $n$ and is also holomorphic when $|\mathrm{Im}(z)| < \frac{\pi}{2}$, that is, in the strip $\mathrm{Im}(z) \in \left(-\frac{1}{2}, \frac{1}{2}\right)$, that is, there are no poles in this region.*

**Remark 8.** The unrelaxed Newton's method fails to converge to its closest zero for the first time at $\tilde{y}_{126}$ which is also the point where $S(t)$ at this zero is not on the principal branch. The case of iterating real-valued $x = \tanh(x)$ in [KBM00, Proposition 2] is a much more compact and easy to understand proof and is easily generalized to complex valued iterates as well.

**Note 9.** The function $M_Z(t)_h$ is discontinuous on a zero of $\dot{Z}(t)$ where it equals plus or minus one and jumps by plus or minus two and has the mean value of the left and right limits approaching the discontinuity being equal to 0. Precisely at these points of discontinuty (the zeros of $Z'$) is where the function becomes non-expansive since the Lipschitz constant at these points approaches the limiting range of the hyperbolic tangent at plus or minus one which has absolute value one approaching from either side, at all other points of the $n$-th Fatou domain the function $M_Z(t)$ is a contraction mapping meaning that its Lipschitz constant is strictly less than one and converges smoothly to zero precisely at a zero of $Z(t)$. The contraction mapping properties of the hyperbolic tangent function are proved in the case of the the real line in [KBM00] where it is shown that the fixed-point iteration of tanh enjoys linear convergence.

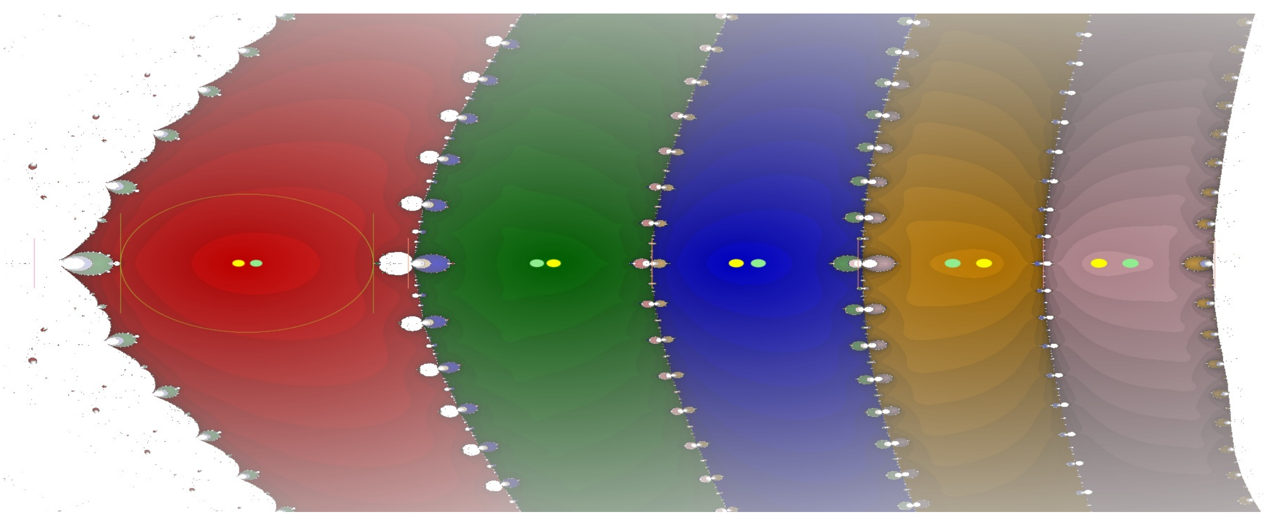

**Figure 2.** Iterated mappings of $t \to t - \frac{Z(t)}{\dot{Z}(t)}$ over a rectangle $A \subset \mathbb{C}$ containing the first 5 roots

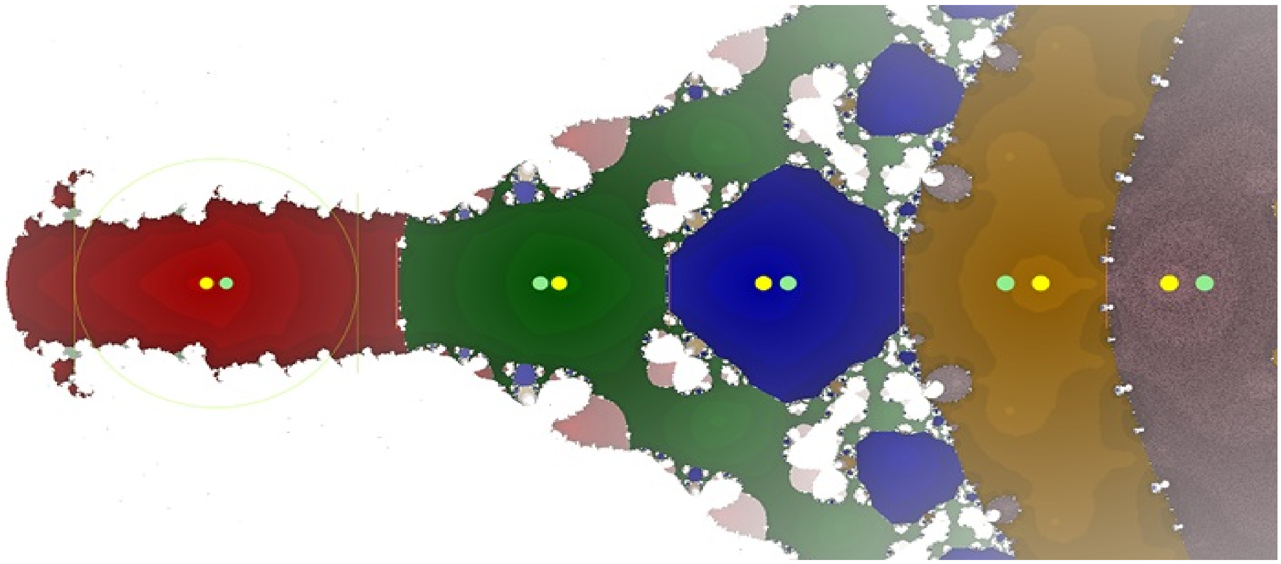

**Figure 3.** Iterated mappings of $t \to t - \tanh\left(\frac{Z(t)}{\dot{Z}(t)}\right)$ over a rectangle $A \subset \mathbb{C}$ containing the first 5 roots

# 2 Proof of the Riemann Hypothesis

## 2.1 Cauchy Sequences and Newton's Method

**Proposition 10.** *The relaxation constant, $h > 0$, of Newton's method can always be chosen small enough such that*

$$\left\{\lim_{k\to\infty} M_Z^{\circ k}(\tilde{y}_n)_h = y_{n,h} \colon N(\tilde{y}_n, y_{n,h}) = 1\right\} \tag{27}$$

*where $y_{n,h}$ is independent of the value of $h$ chosen which can vary independently of $n$, as long as it is greater than 0 and small enough that the resulting trajectory doesn't converge to another point outside of its immediate basin, a situation which would be indicated by $N(\tilde{y}_n, y_{n,h}) \neq 1$*

**Proof.** Let $n$ by any integer. Set $h = 1$. If $N(\tilde{y}_n, y_{n,h}) > 1$ then set $h = \frac{h}{2}$, recalculate the accumulation point $y_{n,h}$ and check again if $N(\tilde{y}_n, y_{n,h}) = 1$, repeating the process if not, and terminating when $h$ becomes small enough such that $N(\tilde{y}_n, y_{n,h}) = 1$. If there were any roots off of the critical line their influences on trajectories of the Newton map would cancel exactly, leaving the only location for which attractive critical points to exist being the real line, that is, with vanishing imaginary part corresponding to the "critical line" $\zeta\left(\frac{1}{2} + it\right)$ via Mobius inversion as in Equation (8) and related equations. The zero counting function $N(t,s)$ is given by exactly

$$N(t,s) = N_s - N_t = \operatorname{Im}\left(\left(\frac{\ln\left(\zeta\left(\frac{1}{2} + is\right)\right)}{\pi} + \frac{i}{\pi}\vartheta(s)\right) - \left(\frac{\ln\left(\zeta\left(\frac{1}{2} + it\right)\right)}{\pi} + \frac{i}{\pi}\vartheta(t)\right)\right) \tag{28}$$

where

$$N_t = \operatorname{Im}\left(i + \frac{1}{\pi}\int \frac{\dot{Z}(t)}{Z(t)}\mathrm{d}t\right) = \operatorname{Im}\left(\frac{\pi i + \ln\left(\zeta\left(\frac{1}{2} + it\right)\right) + \vartheta(t)}{\pi}\right) \tag{29}$$

□

**Remark 11.** If $|M_Z^{\circ k}(\tilde{y}_n)_h|$ was iterated in a neighborhood of a hypothetical zero off of the critical line, that would mean there was a root of $\zeta(s)$ with a real part $\mathrm{Re}(s)$ other than $\frac{1}{2}$ and not among the trivial zeros at $\zeta(-2n)$ which would correspond to a root of $Z(s)$ with imaginary part $\mathrm{Im}(Z(s))$ not equal to zero. Since by definition one can take $y_n$ (the $n$-th zero of $Z$) to be defined as the limit of $\lim_{k\to\infty} M_Z^{\circ k}(\tilde{y}_n)_h$ with small enough $h$ the function $Z(s)$ cannot have a zero with imaginary part not equal to 0 and not among the trivial zeros of $Z(t)$ corresponding to $\zeta(-2n)$. Newton's method converges quadratically and quadratic convergence implies simplicity of the the $n$-th zero which is the accumulation point of the $n$-th Cauchy sequence. If there were any complex zeros of $Z(t)$ they would come in complex conjugate pairs manifesting as a zero with multiplicity 2 and linear, not sub-quadratic convergence; yet the Cauchy sequence $|M_Z^{\circ k}(\tilde{y}_n)_h|$ still converges quadratically, a fact which would negate itself if it occured and therefore is not actually possible.

A complex non-trival zero of $Z$ would manifest itself as a zero of multiplicity 2 at the real part of the complex zero under the absolute value iteration $|M_Z^{\circ k}(\tilde{y}_n)_h|$ and result in non-quadratic convergence.

**Remark 12.** The lower the value of $h$ required to prevent the trajectory from leaving its immediate basin, the more iterations required to converge since each iteration is having smaller impact. The crucial thing as far as the Riemann hypothesis is concerned is that convergence of the Cauchy sequences formed by iterated Newton derivative to unique accumulations is guaranteed by means of the Caccioppoli-Banach and contraction mapping theorems since $M_Z^{\circ k}(\tilde{y}_n)_h$ is a contraction mapping for any value $s \in \mathbb{C}$ not exactly equal to a zero of $\dot{Z}(s)$. If a point in the Cauchy sequence lands precisely on a zero of $\dot{Z}(s)$ then the value is actually 0 there since $-1$ and $+1$ has a mean value of 0. It should be possible to show that the set of possible points which under iteration lead to a zero of $\dot{Z}(t)$ has measure 0 and the LambertW starting points provided by $\tilde{y}_n$ are located well-away from these points towards which trajectories never lead due to the contraction mapping theorem applicable to the entire $n$-th Fatou domain by way of the hyperbolic tangent function mapping $(-\infty,\infty) \to (-1,1)$.

**Corollary 13.** *The LeClair-França exact transcendental equation for the n-th Riemann zero, Equation (15) has a unique solution for each n equal to the accumulation points of the n-th Cauchy sequence* $\{M_Z^{\circ 1}(\tilde{y}_n)_h, M_Z^{\circ 2}(\tilde{y}_n)_h, M_Z^{\circ 3}(\tilde{y}_n)_h, \ldots, M_Z^{\circ\infty}(\tilde{y}_n)_h\}$ *as defined in Proposition 10.*

$$\left\{ S(y_n) = \left(n - \frac{3}{2}\right)\pi - \vartheta(M_Z^{\circ\infty}(\tilde{y}_n)_h) \colon N(\tilde{y}_n, M_Z^{\circ\infty}(\tilde{y}_n)_h) = 1 \right\} \tag{30}$$

*where*

$$M_Z^{\circ\infty}(\tilde{y}_n)_h = \lim_{k\to\infty} M_Z^{\circ k}(\tilde{y}_n)_h \tag{31}$$

*and*

$$M_Z^{\circ\infty}(\tilde{y}_n)_h = y_n \text{ for small enough } h \tag{32}$$

This statement is similiar in form to

**Theorem 14.** *The Predictor-Corrector Convergence Theorem. [CB80, Theorem 8.2] If $f(x,y)$ and $\partial x/\partial y$ are continuous in $x$ and $y$ and on the closed interval $[a,b]$ the inner-iteration defined by*

$$y_{n+1}^{(k)} = y_n + \frac{h}{2}\left( f(x_n, y_n) + f\left(x_{n+1}, y_{n+1}^{(k-1)}\right)\right) \tag{33}$$

*where $k = 1, 2, \ldots$ will converge to a fixed-point in the interval, provided $h$ is chosen small enough so that*

$$\left|\frac{\partial f}{\partial y}\right| h < 2 \forall x = x_n \text{ and } \left\{ y \colon |y - y_{n+1}| \leqslant \left| y_{n+1}^{(0)} - y_{n+1} \right| \right\} \tag{34}$$

**Theorem 15.** *The LeClair-França Theorem. If there is a unique solution to (138) for each n then the zeros can be counted along the critical line since they are enumerated by the integer n. [FL14, VII.B]*

**Corollary 16.** *The LeClair-França Theorem and thus The Riemann Hypothesis are true since Corollary 9 and Proposition 10 prove that a unique solution of the LeClair-França exact equation exists and thus the equality of the $\zeta$-zero counting functions on the critical line and critical strip respectively means that it is impossible that any more zeros off the critical line could be present, and thus the counting functions are precisely equal. [FL14, VII.B]*

## 2.2 Possible Objections

### 2.2.1 Newtons Method and Complex Roots

It has been said that Newton's method might not converge to complex roots give real starting points. If this is true then Muller's method could be applied using the Newton iteration to generate the first two points. Also, the Caccioppoli-Banach Theorem [KA, XVI.1] is a more extensive way to understand how convergence is guaranteed in complete metric spaces and not just the real line, which will be needed to allow the apriori possibility of zeros off of the critical line.

# 3 Appendix

## 3.1 Complex Dynamics

**Definition 17.** *If $f$ is a complex function defined in $\Omega$ and the derivative of $f$ at $z_0$ defined by*

$$\dot{f}(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \forall z_0 \in \Omega \tag{35}$$

*exists then $f$ is said to be holomorphic or analytic in $\Omega$. The class of all holomorphic functions is denoted $H(\Omega)$. A function $f$ is known as an entire function when $\Omega$ is the entire complex plane $\mathbb{C}$. [Rud06, Definition 10.2]*

**Definition 18.** *The* `multiplier` *of $F$ at the fixed-point $\alpha$ is the derivative of the Newton map $N_F(t)$ evaluated at $\alpha$, $N_f(t)|_{t=\alpha}$ which can be written*

$$\dot{N}_F(t) = \frac{F(t)\ddot{F}(t)}{\dot{F}(t)^2} \tag{36}$$

**Definition 19.** *A point $z_0 \in \mathbb{C}$ is called a periodic point of $f$ if $f^n(z_0) = z_0$ for some $n \in \mathbb{N}$. A fixed-point is a 1-periodic point. [93, 3.1]*

**Definition 20.** *A family $(f_k)$ of holomorphic maps $U \to \bar{\mathbb{C}}$ where $U \subset \mathbb{C}$ is a domain is called a normal family if every sequence $(f_k)$ contains a subsequence that converges locally uniformly to a holomorphic limit function $f: U \to \bar{\mathbb{C}}$. [Mil06, 1.3 p30]*

**Definition 21.** *The (stable) Fatou set $F(f(z))$ of a function $f: \mathbb{C} \to \mathbb{C}$ is defined by*

$$F(f(z)) = \{z \in \hat{\mathbb{C}}: \{f^{\circ n}(z): n \in \mathbb{N}\} \text{ is defined and constitutes } a \text{ normal family in some neighborhood of } z\} \tag{37}$$

*which is the set of points for which $\lim_{n\to\infty} f^{\circ n}(z)$ converges to a fixed-point where $f^{\circ n}(z)$ is composition of $f(x)$ with itself $n$ times, e.g. $f^{\circ 3}(z) = f(f(f(z)))$. A Fatou set $F_f$ of a meromorphic function $f$ is said to be completely invariant, that is, $z \in F_f$ if and only if $f(z) \in F_f$. [BKY][Dom98]*

**Definition 22.** *The (chaotic) Julia set $J(f(z))$ of a function $f: \mathbb{C} \to \mathbb{C}$ is defined as the complement of its Fatou set*

$$J(f(z)) = \hat{\mathbb{C}} \backslash F(f(z)) \tag{38}$$

*which is the set of points for which iterated self-maps do not converge.*

### 3.1.1 Holomorphic Index

**Definition 23.** *The holomorphic index of a (holomorphic) function $g: \Omega \to \mathbb{C}$ is defined by*

$$\iota(g,\alpha) = \frac{1}{2\pi i}\int_C \frac{1}{1-g(z)}\mathrm{d}z = \left\{\frac{1}{1-\lambda} : \lambda \neq 1\right\} \tag{39}$$

*where $C$ is a small circle around $\alpha$ with counterclockwise direction. [16, Proposition 3]*

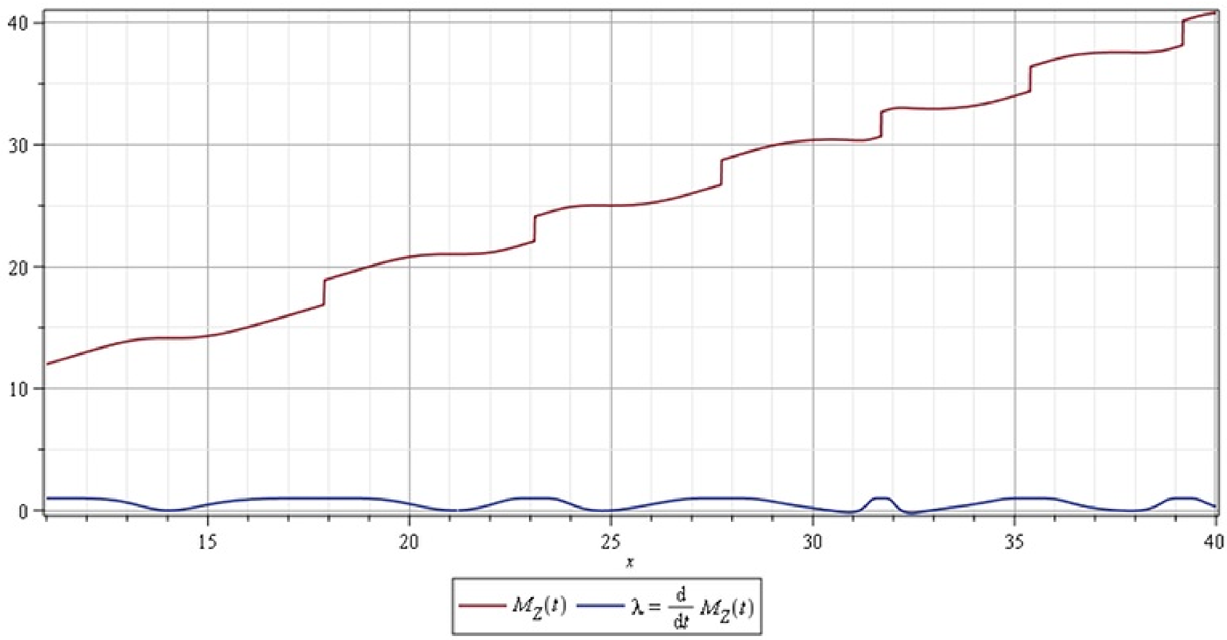


**Figure 4.** $M_Z(t)$ and the multiplier $\lambda = \frac{\mathrm{d}M_Z(t)}{\mathrm{d}t}$

## 3.2 Convergence of Newton's Method, Lipschitz Functions, and The Mean Value Theorem

[Wan00][V.02][Pol06][Han79][CB80, Ch3p.72] See [Gra46, VIII.2] on the topic of solutions which are implicitly defined near an initial solution. [WL03]

**Theorem 24.** `Theorem of the Mean`*. Suppose $f$ is continuous on the finite closed interval $[a,b]$ and has a derivative either finite or infinite at each point of the open interval $(a,b)$. Then there exists a point $c$ in the open interval $(a,b)$ such that*

$$f(b) = f(a) + f'(c)(b-a) \tag{40}$$

*[Gra46, V.I Theorem 4]*

Since $\dot{N}_Z(y_n) = 0$ where $y_n$ is $n$-th approximation zero would prove that all of these roots are simple since by definition the Tayor expansion about the root has the form

$$Z_n(t) = L_n(t-y_n)^{m_n} \tag{41}$$

where $m_n$ is the multiplicity of the $n$-th root

$$m_n = 1 - \frac{1}{1-\dot{N}(y_n)} \tag{42}$$

[Mil06, Problem 4-g][Gil88] [Lor90] [MS06, 2.3 Theorem 2.8 p.330]

## 3.3 Gram Points and Lehmer's Phenomena

### 3.3.1 Gram Points are in the Julia Set of $N_Z$

**Proposition 25.** *The approximate Gram points*

$$\tilde{g}_n = \frac{(8n-7)\pi}{4W\left(\frac{8n-7}{8e}\right)} \tag{43}$$

*are solutions to*

$$\tilde{\vartheta}(\tilde{g}_n) = (n-1)\pi \tag{44}$$

*which are approximations of the exact equations*

$$\vartheta(g_n) = (n-1)\pi \tag{45}$$

*whose solutions $g_n$ lie in the Julia sets which seperate the neighboring basins of attraction.*

**Corollary 26.**

### 3.3.2 Lehmer's Phenomena

D. H. Lehmer discovered cases where the Riemann zeta function has zeros that are "only just" on the critical line: two zeros of the zeta function are so close together that it is unusually difficult to find a sign change between them. This is called "Lehmer's phenomenon", and first occurs at the zeros with imaginary parts 7005.063 and 7005.101, which differ by only .04 while the average gap between other zeros near this point is about 1. The discovery of non-simple roots, or the discovery of a local positive minimum, or a local negative maximum, implies a violation of the Riemann hypothesis. [Edw74, 8.3]

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