Superattractive Fixed-points of a Newton-like Mappping of the Hardy Z Function

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There is shown to exist a unique solutions to the LeClaire-França exact equation for the first 98,020 of the first 100,000 zeros of the Hardy Z Function via the construction of Cauchy sequences whose accumulation points are guaranteed via an application of the Newton-Kantorovich theorem applied to a Newton-like map. [1]

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1 Derivations

1.1 Preliminaries

Let $\zeta(t)$ be the Riemann zeta function

$$\begin{aligned} \zeta(t) &= \sum_{n=1}^{\infty} n^{-s} && \forall \mathrm{Re}(s) > 1 \\ &= (1-2^{1-s}) \sum_{n=1}^{\infty} n^{-s}(-1)^{n-1} && \forall \mathrm{Re}(s) > 0 \end{aligned} \tag{1}$$

and $\vartheta(t)$ be Riemann-Siegel vartheta function $\vartheta(t)$

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$$\vartheta(t) = -\frac{i}{2}\left(\ln\Gamma\left(\frac{1}{4}+\frac{it}{2}\right) - \ln\Gamma\left(\frac{1}{4}-\frac{it}{2}\right)\right) - \frac{\ln(\pi)\,t}{2} \tag{2}$$

The exact equation for the n-th Riemann zero t_n is given by [FL15, Equation 20]

$$\vartheta(t_n) + S(t_n) = \left(n - \frac{3}{2}\right)\pi \tag{3}$$

therefore the normalized argument of ζ is given by

$$\begin{aligned} S(t) &= \pi^{-1}\arg\left(\zeta\left(\frac{1}{2}+it\right)\right) \\ &= -\frac{i}{2\pi}\left(\ln\zeta\left(\frac{1}{2}+it\right) - \ln\zeta\left(\frac{1}{2}-it\right)\right) \\ &= \frac{1}{\pi}\lim_{\varepsilon\to 0}\operatorname{Im}\left(\ln\zeta\left(\frac{1}{2}+it+\varepsilon\right)\right) \end{aligned} \tag{4}$$

which is discontinuous at the the n-th zero where it is equal to

$$S_n = S(y_n) = \left(n - \frac{3}{2}\right)\pi - \vartheta(y_n) \tag{5}$$

which is approximated by

$$S_n = S(y_n) = \left(n - \frac{3}{2}\right)\pi - (\tilde{\vartheta}(\tilde{y}_n) + \Delta()) = S(y_n) + \Delta(y_n) \tag{6}$$

where

$$\begin{aligned} \Delta(t) &= \vartheta(t) - \tilde{\vartheta}(t) = O\left(\frac{1}{t}\right) \\ &= \frac{t}{4}\log\left(1+\frac{1}{4t^2}\right) + \frac{1}{4}\arctan\left(\frac{1}{2t}\right) + \frac{t}{2}\int_0^\infty \frac{\lfloor u\rfloor - \frac{1}{2}}{(u+\frac{1}{4})^2 + (\frac{t}{2})^2}\mathrm{d}u \\ &= \frac{t}{4}\log\left(1+\frac{1}{4t^2}\right) + \frac{1}{4}\arctan\left(\frac{1}{2t}\right) + \frac{t}{2}\sum_{n=0}^{\infty} -\frac{(2n-1)\left(\arctan\left(\frac{4n+1}{2t}\right) - \arctan\left(\frac{4n+5}{2t}\right)\right)}{t} \end{aligned} \tag{7}$$

where $\lfloor u\rfloor$ is the floor function which truncates the fractional part of its argument [Ivi13, Lemma 5.1] and the integral and has been re-expressed as an infinite sum since

$$\int_n^{n+1} \frac{\lfloor u\rfloor - \frac{1}{2}}{(u+\frac{1}{4})^2 + (\frac{t}{2})^2}\mathrm{d}u = -\frac{(2n-1)\left(\arctan\left(\frac{4n+1}{2t}\right) - \arctan\left(\frac{4n+5}{2t}\right)\right)}{t} \tag{8}$$

which has a removable singularity at $t=0$ where it is equalt to the limit

$$\lim_{t\to 0}\int_n^{n+1} \frac{\lfloor u\rfloor - \frac{1}{2}}{(u+\frac{1}{4})^2 + (\frac{t}{2})^2}\mathrm{d}u = \frac{16n-8}{16n^2+24n+5} = \frac{8(2n-1)}{(4n+1)(4n+5)} \tag{9}$$

Definition 1. *The approximate equation for the n-th Riemann zero is*

$$\tilde{\vartheta}(\tilde{y}_n) = \left(n - \frac{3}{2}\right)\pi \tag{10}$$

which is solved by

$$\tilde{y}_n = t_n^{**} = \frac{2\pi\left(n - \frac{11}{8}\right)}{W\left(\frac{n-\frac{11}{8}}{e}\right)} \tag{11}$$

where $W(x)$ is the Lambert W function defined as the solution to the equation $W(x)e^{W(x)}=x$.[FL15] The notation t_n^{**} is the notation used in [MBM09] where this exact same function appeared. The approximate Gram points given by the exact solution of the first-order Stirling expansion of $\vartheta(t)$ in terms of the LambertW *function (which is itself-defined in terms of a certain fixed-point iteration) take values that are very close to the true Gram points. In fact, this exact solution of the approximate equation gives an accuracy of* 2.2×10^{-3} *for $n=0$ decreasing to 3.5×10^{-4} by $n=10$. Let the approximate Riemann-Siegel vartheta function be defined*

$$\tilde{\vartheta}(t)=\frac{t}{2}\ln\left(\frac{t}{2\pi e}\right)-\frac{\pi}{8} \tag{12}$$

which by definition

$$\vartheta(t)=\tilde{\vartheta}(t)+\Delta(t) \tag{13}$$

and $\Delta(t)$ is conjectured to be $O(1)$ due to compelling evidence offered in [LeC16] where it is also hypothesiszed that the suitably normalized distribution of δy_n is a Gaussian distribution with mean 0 and variance $\sqrt{\frac{\pi}{32}}$.

The Hardy Z function[Ivi13] can then be written as

$$Z(t)\ =e^{i\vartheta(t)}\,\zeta\left(\frac{1}{2}+it\right) \tag{14}$$

which can be mapped isometrically back to the ζ function

$$\zeta(t)=e^{-i\vartheta\left(\frac{i}{2}-it\right)}Z\left(\frac{i}{2}-it\right) \tag{15}$$

due to the isometry

$$t=\frac{1}{2}+i\left(\frac{i}{2}-it\right) \tag{16}$$

of the Mobius transforms $f(t)=\frac{at+b}{ct+d}$ with

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix}=\begin{pmatrix} -i & \frac{i}{2}\\ 0 & 1\end{pmatrix}\text{and its inverse}\begin{pmatrix} a & b\\ c & d\end{pmatrix}^{-1}=\begin{pmatrix} i & \frac{1}{2}\\ 0 & 1\end{pmatrix} \tag{17}$$

making possible the Riemann-Siegel-Hardy correspondence. The Bäcklund counting formula gives the exact number of zeros on the critical strip up to level t, not just on the criticial line $\mathrm{Re}(t)=\frac{1}{2}$, [Bor08, 3.2]

$$N(t)=\ \#\{\zeta(x+iy)=0\colon 0\leqslant x\leqslant 1, 0\leqslant y\leqslant t\}=\langle N(t)\rangle+S(t) \tag{18}$$

where $\langle N(t)\rangle$ is the smooth part of the counting function

$$\langle N(t)\rangle=\pi^{-1}\vartheta(t)+1 \tag{19}$$

[FL14] The relationship between the functions $N(t)$, $S(t)$, and $Z(t)$ is demonstrated by

$$\ln\zeta\left(\frac{1}{2}+it\right)=\ln|Z(t)|+i\pi S(t) \tag{20}$$

These formulas are true independent of the Riemann hypothesis which posits that all complex zeros of $\zeta(s+it)$ have real part $s=\frac{1}{2}$. [Ivi13, Corrollary 1.8 p.13]

1.2 The Newton Map $N_Z(t)$ of $Z(t)$

Let the new Newton map [93, 6.1][Bro04] of $Z(t)$ be defined by the meromorphic function

$$\begin{aligned} N_Z(t)\ &=t-\frac{Z(t)}{\dot{Z}(t)}\\ &=t+\frac{i\zeta\left(\frac{1}{2}+it\right)}{\dot{\vartheta}(t)\,\zeta\left(\frac{1}{2}+it\right)+\dot{\zeta}\left(\frac{1}{2}+it\right)}\end{aligned} \tag{21}$$

which has poles at the zeros of $\dot{Z}(s)$.

Proposition 2. *The Newton map $N_Z(t)$ of $Z(t)$ is maximally flat in a neighborhood of its superattractive fixed points which are separated by poles which repel trajectories away from the points of maximum curvature between the fixed points and towards those ultimately leading back to some point of minimal curvature.*

1.2.1 The LeClair-França Formula and Stirling's Approximation of lnΓ

Definition 3. *The exact equation for the Riemann Zeros is*

$$\vartheta(y_n) + S(y_n) = \left(n - \frac{3}{2}\right)\pi \tag{22}$$

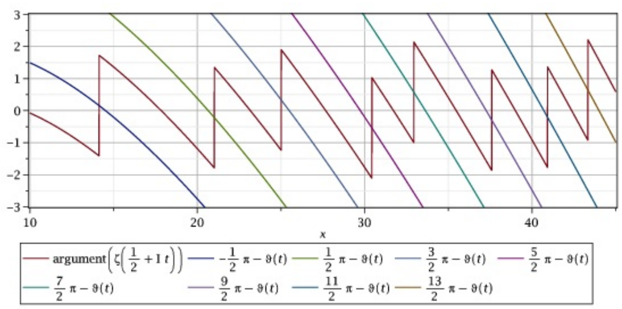

Figure 1. $S(t)$ and $\left\{\left(n - \frac{3}{2}\right)\pi - \vartheta(t) \colon n = 1...8\right\}$

TODO: typeset the equations

Remark 4. Convergence of the relaxed or unrelaxed iteration of the Newton's is assured by having a coeffecient $h_n \leqslant 0.5$ however it does not necessarily to the nearest zero y_n corresponding to the initial approximation zero $\tilde{y}_n$ and instead in many instances converges to a zero several indices away from the matching zero y_n and gets attracted to an accumulation point in a neighboring basin. A slight modification to N_Z is introduced based on the hyperbolic tangent function in the next section which has a much wider Fatou domains on which the Newton map iterates form normal families

1.2.2 A Newton-like Map $M_Z(t)$ Which Is Holomorphic In The Critical Strip

Definition 5. *A function between normed spaces with the property that the distance between function values is bounded by a constant multiple of the distance between the arguments is known as a* Lipschitz function*. If the function satisfies the Lipschitz condition that*

$$\|f(x) - f(y)\| \leqslant L\|x - y\| \forall x, y \in A \tag{23}$$

where possibly A is a single point $A = \{x_0\}$ then $f(x)$ is said to be k-Lipschitz on A or at x_0. When $k = 1$ the function is a `non-expansive mapping`*, and is a* ***contraction mapping*** *when $k < 1$. [Map]*

Definition 6. *Let the modified Newton map [93, 6.1][Bro04] of $Z(t)$ be defined by the meromorphic function*

$$\begin{aligned} M_Z(t)_h &= t - h \cdot \tanh\left(\frac{Z(t)}{\dot{Z}(t)}\right) \\ &= t + h \cdot \tanh\left(\frac{i\zeta\left(\frac{1}{2} + it\right)}{\dot{\vartheta}(t)\,\zeta\left(\frac{1}{2} + it\right) + \dot{\zeta}\left(\frac{1}{2} + it\right)}\right) \end{aligned} \tag{24}$$

where the hypebolic tangent function $\tanh(z) = \frac{e^z e^z - 1}{e^z e^z + 1}$ *maps* $(-\infty, \infty) \to (-1, 1)$ *and the parameter* h *is the relaxation parameter. The* tanh *function has an infinite number of simple poles with residue 1 at* $\frac{i}{2}\pi(2n+1)$ *for all integers* n *and is also holomorphic when* $|\mathrm{Im}(z)| < \frac{\pi}{2}$*, that is, in the strip* $\mathrm{Im}(z) \in \left(-\frac{1}{2}, \frac{1}{2}\right)$*, that is, there are no poles in this region.*

Definition 7. *A function* f *that is Fréchet differentiable for any point of* U *is said to be* C^1 *if the function*

$$\mathrm{D}f = U \to B(V, W); x \mapsto \mathrm{D}f(x) \tag{25}$$

is continuous.

Corollary 8. *The hyperbolic tangent function is continuous on* $\bar{\mathbb{R}} = (-\infty, \infty)$ *and thereore Fréchet differentiable on* $\bar{\mathbb{R}} = (-\infty, \infty)$*.*

Note 9. The function $M_Z(t)_h$ is discontinuous on a zero of $\dot{Z}(t)$ where it equals ± 1 and jumps by ± 2 and has the average value of the left and right limits approaching the discontinuity being equal to 0. The zeros of $\dot{Z}(t)$ are points of discontinuty with finite jumps of magnitude 2 where the function becomes non-expansive since the Lipschitz constant at these points approaches the limiting range of the hyperbolic tangent at ± 1 which has absolute value 1 approaching from either the left or right, at all other points of the n-th Fatou domain the function $M_Z(t)$ is a contraction mapping meaning that its Lipschitz constant is strictly less than one and converges smoothly to zero precisely at a zero of $Z(t)$. The contraction mapping properties of the hyperbolic tangent function are proved in the case of the the real line in [KBM00] where it is shown that the fixed-point iteration converges at a linear rate.

2 Newton's Method and the Hardy Z Function

2.1 Cauchy Sequences and Newton's Method

Example 10. The first point(smallest value of n) where Newton's method fails to converge is

$$\begin{aligned} \lim_{k\to\infty} M_Z^{\circ k}(\tilde{y}_{1443})_1 &= 1918.786499\ldots \\ \lim_{k\to\infty} M_Z^{\circ k}(\tilde{y}_{1444})_1 &= 1918.786499\ldots \\ \lim_{k\to\infty} M_Z^{\circ k}(\tilde{y}_{1444})_{0.8} &= 1919.207779\ldots \end{aligned} \tag{26}$$

Remark 11. The lower the value of h required to prevent the trajectory from leaving its immediate basin, the more iterations required to converge since each iteration is having smaller impact. If a point in the Cauchy sequence lands precisely on a zero of $\dot{Z}(s)$ then the value is actually 0 there since -1 and $+1$ has a mean value of 0.

Example 12. There are values of n which do not converge no matter how small the relaxation factor is taken. This occurs when the starting point given by the Lambert W approximization zero closed-form expression is evaluated for $n = 6400$ where it just to the left of the boundary determined by the zero of Z' just to the right of it

The smallest index n for which

$$\lim_{m\to\infty} M_Z^{\circ m}(\tilde{y}_n^{(-)}) \neq y_n^{(-)} \tag{27}$$

is $n = 1444$ where

$$\lim_{m\to\infty} M_Z^{\circ m}(\tilde{y}_{1444}^{(-)}) = y_{1443}^{(-)} \neq y_{1444}^{(-)} \tag{28}$$

And the 21-st point

$$\lim_{m\to\infty} M_Z^{\circ m}(\tilde{y}_{6400}^{(-)}) = y_{6399}^{(-)} \neq y_{6400}^{(-)} \tag{29}$$

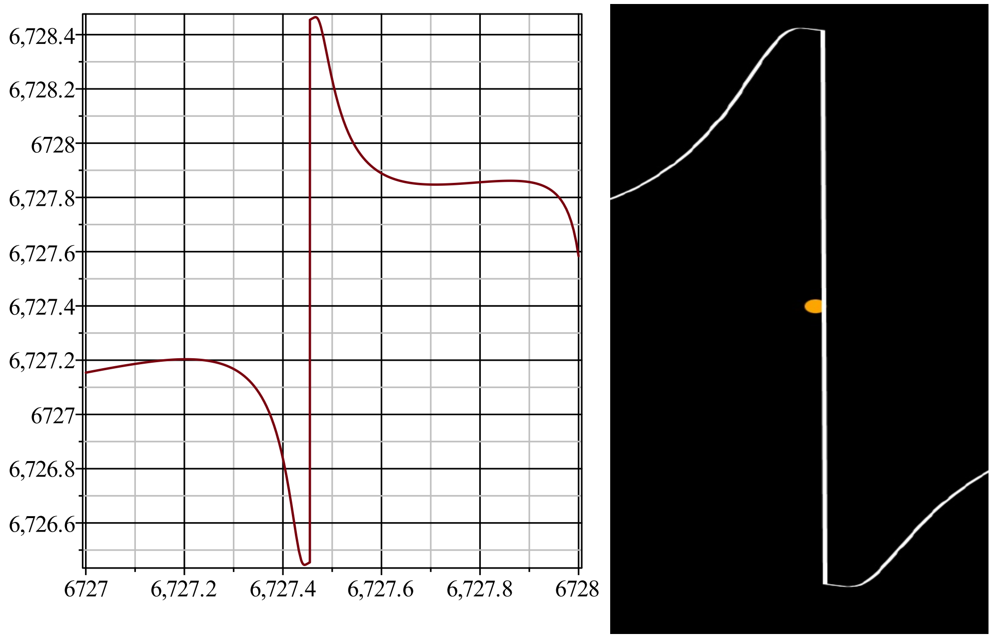

Figure 2. $\{M_Z(t) : t \in 6727...6728\}$ near $t = y_{6400}^{(-)} = 6727.861353...$

The first few values n for which $t \to t - \tanh\left(\frac{Z(\tilde{y}_n^{(-)})}{\dot{Z}(\tilde{y}_n^{(-)})}\right)$ does not converge to $y_n^{(-)}$ is

$$n = \{1444, 2041, 2813, 3359, 3474, 3648, 3841, 3855, 4330, 4408, 4527, 5106, ...\} \tag{30}$$

and it has been calculated that there are 1,980 of these points up to $n = 100,000$ which is 98.02% of the integers n up to that level.

Theorem 13. *The LeClair-França Theorem. If there is a unique solution to (138) for each n then the zeros can be counted along the critical line since they are enumerated by the integer n. [FL14, VII.B]*

2.2 Semi-Local Convergence of the Relaxed Newton's Method

In [AGMR14, 3.] semi-local convergence theorems for the relaxed Newton's method are used to find estimates on the radii of the spheres of convergence surrounding a given starting point x_o. Local convergence theorems give estimates of the radii of the attractors (the convergence balls which are subsets of the Fatou sets) surrounding the precisely known location of a root. Semi-local theorems give estimates of the convergence radii centered around the initial points of the sequence rather than the accumulation point. Semi-local convergence theorems are thus more applicible to the situation here where the initial points are given by the $\tilde{y}_n$ exact Lambert-W function solutions to the Stirling approximation equation and the goal is to show that the ball is large enough to always contain the relaxed Newton iterates given a small enough step size h).

3 Appendix

3.1 Complex Dynamics

Definition 14. *If f is a complex function defined in Ω and the derivative of f at z_0 defined by*

$$\dot{f}(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \forall z_0 \in \Omega \tag{31}$$

exists then f is said to be holomorphic or analytic in Ω. The class of all holomorphic functions is denoted $H(\Omega)$. A function f is known as an entire function when Ω is the entire complex plane $\mathbb{C}$. [Rud06, Definition 10.2]

Definition 15. *The* `multiplier` *of F at the fixed-point α is the derivative of the Newton map $N_F(t)$ evaluated at α, $N_f(t)|_{t=\alpha}$ which can be written*

$$\dot{N}_F(t) = \frac{F(t)\ddot{F}(t)}{\dot{F}(t)^2} \tag{32}$$

Definition 16. *A point $z_0 \in \mathbb{C}$ is called a periodic point of f if $f^n(z_0) = z_0$ for some $n \in \mathbb{N}$. A fixed-point is a 1-periodic point. [93, 3.1]*

Definition 17. *A family (f_k) of holomorphic maps $U \to \bar{\mathbb{C}}$ where $U \subset \mathbb{C}$ is a domain is called a normal family if every sequence (f_k) contains a subsequence that converges locally uniformly to a holomorphic limit function $f: U \to \bar{\mathbb{C}}$. [Mil06, 1.3 p30]*

Definition 18. *The (stable) Fatou set $F(f(z))$ of a function $f: \mathbb{C} \to \mathbb{C}$ is defined by*

$$F(f(z)) = \{z \in \hat{\mathbb{C}}: \{f^{\circ n}(z): n \in \mathbb{N}\} \text{ is defined and constitutes } a \text{ normal family in some neighborhood of } z\} \tag{33}$$

which is the set of points for which $\lim_{n\to\infty} f^{\circ n}(z)$ converges to a fixed-point where $f^{\circ n}(z)$ is composition of $f(x)$ with itself n times, e.g. $f^{\circ 3}(z) = f(f(f(z)))$. A Fatou set F_f of a meromorphic function f is said to be completely invariant, that is, $z \in F_f$ if and only if $f(z) \in F_f$. [BKY][Dom98]

Definition 19. *The (chaotic) Julia set $J(f(z))$ of a function $f: \mathbb{C} \to \mathbb{C}$ is defined as the complement of its Fatou set*

$$J(f(z)) = \hat{\mathbb{C}} \backslash F(f(z)) \tag{34}$$

which is the set of points for which iterated self-maps do not converge.

3.1.1 Holomorphic Index

Definition 20. *The holomorphic index of a (holomorphic) function $g: \Omega \to \mathbb{C}$ is defined by*

$$\iota(g, \alpha) = \frac{1}{2\pi i}\int_C \frac{1}{1 - g(z)} \mathrm{d}z = \left\{\frac{1}{1-\lambda}: \lambda \neq 1\right\} \tag{35}$$

where C is a small circle around α with counterclockwise direction. [16, Proposition 3]

3.2 Convergence of Newton's Method, Lipschitz Functions, and The Mean Value Theorem

[Wan00][V.02][Pol06][Han79][CB80, Ch3p.72] See [Gra46, VIII.2] on the topic of solutions which are implicitly defined near an initial solution. [WL03]

Theorem 21. `Theorem of the Mean`*. Suppose f is continuous on the finite closed interval $[a, b]$ and has a derivative either finite or infinite at each point of the open interval (a, b). Then there exists a point c in the open interval (a, b) such that*

$$f(b) = f(a) + f'(c)(b - a) \tag{36}$$

[Gra46, V.I Theorem 4]

Since $\dot{N}_Z(y_n) = 0$ where y_n is n-th approximation zero would prove that all of these roots are simple since by definition the Tayor expansion about the root has the form

$$Z_n(t) = L_n(t - y_n)^{m_n} \tag{37}$$

where m_n is the multiplicity of the n-th root

$$m_n = 1 - \frac{1}{1 - \dot{N}(y_n)} \tag{38}$$

[Mil06, Problem 4-g][Gil88] [Lor90] [MS06, 2.3 Theorem 2.8 p.330]

3.2.1 Lehmer's Phenomena

D. H. Lehmer discovered cases where the Riemann zeta function has zeros that are "only just" on the critical line: two zeros of the zeta function are so close together that it is unusually difficult to find a sign change between them. This is called "Lehmer's phenomenon", and first occurs at the zeros with imaginary parts 7005.063 and 7005.101, which differ by only .04 while the average gap between other zeros near this point is about 1. The discovery of non-simple roots, or the discovery of a local positive minimum, or a local negative maximum, implies a violation of the Riemann hypothesis. [Edw74, 8.3]

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