On Quadrics and Pseudoquadrics Inversions in Hyper-pseudospheres

by Robert Benjamin Easter

Email: `dirname@yahoo.com`

July 21, 2016

Abstract

This note on quadrics and pseudoquadrics inversions in hyperpseudospheres shows that the inversions produce different results in a three-dimensional spacetime. Using Geometric Algebra, all quadric and pseudoquadric entities and operations are in the $\mathcal{G}_{4,8}$ Double Conformal Space-Time Algebra (DCSTA). Quadrics at zero velocity are purely spatial entities in xyz-space that are hyper-cylinders in $wxyz$-spacetime. Pseudoquadrics represent quadrics in a three-dimensional (3D) xyw, yzw, or zxw spacetime with the pseudospatial w-axis that is associated with time $w = ct$. The inversion of a quadric in a hyperpseudosphere can produce a Darboux pseudocyclide in a 3D spacetime that is a quartic hyperbolic (infinite) surface, which does not include the point at infinity. The inversion of a pseudoquadric in a hyperpseudosphere can produce a Darboux pseudocyclide in a 3D spacetime that is a quartic finite surface. A quadric and pseudoquadric can represent the same quadric surface in space, and their two different inversions in a hyperpseudosphere represent the two types of reflections of the quadric surface in a hyperboloid.

1 Introduction

In the $\mathcal{G}_{4,8}$ *Double Conformal Space-Time Algebra* (DCSTA) [1], using the DCSTA *extraction elements*, it is possible to form *quadric* surface entities in xyz-space, or *pseudoquadric* surface entities in any one of xyw-spacetime, yzw-spacetime, or zxw-spacetime. For example, a *quadric* ellipsoid entity $\mathbf{E}$ is

$$\mathbf{E} \;=\; T_{x^2}/a^2 + T_{y^2}/b^2 + T_{z^2}/c^2 - T_1.$$

And, for example, a *pseudoquadric* ellipsoid entity $\mathbf{E}^{+}$ in xyw-spacetime is

$$\mathbf{E}^{+z} \;=\; T_{x^2}/a^2 + T_{y^2}/b^2 + T_{w^2}/c^2 - T_1,$$

where the z-axis has been replaced with the pseudospatial w-axis that is associated with time $w = ct$.

The quadric and pseudoquadric both represent the same quadric surface in space, but they are different entities in spacetime that have different properties.

2 Results

2.1 Pseudoquadric inversion in hyperpseudosphere

As an example, let the pseudoquadric ellipsoid be

$$\mathbf{E}^{+z} \;=\; T(T_{x^2}/5^2 + T_{y^2}/20^2 + T_{w^2}/5^2 - T_1)T^{\sim},$$

with translator T for translation by

$$\begin{aligned}\boldsymbol{d} &= w\boldsymbol{\gamma}_0 + x\boldsymbol{\gamma}_1\\ &= 2\boldsymbol{\gamma}_0 + 12\boldsymbol{\gamma}_1,\end{aligned}$$

such that $\mathbf{E}^{+z}$ is centered at position $(w, x, y, z) = (2, 12, 0, 0)$. Let the hyperpseudosphere be at the origin $\mathbf{e}_o$ with radius $r_0 = 8$ as

$$\boldsymbol{\Sigma} \;=\; \boldsymbol{\Sigma}_{\mathcal{C}^1} \wedge \boldsymbol{\Sigma}_{\mathcal{C}^2},$$

where

$$\begin{aligned}\boldsymbol{\Sigma}_{\mathcal{C}^1} &= \mathbf{e}_{o1} + \frac{1}{2}8^2\mathbf{e}_{\infty 1}\\ \boldsymbol{\Sigma}_{\mathcal{C}^2} &= \mathbf{e}_{o2} + \frac{1}{2}8^2\mathbf{e}_{\infty 2}.\end{aligned}$$

The 3D spacetime is xyw-spacetime, where the pseudospatial w-axis can be treated as the new z-axis. The hyperpseudosphere at $z=0$ is a spacetime hyperboloid in xyw-spacetime. The hyperboloid is to be graphed with $z=0$, and the w-axis acts as the new z-axis in the graphing space.

The inversion of $\mathbf{E}^{+z}$ in $\boldsymbol{\Sigma}$ is

$$\boldsymbol{\Omega} = \boldsymbol{\Sigma}\mathbf{E}^{+z}\boldsymbol{\Sigma}.$$

The inversion $\boldsymbol{\Omega}$ is to be graphed at $z=0$, and the w-axis is graphed as if it is the z-axis.

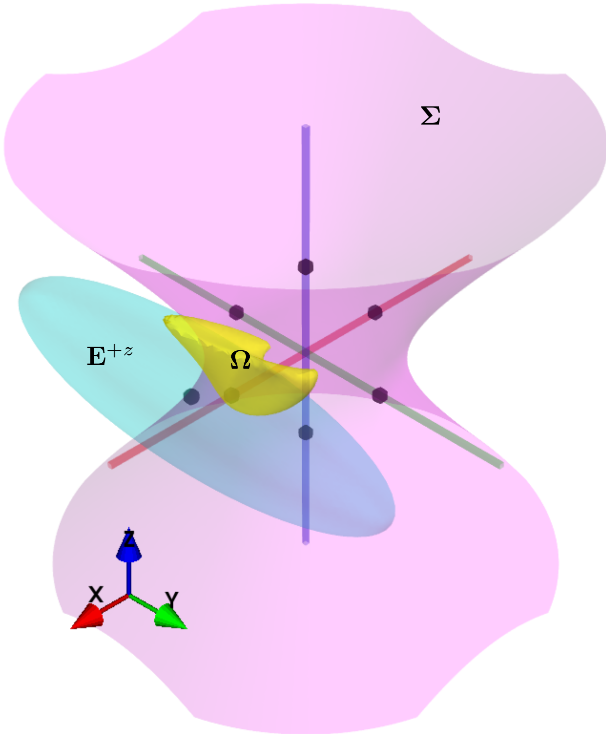

Figure 1. $\boldsymbol{\Omega} = \boldsymbol{\Sigma}\mathbf{E}^{+z}\boldsymbol{\Sigma}$

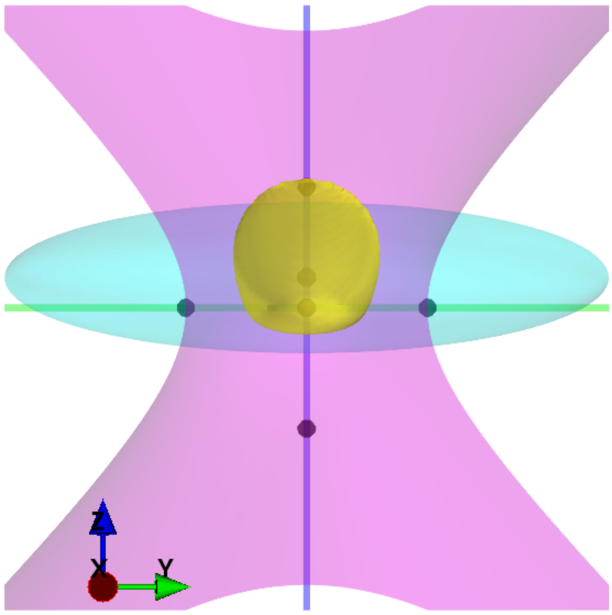

Figure 2. $\boldsymbol{\Omega} = \boldsymbol{\Sigma}\mathbf{E}^{+z}\boldsymbol{\Sigma}$

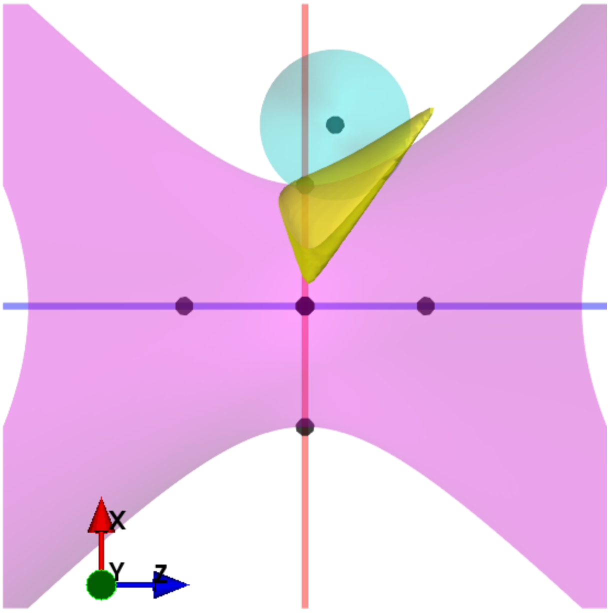

Figure 3. $\mathbf{\Omega} = \mathbf{\Sigma}\mathbf{E}^{+z}\mathbf{\Sigma}$

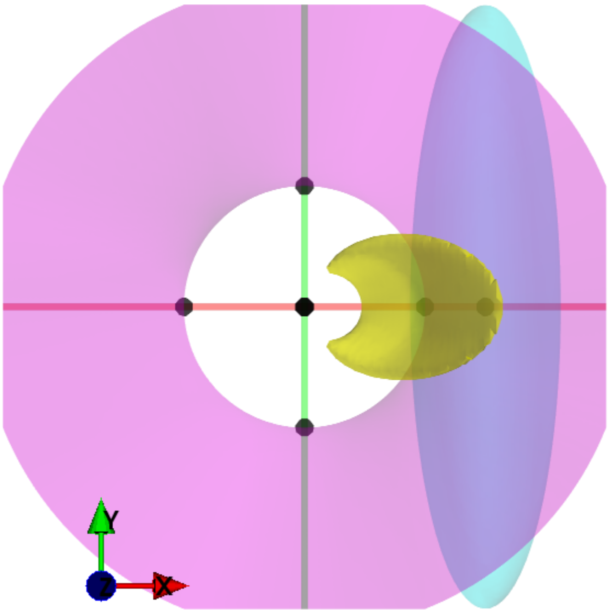

Figure 4. $\mathbf{\Omega} = \mathbf{\Sigma}\mathbf{E}^{+z}\mathbf{\Sigma}$

Figures 1-4 show the inversion $\mathbf{\Omega} = \mathbf{\Sigma}\mathbf{E}^{+z}\mathbf{\Sigma}$ of the pseudoquadric $\mathbf{E}^{+z}$ in the hyperpseudosphere $\mathbf{\Sigma}$. The result is a finite quartic surface $\mathbf{\Omega}$, which has been called a Darboux pseudocyclide, that does *not* include the infinity point $\mathbf{e}_\infty$. This may be a realistic reflection of the ellipsoid in the hyperboloid.

2.2 Quadric inversion in hyperpseudosphere

Similar to the pseudoquadric, let the quadric ellipsoid be

$$\mathbf{E} \;=\; T(T_{x^2}/5^2 + T_{y^2}/20^2 + T_{z^2}/5^2 - T_1)T^{\sim},$$

with translator T for translation by

$$\begin{aligned} \boldsymbol{d} &= x\boldsymbol{\gamma}_1 \\ &= 12\boldsymbol{\gamma}_1, \end{aligned}$$

such that $\mathbf{E}$ is centered at position $(w, x, y, z) = (w, 12, 0, 0)$. The quadric is independent of time w and is a hypercylinder in spacetime. Let the hyperpseudosphere again be at the origin $\mathbf{e}_o$ with radius $r_0 = 8$ as

$$\mathbf{\Sigma} \;=\; \mathbf{\Sigma}_{\mathcal{C}^1} \wedge \mathbf{\Sigma}_{\mathcal{C}^2},$$

where

$$\begin{aligned}\mathbf{\Sigma}_{\mathcal{C}^1} &= \mathbf{e}_{o1} + \frac{1}{2}8^2\mathbf{e}_{\infty 1}\\ \mathbf{\Sigma}_{\mathcal{C}^2} &= \mathbf{e}_{o2} + \frac{1}{2}8^2\mathbf{e}_{\infty 2}.\end{aligned}$$

The 3D spacetime is xyw-spacetime, where the pseudospatial w-axis can be treated as the new z-axis. The hyperpseudosphere at $z=0$ is a spacetime hyperboloid in xyw-spacetime. The hyperboloid is to be graphed with $z=0$, and the w-axis acts as the new z-axis in the graphing space.

The inversion of $\mathbf{E}$ in $\mathbf{\Sigma}$ is

$$\mathbf{\Omega} = \mathbf{\Sigma}\mathbf{E}\mathbf{\Sigma}.$$

The inversion $\mathbf{\Omega}$ is to be graphed at $z=0$, and the w-axis is graphed as if it is the z-axis.

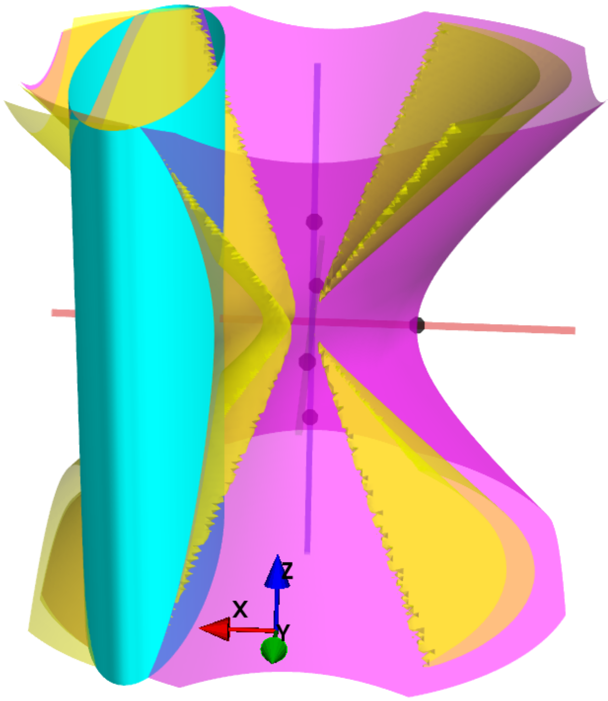

Figure 5. $\mathbf{\Omega} = \mathbf{\Sigma}\mathbf{E}\mathbf{\Sigma}$

As shown in Figure 5, the hyperpseudosphere $\mathbf{\Sigma}$ at $z=0$ is the hyperboloid, and the quadric ellipsoid at $z=0$ is a xy-plane ellipse that is an elliptical cylinder in xyw-space. The result $\mathbf{\Omega}$ is the inversion of the elliptic cylinder in the hyperboloid, all in xyw-spacetime.

3 Conclusion

The inversion of a pseudoquadric $\mathbf{E}^{+z}$, representing a quadric $\mathbf{E}$, in a pseudosphere (spacetime circular hyperboloid), which is the hyperpseudosphere $\mathbf{\Sigma}$ at $z=0$, is the entity $\mathbf{\Omega} = \mathbf{\Sigma}\mathbf{E}^{+z}\mathbf{\Sigma}$ that appears to be the correct inversion of a quadric $\mathbf{E}$ in a circular hyperboloid.

On the other hand, a quadric $\mathbf{E}$ in xyz-space is a kind of cylinder in $wxyz$-spacetime that is symmetrical around the pseudospatial time ($w = ct$)-axis, and its inversion in a spacetime circular hyperboloid is the inversion of an elliptical cylinder in the hyperboloid.

The $\mathcal{G}_{4,8}$ *Double Conformal Space-Time Algebra* (DCSTA) has entities for both spatial *quadrics* in xyz-space, and also similar *pseudoquadric* entities that are formed in a 3D spacetime that uses the pseudospatial time ($w = ct$)-axis as a drop-in replacement for one of the usual x, y, or z spatial axes. The inversion of the pseudoquadrics in a pseudosphere (circular hyperboloid) appears to be a correct reflection or inversion. DCSTA provides new representations of quadrics as pseudoquadrics, and a new inversion operation for their inversions in circular hyperboloids.

Bibliography

[1] Robert B. Easter. Double Conformal Space-Time Algebra. viXra.org, 2016. Preprint: vixra.org/abs/1602.0114.