The Cartan Model for Equivariant Cohomology

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Abstract

In this article, we will discuss a new operator d_C on $W(\mathfrak{g})\otimes\Omega^*(M)$ and to construct a new Cartan model for equivariant cohomology. We use the new Cartan model to construct the corresponding BRST model and Weil model, and discuss the relations between them.

1 Introduction

The standard Cartan model for equivariant cohomology is construct on the algebra $W(\mathfrak{g})\otimes \Omega^*(M)$ with operator

$$d_C\phi^i = 0, \phi^i \in S(\mathfrak{g}^*), i = 1, \cdots, n;$$

$$d_C\eta = (1 \otimes d - \sum_{b=1}^{n} \phi^b \otimes \iota_b)\eta, \eta \in \Omega^*(M),$$

where ι_b is ι_{e_b}(see [4],[5],[7],[8]). We can also introduce a new operator on $W(\mathfrak{g}) \otimes \Omega^*(M)$ by

$$d_C\phi^i = 0, \phi^i \in S(\mathfrak{g}^*), i = 1, \cdots, n;$$

$$d_C\eta = (1 \otimes d - \sum_{b=1}^{n} \phi^b \otimes (\iota_b + \sqrt{-1}f_b^a\iota_a))\eta, \eta \in \Omega^*(M) \otimes \mathbb{C},$$

where ι_b is ι_{e_b}. In this article we construct the new model for equivariant cohomology which also called Cartan model. The idea comes form the article [3]. We also use the new Cartan model to construct the corresponding BRST model and Weil model.

2 Cartan model

Let G ba a compact Lie group with Lie algebra $\mathfrak{g}$, $\mathfrak{g}^*$ be the dual of $\mathfrak{g}$. We known the Weil algebra is

$$W(\mathfrak{g}) = \wedge(\mathfrak{g}^*) \otimes S(\mathfrak{g}^*).$$

The contraction i_X and the exterior derivative d_W on $W(\mathfrak{g})$ defined as follow:

Choose a basis $e_1, \cdots, e_n$ for $\mathfrak{g}$ and let $e_1^*, \cdots, e_n^*$ be the dual basis of $\mathfrak{g}^*$. Let $\theta^1, \cdots, \theta^n$ be the dual basis of $\mathfrak{g}^*$ generating the exterior algebra $\wedge(\mathfrak{g}^*)$ and let $\phi^1, \cdots, \phi^n$ be the dual basis of $\mathfrak{g}^*$ generating the symmetric algebra $S(\mathfrak{g}^*)$. Let c_{jk}^i be the structure constants of

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$\mathfrak{g}$(see [6]), that is $[e_j, e_k] = \Sigma_{i=1}^n c_{jk}^i e_i$. We kown that $S(\mathfrak{g}^*)$ is identified with the polynomial ring $\mathbb{C}[\phi^1, \cdots, \phi^n]$.

Define the contraction i_X on $W(\mathfrak{g})$ for any $X \in \mathfrak{g}$ by

$$i_{e_r}(\theta^s) = \delta_r^s, \ i_{e_r}(\phi^s) = 0$$

for all $r, s = 1, \cdots, n$ and extending by linearity and as a derivation.

Define d_W by

$$d_W \theta^i = -\frac{1}{2} \sum_{j,k} c_{jk}^i \theta^j \wedge \theta^k + \phi^i$$

and

$$d_W \phi^i = -\sum_{j,k} c_{jk}^i \theta^j \phi^k$$

and extending d_W to $W(\mathfrak{g})$ as a derivation.

The Lie derivative on $W(\mathfrak{g})$ is defined by

$$L_X = d_W \cdot i_X + i_X \cdot d_W.$$

Lemma 1. *$L_{e_i}\theta^j = -\sum_k c_{ik}^j \theta^k$ and $L_{e_i}\phi^j = -\sum_k c_{ik}^j \phi^k$.*

Proof. Because

$$L_{e_i}\theta^j = (d_W \cdot i_{e_i} + i_{e_i} \cdot d_W)\theta^j = i_{e_i}(-\frac{1}{2}\sum_{i,k} c_{ik}^j \theta^i \wedge \theta^k + \phi^j) = -\sum_k c_{ik}^j \theta^k,$$

$$L_{e_i}\phi^j = (d_W \cdot i_{e_i} + i_{e_i} \cdot d_W)\phi^j = i_{e_i}(-\sum_{i,k} c_{ik}^j \theta^i \phi^k) = -\sum_k c_{ik}^j \phi^k$$

□

Lemma 2. *The operators i_X, d_W, L_X on $W(\mathfrak{g})$ satisfy the following identities:*

(1) $d_W^2 = 0$;

(2) $L_X \cdot d_W - d_W \cdot L_X = 0$, *for any* $X \in \mathfrak{g}$;

(3) $i_X i_Y + i_Y i_X = 0$, *for any* $X, Y \in \mathfrak{g}$;

(4) $L_X i_Y - i_Y L_X = i_{[X,Y]}$, *for any* $X, Y \in \mathfrak{g}$;

(5) $L_X L_Y - L_Y L_X = L_{[X,Y]}$, *for any* $X, Y \in \mathfrak{g}$;

(6) $d_W i_X + i_X d_W = L_X$, *for any* $X \in \mathfrak{g}$.

Proof. see [4]. □

So, there is a complex $(W(\mathfrak{g}), d_W)$, the cohomology of $(W(\mathfrak{g}), d_W)$ is trivial (see [5]), i.e. $H^*(W(\mathfrak{g})) \cong \mathbb{R}$.

Let M be a smooth closed manifold with G acting smoothly on the left. Let X^M be the vector field generated by the Lie algebra element $X \in \mathfrak{g}$ given by

$$(X^M f)(x) = \frac{d}{dt} f(\exp(-tX) \cdot x) \mid_{t=0} .$$

Set $d, \iota_{X^M}, \mathcal{L}_{X^M}$ be the exterior derivative, contraction and Lie derivative on $\Omega^*(M)$. Denote $\iota_X = \iota_{X^M}$ and $\mathcal{L}_X = \mathcal{L}_{X^M}$ acting on $\Omega^*(M)$.

Definition 1. *The Cartan model is defined by the algebra*

$$S(\mathfrak{g}^*) \otimes \Omega^*(M)$$

and the differential

$$d_C\phi^i = 0, \phi^i \in S(\mathfrak{g}^*), i = 1, \cdots, n;$$

$$d_C\eta = (1 \otimes d - \sum_{i=1}^{n} \phi^i \otimes (\iota_i + \sqrt{-1} f_i^j \iota_j))\eta, \eta \in \Omega^*(M) \otimes \mathbb{C},$$

where ι_i *is* ι_{e_i} *and* $f_i^j \in \mathbb{R}$*. The operator* d_C *is called the equivariant exterior derivative.*

Its action on forms $\alpha \in S(\mathfrak{g}^*) \otimes \Omega^*(M)$ is

$$(d_C\alpha)(X) = (d - \iota_{X^M} - \sqrt{-1}\iota_{Y^M})(\alpha(X))$$

where $X^M = c^i X_i^M$ is the vector field on M generated by the Lie algebra element $X = c^i e_i \in \mathfrak{g}$,$Y^M = f_j^i c^j X_i^M$(see [2]). In the artile [3] we use the operator $d + \iota_{X^M} + \sqrt{-1}\iota_{Y^M}$ to construct an complex $(\Omega^*(M) \otimes \mathbb{C}, d + \iota_{X^M} + \sqrt{-1}\iota_{Y^M})$ and cohomology group $H^*_{X+\sqrt{-1}Y}(M)$, we can do it in the same way by the operator $d - \iota_{X^M} - \sqrt{-1}\iota_{Y^M}$.

Lemma 3.

$$d_C^2 = -\sum_{i=1}^{n} \phi^i \otimes (\mathcal{L}_i + \sqrt{-1} f_i^j \mathcal{L}_j)$$

Proof. By the lemma 2. we have

$$\begin{aligned} d_C^2 &= (1 \otimes d - \sum_{i=1}^{n} \phi^i \otimes (\iota_i + \sqrt{-1} f_i^j \iota_j))(1 \otimes d - \sum_{i=1}^{n} \phi^i \otimes (\iota_i + \sqrt{-1} f_i^j \iota_j)) \\ &= -\sum_{i=1}^{n} \phi^i \otimes [d(\iota_i + \sqrt{-1} f_i^j \iota_j)) + (\iota_i + \sqrt{-1} f_i^j \iota_j))d] \\ &= -\sum_{i=1}^{n} \phi^i \otimes (\mathcal{L}_i + \sqrt{-1} f_i^j \mathcal{L}_j) \end{aligned}$$

□

Let $(S(\mathfrak{g}^*) \otimes \Omega^*(M))^{\widetilde{G}}$ be the subalgebra of $S(\mathfrak{g}^*) \otimes \Omega^*(M)$ which satisfied

$$(\sum_{i=1}^{n} \phi^i \otimes (\mathcal{L}_i + \sqrt{-1} f_i^j \mathcal{L}_j))\alpha = 0, \forall \alpha \in (S(\mathfrak{g}^*) \otimes \Omega^*(M))^{\widetilde{G}}$$

So we get the complex $((S(\mathfrak{g}^*) \otimes \Omega^*(M))^{\widetilde{G}}, d_C)$. The equivariantly closed form is $\forall \alpha \in (S(\mathfrak{g}^*) \otimes \Omega^*(M))^{\widetilde{G}}$ with $d_C\alpha = 0$, the equivariantly exact form is $\forall \alpha \in (S(\mathfrak{g}^*) \otimes \Omega^*(M))^{\widetilde{G}}$ there is $\beta \in (S(\mathfrak{g}^*) \otimes \Omega^*(M))^{\widetilde{G}}$ with $\alpha = d_C\beta$.

As in [8] we can define the equivariant connection

$$\nabla_{\mathfrak{g}} = 1 \otimes \nabla - \sum_{i=1}^{n} \phi^i \otimes (\iota_i + \sqrt{-1} f_i^j \iota_j)$$

and the equivariant curvature of the connection

$$F_{\mathfrak{g}} = (\nabla_{\mathfrak{g}})^2 + \sum_{i=1}^{n} \phi^i \otimes (\mathcal{L}_i + \sqrt{-1} f_i^j \mathcal{L}_j)$$

3 BRST model

This section is inspired by [5]. First, we will to construct the BRST differential algebra. The algebra is

$$B = W(\mathfrak{g}) \otimes \Omega^*(M).$$

The BRST operator is

$$\delta = d_W \otimes 1 + 1 \otimes d + \sum_{i=1}^{n} \theta^i \otimes (\mathcal{L}_i + \sqrt{-1} f_i^j \mathcal{L}_j) - \sum_{a=1}^{n} \phi^a \otimes (\iota_a + \sqrt{-1} f_a^b \iota_b) + \frac{1}{2} \sum_{j,k} c_{jk}^i \theta^j \theta^k \otimes (\iota_i + \sqrt{-1} f_i^j \iota_j)$$

$$- \sum_{j<k} \theta^j \theta^k \otimes ((\mathcal{L}_j + \sqrt{-1} f_j^h \mathcal{L}_h)(\iota_k + \sqrt{-1} f_k^g \iota_g) - (\iota_j + \sqrt{-1} f_j^h \iota_h)(\mathcal{L}_k + \sqrt{-1} f_k^g \mathcal{L}_g))$$

where $\mathcal{L}_i$ is $\mathcal{L}_{e_i}$ and ι_a is ι_{e_a}.

Lemma 4. *On the algebra $W(\mathfrak{g}) \otimes \Omega^*(M)$, we have $\delta^2 = 0$.*

Proof. By computation, we have

$$\delta = \exp(\sum_{i=1}^{n} \theta^i \otimes (\iota_i + \sqrt{-1} f_i^j \iota_j))(d_W \otimes 1 + 1 \otimes d) \exp(-\sum_{i=1}^{n} \theta^i \otimes (\iota_i + \sqrt{-1} f_i^j \iota_j))$$

where ι_a is ι_{e_a}. So we have

$$\delta^2 = \exp(\sum_{i=1}^{n} \theta^i \otimes (\iota_i + \sqrt{-1} f_i^j \iota_j))(d_W \otimes 1 + 1 \otimes d) \exp(-\sum_{i=1}^{n} \theta^i \otimes (\iota_i + \sqrt{-1} f_i^j \iota_j)) \cdot$$

$$\exp(\sum_{i=1}^{n} \theta^i \otimes (\iota_i + \sqrt{-1} f_i^j \iota_j))(d_W \otimes 1 + 1 \otimes d) \exp(-\sum_{i=1}^{n} \theta^i \otimes (\iota_i + \sqrt{-1} f_i^j \iota_j))$$

$$= \exp(\sum_{i=1}^{n} \theta^i \otimes (\iota_i + \sqrt{-1} f_i^j \iota_j))(d_W \otimes 1 + 1 \otimes d)^2 \exp(-\sum_{i=1}^{n} \theta^i \otimes (\iota_i + \sqrt{-1} f_i^j \iota_j))$$

$$= 0$$

□

So we get the BRST differential algebra $(W(\mathfrak{g}) \otimes \Omega^*(M), \delta)$.

Lemma 5. *Fixing the index i and k*

$$(\theta^i \otimes (\iota_i + \sqrt{-1} f_i^j \iota_j))(\theta^k \otimes (\iota_k + \sqrt{-1} f_k^l \iota_l)) = (\theta^k \otimes (\iota_k + \sqrt{-1} f_k^l \iota_l))(\theta^i \otimes (\iota_i + \sqrt{-1} f_i^j \iota_j))$$

Proof. If $i = k$, we have

$$(\theta^i \otimes (\iota_i + \sqrt{-1} f_i^j \iota_j))(\theta^k \otimes (\iota_k + \sqrt{-1} f_k^l \iota_l)) = 0 = (\theta^k \otimes (\iota_k + \sqrt{-1} f_k^l \iota_l))(\theta^i \otimes (\iota_i + \sqrt{-1} f_i^j \iota_j))$$

If $i \neq k$, then because

$$(\theta^i \otimes \iota_i)(\theta^k \otimes \iota_k) = -\theta^i \wedge \theta^k \otimes \iota_i \iota_k = -\theta^k \wedge \theta^i \otimes \iota_k \iota_i = (\theta^k \otimes \iota_k)(\theta^i \otimes \iota_i)$$

$$(\theta^i \otimes (\sqrt{-1} f_i^j \iota_j))(\theta^k \otimes \iota_k) = -\theta^i \wedge \theta^k \otimes (\sqrt{-1} f_i^j \iota_j) \iota_k = -\theta^k \wedge \theta^i \otimes \iota_k (\sqrt{-1} f_i^j \iota_j) = (\theta^k \otimes \iota_k)(\theta^i \otimes (\sqrt{-1} f_i^j \iota_j))$$

So we get the result. □

Let $\psi : W(\mathfrak{g}) \otimes \Omega^*(M) \to W(\mathfrak{g}) \otimes \Omega^*(M)$ be the map

$$\psi = \prod_i (1 - \theta^i \otimes (\iota_i + \sqrt{-1} f_i^j \iota_j)).$$

By computation

$$(1 - \theta^1 \otimes (\iota_1 + \sqrt{-1} f_1^j \iota_j))(1 - \theta^2 \otimes (\iota_2 + \sqrt{-1} f_2^j \iota_j)) \cdots (1 - \theta^n \otimes (\iota_n + \sqrt{-1} f_n^j \iota_j))$$

we have

$$\psi = \exp(-\sum_{i=1}^{n} \theta^i \otimes (\iota_i + \sqrt{-1} f_i^j \iota_j)).$$

In the section 5. we will discuss the map ψ.

4 Weil model

The exterior derivative operator on $W(\mathfrak{g}) \otimes \Omega^*(M)$ is defined by

$$D \doteq d_W \otimes 1 + 1 \otimes d,$$

the contraction operator is defined by

$$\widetilde{i}_X \doteq i_X \otimes 1 + 1 \otimes \iota_X$$

and Lie derivative be defined by

$$\widetilde{L}_X \doteq L_X \otimes 1 + 1 \otimes \mathcal{L}_X$$

Lemma 6. *The operators $\widetilde{i}_X, D, \widetilde{L}_X$ on $W(\mathfrak{g}) \otimes \Omega^*(M)$ satisfy the following identities:*

(1) $D^2 = 0$;

(2) $\widetilde{L}_X \cdot D - D \cdot \widetilde{L}_X = 0$, *for any* $X \in \mathfrak{g}$;

(3) $\widetilde{i}_X \widetilde{i}_Y + \widetilde{i}_Y \widetilde{i}_X = 0$, *for any* $X, Y \in \mathfrak{g}$;

(4) $\widetilde{L}_X \widetilde{i}_Y - \widetilde{i}_Y \widetilde{L}_X = \widetilde{i}_{[X,Y]}$, *for any* $X, Y \in \mathfrak{g}$;

(5) $\widetilde{L}_X \widetilde{L}_Y - \widetilde{L}_Y \widetilde{L}_X = \widetilde{L}_{[X,Y]}$, *for any* $X, Y \in \mathfrak{g}$;

(6) $\widetilde{L}_X = D \cdot \widetilde{i}_X + \widetilde{i}_X \cdot D$, *for any* $X \in \mathfrak{g}$.

Proof. see [4]. □

Set

$$\widetilde{i}_{X+\sqrt{-1}Y} \doteq i_X \otimes 1 + 1 \otimes (\iota_X + \sqrt{-1}\iota_Y)$$

be the contraction operator on $W(\mathfrak{g}) \otimes \Omega^*(M)$ induced by the contraction of $X + \sqrt{-1}Y$.

Set

$$\widetilde{L}_{X+\sqrt{-1}Y} \doteq L_X \otimes 1 + 1 \otimes (\mathcal{L}_X + \sqrt{-1}\mathcal{L}_Y)$$

be the Lie derivative on $W(\mathfrak{g}) \otimes \Omega^*(M)$ about $X + \sqrt{-1}Y$.

Lemma 7.
$$\widetilde{L}_{X+\sqrt{-1}Y} = D \cdot \widetilde{i}_{X+\sqrt{-1}Y} + \widetilde{i}_{X+\sqrt{-1}Y} \cdot D$$
for any $X, Y \in \mathfrak{g}$.

Proof.
$$\begin{aligned} D \cdot \widetilde{i}_{X+\sqrt{-1}Y} + \widetilde{i}_{X+\sqrt{-1}Y} \cdot D &= (d_W \otimes 1 + 1 \otimes d) \cdot \widetilde{i}_{X+\sqrt{-1}Y} + \widetilde{i}_{X+\sqrt{-1}Y} \cdot (d_W \otimes 1 + 1 \otimes d) \\ &= d_W i_X \otimes 1 + i_X d_W \otimes 1 + 1 \otimes d(\iota_X + \sqrt{-1}\iota_Y) + 1 \otimes (\iota_X + \sqrt{-1}\iota_Y)d \\ &= L_X \otimes 1 + 1 \otimes (\mathcal{L}_X + \sqrt{-1}\mathcal{L}_Y) \\ &= \widetilde{L}_{X+\sqrt{-1}Y} \end{aligned}$$

□

Definition 2. *An element* $\eta \in W(\mathfrak{g}) \otimes \Omega^*(M)$ *is* **basic** *if it satisfies* $\widetilde{i}_{X+\sqrt{-1}Y}\eta = 0$, $\widetilde{L}_{X+\sqrt{-1}Y}\eta = 0$ *for any* $X, Y \in \mathfrak{g}$. *Set* $(W(\mathfrak{g}) \otimes \Omega^*(M))_{bas}$ *be the set of basic elements.*

Lemma 8. *The operator* D *preserves* $(W(\mathfrak{g}) \otimes \Omega^*(M))_{bas}$.

Proof. Set $\eta \in (W(\mathfrak{g}) \otimes \Omega^*(M))_{bas}$, then $\widetilde{i}_{X+\sqrt{-1}Y}\eta = 0$ and $\widetilde{L}_{X+\sqrt{-1}Y}\eta = 0$ for any $X, Y \in \mathfrak{g}$. So by Lemma 7., we have
$$(\widetilde{i}_{X+\sqrt{-1}Y} \cdot D)\eta = \widetilde{i}_{X+\sqrt{-1}Y}(D\eta) = \widetilde{L}_{X+\sqrt{-1}Y}\eta - D(\widetilde{i}_{X+\sqrt{-1}Y}\eta) = 0$$
for any $X, Y \in \mathfrak{g}$.

And
$$\widetilde{L}_{X+\sqrt{-1}Y}(D\eta) = D(\widetilde{i}_{X+\sqrt{-1}Y} \cdot D)\eta + \widetilde{i}_{X+\sqrt{-1}Y}(D^2)\eta = 0$$
for any $X, Y \in \mathfrak{g}$.

Then we get
$$D\eta \in (W(\mathfrak{g}) \otimes \Omega^*(M))_{bas}.$$

□

Now we can construct the cohomology group as following:

By the complex $((W(\mathfrak{g}) \otimes \Omega^*(M))_{bas}, D)$, we can define the cohomology group as follow,
$$H^*_G(M) \doteq \frac{\mathrm{KerD}|_{(\mathrm{W}(\mathfrak{g})\otimes\Omega^*(\mathrm{M}))_{\mathrm{bas}}}}{\mathrm{ImD}|_{(\mathrm{W}(\mathfrak{g})\otimes\Omega^*(\mathrm{M}))_{\mathrm{bas}}}}.$$

Definition 3. *The cohomology group* $H^*_G(M)$ *is called the equivariant cohomology groups of* M. *The equivariant cohomology construct by this way is called* **Weil model**.

5 The main results

In this section we explain the precise relation between the Weil model and the Cartan model for equivariant cohomology defined earlier.

Theorem 1. ψ *is an isomorphism of differential algebra, i.e., the diagram*
$$\begin{array}{ccc} W(\mathfrak{g}) \otimes \Omega^*(M) & \xrightarrow{\psi} & W(\mathfrak{g}) \otimes \Omega^*(M) \\ \delta \downarrow & & \downarrow D \\ W(\mathfrak{g}) \otimes \Omega^*(M) & \xrightarrow[\psi]{} & W(\mathfrak{g}) \otimes \Omega^*(M) \end{array}$$
commutes.

Proof. By computation in lemma 4., we have

$$\delta = \psi \cdot D \cdot \psi^{-1}$$

□

Theorem 2. *We have the following commutative diagram:*

$$\begin{array}{ccc} (W(\mathfrak{g}) \otimes \Omega^*(M), \delta) & \xrightarrow{\psi} & (W(\mathfrak{g}) \otimes \Omega^*(M), D) \\ {\scriptstyle id}\uparrow & & \uparrow{\scriptstyle id} \\ (S(\mathfrak{g}^*) \otimes \Omega^*(M))^{\widetilde{G}} & \xrightarrow[\psi]{} & (W(\mathfrak{g}) \otimes \Omega^*(M))_{bas} \end{array}$$

Proof. For $\forall \alpha \in (S(\mathfrak{g}^*) \otimes \Omega^*(M))^{\widetilde{G}}$,by

$$\prod_a (1 - \theta^a \otimes (\iota_a + \sqrt{-1} f_a^b \iota_b)) \cdot (i_k \otimes 1) = (i_k \otimes 1 + 1 \otimes (\iota_k + \sqrt{-1} f_k^j \iota_j)) \cdot \prod_a (1 - \theta^a \otimes (\iota_a + \sqrt{-1} f_a^b \iota_b))$$

we have

$$(i_k \otimes 1 + 1 \otimes (\iota_k + \sqrt{-1} f_k^j \iota_j))(\psi(\alpha)) = 0.$$

Because

$$[\delta, i_k \otimes 1] = L_k \otimes 1 + 1 \otimes (\mathcal{L}_k + \sqrt{-1} f_k^j \mathcal{L}_j)$$

and

$$\prod_a (1 - \theta^a \otimes (\iota_a + \sqrt{-1} f_a^b \iota_b)) \cdot (L_k \otimes 1 + 1 \otimes (\mathcal{L}_k + \sqrt{-1} f_k^j \mathcal{L}_j))$$
$$= (L_k \otimes 1 + 1 \otimes (\mathcal{L}_k + \sqrt{-1} f_k^j \mathcal{L}_j)) \cdot \prod_a (1 - \theta^a \otimes (\iota_a + \sqrt{-1} f_a^b \iota_b))$$

so we have

$$(L_k \otimes 1 + 1 \otimes (\mathcal{L}_k + \sqrt{-1} f_k^j \mathcal{L}_j))(\psi(\alpha)) = 0$$

Then we get $\psi(\alpha) \in (W(\mathfrak{g}) \otimes \Omega^*(M))_{bas}$. So we get the commutative diagram. □

The theorem 2. tell us the relation about BRST model and Cartan model.

Theorem 3.

$$(S(\mathfrak{g}^*) \otimes \Omega^*(M))^{\widetilde{G}} \xrightarrow{\psi} (W(\mathfrak{g}) \otimes \Omega^*(M))_{bas}$$

is a isomorphism.

Proof. For $\forall \eta \in (W(\mathfrak{g}) \otimes \Omega^*(M))_{bas}$, $\psi^{-1}\eta = \prod_a (1 + \theta^a \otimes (\iota_a + \sqrt{-1} f_a^b \iota_b))\eta$. By

$$\prod_a (1 + \theta^a \otimes (\iota_a + \sqrt{-1} f_a^b \iota_b))|_{(W(\mathfrak{g}) \otimes \Omega^*(M))_{bas}} = \prod_a (1 - \theta^a i_a \otimes 1)|_{(W(\mathfrak{g}) \otimes \Omega^*(M))_{bas}}$$

and

$$Im(1 - \theta^a i_a \otimes 1) = Ker(i_a \otimes 1)$$

So

$$\psi^{-1}\eta \in (S(\mathfrak{g}^*) \otimes \Omega^*(M))_{bas}.$$

Then

$$(\sum_{i=1}^n \phi^i \otimes (\mathcal{L}_i + \sqrt{-1} f_i^j \mathcal{L}_j))\psi^{-1}\eta = 0$$

i.e.,$\psi^{-1}\eta \in (S(\mathfrak{g}^*) \otimes \Omega^*(M))^{\widetilde{G}}$. And by the proof in theorem 2.we get that ψ is a isomorphism.

□

The theorem 3. tell us the relation about Cartan model and Weil model.

References

[1] M. Atiyah, R. Bott. The moment map and equivariant cohomology. Topology 23, 1, 1984.

[2] N. Berline, E. Getzler and M. Vergne. *Heat Kernels and Dirac Operators*. Germany: Springer-Verlag, 1992.

[3] X. Chen. Localization formulas about two Killing vector Fields. http://arxiv.org/abs/1304.3806v1

[4] V. W. Guillemin and S. Sternberg. *Supersymmetry and equivariant de Rham theory.* Springer-Verlag, Berlin, 1999.

[5] J. Kalkman. BRST Model for Equivariant Cohomology and Representatives for the Equivariant Thom Class. Commun. Math. Phys., 153: 447-463, 1993.

[6] S. Kobayashi and K. Nomizu. *Foundations of differential geometry.* Vol. I. John Wiley and Sons, New York(1963).

[7] V. Mathai, D. Quillen. Thom classes, superconnections and equivariant differential forms. Topology 25, 85, 1986.

[8] Richard J. Szabo. *Equivariant cohomology and localzition of Path Integrals*. Springer-Verlag, Berlin, 2000.