The topology on a complete semilattice

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Abstract

We define the topology $atop(\chi)$ on a complete upper semilattice $\chi = (M, \leq)$. The limit points are determined by the formula

$$\lim_D(X) = sup\{a \in M| \, \{x \in X| \, a \leq x\} \in D\},$$

where $X \subseteq M$ is an arbitrary set, D is an arbitrary non-principal ultrafilter on X. We investigate $\lim\limits_D(X)$ and topology $atop(\chi)$ properties. In particular, we prove the compactness of the topology $atop(\chi)$.

1. Preliminaries

For any set X we use $P(X)$ to denote the set of all subsets of X. For an arbitrary collection S of sets we use $\cup S$ and $\cap S$ to denote the union and the intersection of all sets of S respectively.

A cardinal will be identified with the corresponding lowest ordinal. The cardinality (size) of a set will be identified with the corresponding cardinal. Example: $|\omega| = \omega = \omega_0$. We assume the axiom of choice.

Let D be an ultrafilter on X and $Y \in D$. $D|_Y$ is the ultrafilter on Y, where $D|_Y = \{Z \cap Y \,|\, Z \in D\}$.

A principal ultrafilter is an ultrafilter containing a least element. An ultrafilter is a non-principal, if it does not contain finite sets.

A complete upper semilattice is a partially ordered set in which every not empty subset has a least upper bound (sup). We assume that any complete upper semilattice of this article have the infimum element (zero).

A complete lattice is a partially ordered set in which every subset has a least upper bound (sup) and a greatest lower bound (inf).

2. The limit $\lim\limits_{D}(X)$ and the associated topology

Definition 1. Let $\chi = (M, \leq)$ be a complete upper semilattice, $X \subseteq M$, $X \neq \emptyset$ and let D be an arbitrary ultrafilter on X. We denote

$$\lim_{D}(X) = sup\{a \in M | \, \{x \in X | \, a \leq x\} \in D\},$$

Definition 2. Let $\chi = (M, \leq)$ be a complete upper semilattice. A set $\Delta \subseteq M$ is **an approximation base**, if for every $x \in M$ we have

$$X = sup\{\alpha \in \Delta | \, \alpha \leq x\}.$$

Definition 3. Let $\chi = (M, \leq)$ be a complete upper semilattice, let Δ be an approximation base, $X \subseteq M$, $X \neq \emptyset$ and let D be an arbitrary ultrafilter on X. We denote

$$lim_1(\Delta, D, X) = sup\{\alpha \in \Delta | \, \{x \in X | \, \alpha \leq x\} \in D\}$$

.

Remark 1. Definitions 1,2,3 are correct, because in accordance with the agreement, each complete upper semilattice of this article has the infimum element (zero) $inf(M) \in M$. Thus, $inf(M) \leq x$ for any $x \in M$.

The approximation base $\Delta = M$ is associated with every complete upper semilattice $\chi = (M, \leq)$. In this case we have

$$\lim_{D}(X) \equiv lim_1(\Delta, D, X)$$

Definition 4. Let $\chi = (M, \leq)$ be a complete lattice, $X \subseteq M$, $X \neq \emptyset$ and let D be an arbitrary ultrafilter on X. We denote

$$lim_2(D, X) = sup\{inf(Y) | \, Y \in D\},$$

$$lim_3(D, X) = inf\{sup(Y) | \, Y \in D\}.$$

Remark 2. Let $\chi = (M, \leq)$ be a complete lattice and $\chi^* = (M, \leq^*)$, where $\leq^*$ coincide with $\geq$. Let $X \subseteq M$, $X \neq \emptyset$ and let D be an arbitrary ultrafilter on X. We see that χ^* is a complete semilattice and

$$lim_2(D, X) = lim_3^*(D, X),$$

$$lim_3(D, X) = lim_2^*(D, X),$$

where $lim_2^*(D,X), lim_3^*(D,X)$ are correspond to $lim_2(D,X), lim_3(D,X)$ for lattice χ^*.

Proposition 1. Let $\chi = (M, \leq)$ be a complete upper semilattice, let Δ be an approximation base, $X \subseteq M$, $X \neq \emptyset$ and let D be an ultrafilter on X. We have

$$\lim_D(X) = lim_1(\Delta, D, X).$$

Proof. Since $\lim_D(X) \equiv lim_1(M, D, X)$, it is sufficiently to prove that $lim_1(\Delta, D, X) = lim_1(M, D, X)$. It is easy to see that

$$lim_1(\Delta, D, X) \leq lim_1(M, D, X).$$

Let $a_0 \in \{a \in M | \{x \in X | a \leq x\} \in D\}$. Since Δ is an approximation base,

$$a_0 = sup\{\delta \in \Delta | \delta \leq a_0\}.$$

Let $\delta_0 \in \{\delta \in \Delta | \delta \leq a_0\}$, i.e. $\delta_0 \leq a_0$ and

$$\{x \in X | a_0 \leq x\} \subseteq \{x \in X | \delta_0 \leq x\}.$$

Since $\{x \in X | a_0 \leq x\} \in D$, we have $\{x \in X | \delta_0 \leq x\} \in D$ and

$$\delta_0 \in \{\delta \in \Delta | \{x \in X | \delta \leq x\} \in D\},$$

$$\{\delta \in \Delta | \delta \leq a_0\} \subseteq \{\delta \in \Delta | \{x \in X | \delta \leq x\} \in D\},$$

$$sup\{\delta \in \Delta | \delta \leq a_0\} \leq sup\{\delta \in \Delta | \{x \in X | \delta \leq x\} \in D\},$$

$$a_0 \leq sup\{\delta \in \Delta | \{x \in X | \delta \leq x\} \in D\}.$$

Since $a_0 \in \{a \in M | \{x \in X | a \leq x\} \in D\}$ is arbitrary, we have

$$sup\{a \in M | \{x \in X | a \leq x\} \in D\} \leq sup\{\delta \in \Delta | \{x \in X | \delta \leq x\} \in D\},$$

$$lim_1(M, D, X) \leq lim_1(\Delta, D, X).$$

Proposition 1 is proved.

Remark 3. From Proposition 1 it follows that $lim_1(\Delta, D, X)$ does not depend on Δ actually. But we will use it in the future, because it is easier to prove certain properties associated with $\lim_D(X)$.

Proposition 2. Let $\chi = (M, \leq)$ be a complete lattice, let Δ be an approximation base, $X \subseteq M$, $X \neq \emptyset$ and let D be an ultrafilter on X. We have

$$\lim_D(X) = lim_1(\Delta, D, X) = lim_2(D, X).$$

Proof. First equation $\lim_D(X) = lim_1(\Delta, D, X)$ follows from Proposition 1. It is necessary to prove that

$$sup\{inf(Y)|\, Y \in D\} = sup\{\alpha \in \Delta|\, \{x \in X|\, \alpha \leq x\} \in D\}.$$

Let $Y \in D$. Since Δ is an approximation base,

$$inf(Y) = sup\{\alpha \in \Delta|\, \alpha \leq inf(Y)\}.$$

Let $\alpha_0 \in \{\alpha \in \Delta|\, \alpha \leq inf(Y)\}$, $X_0 = \{x \in X|\, \alpha_0 \leq x\}$ then $\alpha_0 \leq inf(Y)$ and $Y = \{y \in Y|\, \alpha_0 \leq y\} \subseteq \{x \in X|\, \alpha_0 \leq x\} = X_0$. Since $Y \in D$ and $Y \subseteq X_0$ then $X_0 \in D$. We see that

$$\alpha_0 \in \{\alpha \in \Delta|\, \{x \in X|\, \alpha \leq x\} \in D\},$$

$$\{\alpha \in \Delta|\, \alpha \leq inf(Y)\} \subseteq \{\alpha \in \Delta|\, \{x \in X|\, \alpha \leq x\} \in D\},$$

$$inf(Y) = sup\{\alpha \in \Delta|\, \alpha \leq inf(Y)\} \leq sup\{\alpha \in \Delta|\, \{x \in X|\, \alpha \leq x\} \in D\},$$

$$sup\{inf(Y)|\, Y \in D\} \leq sup\{\alpha \in \Delta|\, \{x \in X|\, \alpha \leq x\} \in D\}.$$

Let us prove the opposite direction.
Let $\alpha_0 \in \{\alpha \in \Delta|\, \{x \in X|\, \alpha \leq x\} \in D\}$ then

$$\{x \in X|\, \alpha_0 \leq x\} \in D,$$

$$\alpha_0 \leq \inf\{x \in X|\, \alpha_0 \leq x\} \leq sup\{inf(Y)|\, Y \in D\},$$

$$sup\{\alpha \in \Delta|\, \{x \in X|\, \alpha \leq x\} \in D\} \leq sup\{inf(Y)|\, Y \in D\}.$$

Proposition 2 is proved.

Lemma 1. Let $\chi = (M, \leq)$ be a complete upper semilattice, $X \subseteq M$, $X \neq \emptyset$, let D be an ultrafilter on X. The following are

1) if $\{a\} \in D$ then $\lim_D(X) = a$;

2) if $Y \in D$ then $\lim_D(X) = \lim_F(Y)$, where $F = D|_Y$,
D is a principal ultrafilter $\Leftrightarrow$ F is a principal ultrafilter;

3) if $Z \subseteq M$ and $X \subseteq Z$ then exist an ultrafilter G on Z that $X \in G$, $G|_X = D$, $\lim_D(X) = \lim_G(Z)$,
D is a principal ultrafilter $\Leftrightarrow$ G is a principal ultrafilter;

4) if $X \subseteq M$ is an infinite set, D is a non-principal ultrafilter on X then for any finite set $X' \subseteq X$ we have $\lim_F(X \setminus X') = \lim_D(X)$, where $F = D|_{X \setminus X'}$ is a non-principal ultrafilter.

Proof. Let $\Delta = M$. We prove 1). If $\{a\} \in D$ then for any $Y \in D$ we have $\{a\} \cap Y \neq \emptyset$, $a \in Y$. Hence for any $\delta \in \Delta$ it is

$$\delta \leq a \Leftrightarrow \{x \in X|\, \delta \leq x\} \in D,$$

i.e.

$$a = sup\{\delta \in \Delta |\, \delta \leq a\} = sup\{\delta \in \Delta |\, \{x \in X |\, \delta \leq x\} \in D\} =$$
$$= lim_1(\Delta, D, X) = \lim_D(X).$$

The last equation follows from Proposition 1.

We prove 2). We prove that for every
$\alpha \in \Delta$ it is

$$\{x \in X |\, \alpha \leq x\} \in D \Leftrightarrow \{y \in Y |\, \alpha \leq y\} \in F. \tag{1}$$

Let $\{x \in X |\, \alpha \leq x\} \in D$. Since $Y \subseteq X$,

$$\{y \in Y |\, \alpha \leq y\} = Y \cap \{x \in X |\, \alpha \leq x\} \in D|_Y = F.$$

We prove in the opposite direction. Let $Y_0 = \{y \in Y |\, \alpha \leq y\} \in F$. We suppose the opposite that $\{x \in X |\, \alpha \leq x\} \notin D$, i.e. $\{x \in X |\, \alpha \not\leq x\} \in D$ and

$$Y \setminus Y_0 = \{y \in Y |\, \alpha \not\leq y\} = Y \cap \{x \in X |\, \alpha \not\leq x\} \in D|_Y = F.$$

We obtain the contradictory $Y_0 \in F, Y \setminus Y_0 \in F$. We conclusion that $\{x \in X |\, \alpha \leq x\} \in D$. We have proved the statement (1). From (1) it follows

$$\{\alpha \in \Delta | \{x \in X |\, \alpha \leq x\} \in D\} = \{\alpha \in \Delta | \{y \in Y |\, \alpha \leq y\} \in F\},$$

$$sup\{\alpha \in \Delta | \{x \in X |\, \alpha \leq x\} \in D\} = sup\{\alpha \in \Delta | \{y \in Y |\, \alpha \leq y\} \in F\},$$
$$lim_1(\Delta, D, X) = lim_1(\Delta, F, Y),$$
$$\lim_D(X) = \lim_F(Y).$$

As $F \subseteq D$, if F contains a finite set then D also contains a finite set. On the other hand, if D contains a finite set Z then ultrafilter F contains a finite set $Z \cap Y$. This implies that D is a principal ultrafilter $\Leftrightarrow$ F is a principal ultrafilter.

We prove 3). Let $G = \{Y \subseteq Z | Y \cap X \in D\}$. G is an ultrafilter obviously, $G|_X = D$, $X \in G$. The remaining assertions of the item 3) follow from the item 2).

Let us prove 4). Let D be a non-principal ultrafilter, i.e. $X \setminus X' \in D$. Let $F = D|_{X \setminus X'}$. The ultrafilter F with respect to the item 2) is a non-principal. According to 2),

$$\lim_F(X \setminus X') = \lim_D(X).$$

Definition 3. Let $\chi = (M, \leq)$ be a complete upper semilattice. We define the operation $C()$ on the subsets of M. Let $X \subseteq M$, we define

$$C(X) = X \cup \{\lim_D(X) |\, D \,\text{is a non-principal ultrafilter on X}\}. \tag{2}$$

Lemma 2. Let $\chi = (M, \leq)$ be a complete upper semilattice. The operation $C()$ defind by (2) has the following properties:
1) $C(X_1 \cup X_2) = C(X_1) \cup C(X_2)$, where $X_1, X_2 \subseteq M$;
2) $X \subseteq C(X)$, where $X \subseteq M$;
3) if X is finite then $C(X) = X$, where $X \subseteq M$ (particulaly, $C(\emptyset) = \emptyset$);
4) if $X \subseteq Y$ then $C(X) \subseteq C(Y)$ for all $X, Y \subseteq M$.

Proof. Assertions 2) and 3) follow from the definition of the operation $C()$ and Lemma 1.

We prove 1). If X_1 is a finite set then

$$C(X_1) = X_1 \subseteq X_1 \cup X_2 \subseteq C(X_1 \cup X_2).$$

Assume that X_1 is an infinite set. Let D_1 be an arbitrary non-principal ultrafilter on X_1. By Lemma 1 there exists a non-principal ultrafilter D on $X_1 \cup X_2$ that $X_1 \in D, D|_{X_1} = D_1$ and $\lim\limits_{D}(X_1 \cup X_2) = \lim\limits_{D_1}(X_1)$. We see that $C(X_1) \subseteq C(X_1 \cup X_2)$. Similarly $C(X_2) \subseteq C(X_1 \cup X_2)$, i.e.

$$C(X_1) \cup C(X_2) \subseteq C(X_1 \cup X_2).$$

Let us prove the item 1) in the opposite direction. Let D be an arbitrary non-principal ultrafilter on $X_1 \cup X_2$. It is obvious that $X_1 \in D$ or $X_2 \setminus X_1 \in D$. If $X_2 \setminus X_1 \in D$ then $X_2 \in D$. Hence it is $X_1 \in D$ or $X_2 \in D$. Let $X_1 \in D$ for definiteness. From Lemma 1 we have $\lim\limits_{D}(X_1 \cup X_2) = \lim\limits_{D_1}(X_1)$, where $D_1 = D|_{X_1}$. From this it follows that

$$C(X_1 \cup X_2) \subseteq C(X_1) \cup C(X_2).$$

The item 1) of the Lemma is proved.

Let us prove the item 4). If X is finite then $C(X) = X \subseteq Y \subseteq C(Y)$. Suppose that X is infinite. Let D be an arbitrary non-principal ultrafilter on X. From Lemma 1 we have that there is a non-principal ultrafilter G on Y that $X \in G$, $G|_X = D$, $\lim\limits_{D}(X) = \lim\limits_{G}(Y)$, i.e. $C(X) \subseteq C(Y)$. The item 4) of the the Lemma is proved.

Lemma 3. Let $\chi = (M, \leq)$ be a complete upper semilattice. The set of all sets such that $X \subseteq M$, $C(X) = X$ (we assume that X is closed) is a topology.

Proof. Let $R = \{X \subseteq M | C(X) = X\}$. Obviously $\emptyset, X \in R$. It is sufficient to prove:
1) there is $\cup P \in R$ for every finite $P \subseteq R$;
2) there is $\cap P \in R$ for each $P \subseteq R$.

The associativity of the union of a sets implies that property 1) suffices to prove for two sets in R. Let $X_1, X_2 \in R$. From Lemma 2 it follows that $C(X_1 \cup X_2) = C(X_1) \cup C(X_2) = X_1 \cup X_2$. That is $X_1 \cup X_2 \in R$. The property 1) is proved.

We prove the property 2). Let $P \subseteq R$. We consider an arbitrary set $Y \in P$. It is obviously that $\cap P \subseteq Y$. We obtain by Lemma 2 that $C(\cap P) \subseteq C(Y) = Y$, i.e. $C(\cap P) \subseteq Y$. As $Y \in P$ is arbitrary, we have $C(\cap P) \subseteq \cap P$. Since $\cap P \subseteq C(\cap P)$, we have $C(\cap P) = \cap P$, i.e. $\cap P \in R$.

Definition 4. The topological space is defined by the Lemma 3 will be denoted by $atop(\chi)$.

Lemma 4. Let $\chi = (M, \leq)$ is a complete upper semilattice, κ is an infinite cardinal and $(X_\lambda)_{\lambda<\kappa}$ is a not increasing sequence of closed in the topology $atop(\chi)$ sets, i.e. $X_\lambda \supseteq X_{\lambda+1}, C(X_\lambda) = X_\lambda$ for all ordinals $\lambda < \kappa$. If $X_\lambda \neq \emptyset$ for all $\lambda < \kappa$ then this sequence has a non-empty intersection.

Proof. If κ is't a regular cardinal then we can choose some subsequence of the size of the regular cardinal $cf(\kappa)$ (cofinality κ) that for any ordinal $\lambda_0 < \kappa$ there is an ordinal $\lambda > \lambda_0$ corresponding to the element of the selected subsequence. It is clear that the intersection of the original sequence and the selected subsequence are the some. Therefore, we can assume without loss of generality that κ is a regular cardinal.

Let $h : \kappa \to M$ is an arbitrary mapping for which $h(\lambda) \in X_\lambda$ for all $\lambda < \kappa$.

If $|Rang(h)| < \kappa$ then (in accordance with the regularity of κ) there is an element $a \in Rang(h)$ that $|h^{-1}(a)| = \kappa$. For any ordinal $\lambda_0 < \kappa$ there is an ordinal λ that $\lambda_0 < \lambda < \kappa$ and $h(\lambda) = a$. This means that $a \in X_\lambda$ for all $\lambda < \kappa$, i.e. the sequence $(X_\lambda)_{\lambda<\kappa}$ has a non empty intersection.

Let us assume that $|Rang(h)| = \kappa$. Let D be a non-principal ultrafilter on the set $Rang(h)$ that if $Z \in D$ then $|Z| = \kappa$. We consider an arbitrary ordinal $\lambda_0 < \kappa$. Let $E = \{h(\lambda) | \lambda < \lambda_0\}$. It is obvious that the $Rang(h) \setminus E \in D$. According to Lemma 1, we have

$$\lim_D (Rang(h)) = \lim_F (Rang(h) \setminus E),$$

where $F = D|_{Rang(h) \setminus E}$. We note that $Rang(h) \setminus E \subseteq X_{\lambda_0}$. By lemma 1 there is a non-principal ultrafilter G on the set X_{λ_0} that

$$\lim_G (X_{\lambda_0}) = \lim_F (Rang(h) \setminus E).$$

Since X_{λ_0} is a close set, $C(X_{\lambda_0}) = X_{\lambda_0}$ and

$$\lim_D (Rang(h)) = \lim_F (Rang(h) \setminus E) = \lim_G (X_{\lambda_0}) \in X_{\lambda_0}.$$

Since $\lambda_0 < \kappa$ is an arbitrary ordinal,

$$\lim_D (Rang(h)) \in \bigcap_{\lambda<\kappa} X_\lambda$$

Lemma is proved.

Theorem 1. Let $\chi = (M, \leq)$ be a complete upper semilattice then the topological space $atop(\chi)$ is compact.

Proof. We prove this theorem in two ways.

1) By Lemma 2, every point of $atop(\chi)$ is a close set, i.e. $atop(\chi)$ is a T_1 space. Lemma 4 implies that every well ordered sequence of non-empty closed decreasing sets is non-empty intersection. The theorem (Alexandrov P. S. Uryson P.S.,[1, p.26]) for T_1 topological spaces implies that if every well ordered sequence of non-empty closed decreasing sets is non-empty intersection then the topology is a compact. Thus, $atop(\chi)$ is a compact topology.

2) The second proof uses the methods of the proof of the existence of a finite subcovering of a countable cover of a countably compact (with the modern interpretation) topological space (F. Hausdorff [3, p.141]). In this proof we construct a finite subcovering for a arbitrary covering. This proof is longer, but it is useful for future analysis.

We will show that each open covers of $atop(\chi)$ has a subcovering of a smaller cardinality. Let κ be an infinite cardinal and let $(G_\lambda)_{\lambda<\kappa}$ be an open covering of the cardinality κ. Hence $(F_\lambda)_{\lambda<\kappa}$, where $F_\lambda = G_0 \cup ... \cup G_\lambda$, is a non-decreasing sequence of open sets, which is covered. We consider the corresponding sequence of the close sets $M \setminus F_0 \supseteq ... \supseteq M \setminus F_\lambda \supseteq ...$, where $\lambda < \kappa$. This sequence has empty intersection. By Lemma 4 there exists an ordinal $\lambda_0 < \kappa$ that $M \setminus F_\lambda = \emptyset$ for all $\lambda < \kappa$, which $\lambda > \lambda_0$.

We see that the set of open sets $G_0, ..., G_{\lambda_0}$ is covered and the cardinality of the cover is $|\lambda_0| < \kappa$ ($|\lambda_0 + 1|$ in finite case). If $|\lambda_0|$ is an infinite cardinal then we can repeat the same procedure and we can get an open covering of a cardinality less than

$|\lambda_0|$ and so on. Thus for a finite number of steps we can get a finite covering.

Theorem 2. Let $\chi = (M, \leq)$ be a complete upper semilattice, Δ be an approximation base, $|\Delta| = \omega_0$, $X \subseteq M$ be an infinite, D be a non-principal ultrafilter on X. There is a countable subset $X_0 \subseteq X$ that

$$\lim_D(X) = sup(\Delta_0) = \lim_{D_0}(X_0),$$

where

$$\Delta_0 = \{\alpha \in \Delta | \{x \in X_0 | \alpha \not\leq x\} \text{ is finite}\},$$

$$\Delta_1 = \Delta \setminus \Delta_0 = \{\alpha \in \Delta | \{x \in X_0 | \alpha \leq x\} \text{ is finite}\},$$

D_0 is an arbitrary non-principal ultrafilter on X_0.

Proof.

For anyone $\alpha \in \Delta$ we denote

$$X_\alpha = \{x \in X | \, \alpha \leq x\},$$

$$\bar{X}_\alpha = \{x \in X | \, \alpha \not\leq x\},$$

$$\Delta_0 = \{\alpha \in \Delta | \, X_\alpha \in D\},$$

$$\Delta_1 = \{\alpha \in \Delta | \, \bar{X}_\alpha \in D\}.$$

It is obvious that

$$\lim_D(X) = lim_1(\Delta, D, X) = sup(\Delta_0),$$

$$\Delta_0 \cap \Delta_1 = \emptyset, \Delta_0 \cup \Delta_1 = \Delta.$$

Let the sequence $\alpha_0, \alpha_1, \alpha_2, \dots$ be a list of all elements of Δ. We define the sequence $X^{(0)}, X^{(1)}, X^{(2)}, \dots$ by the induction.
1) If $\alpha_0 \in \Delta_0$ then $X^{(0)} = X_{\alpha_0}$, if $\alpha_0 \in \Delta_1$ then $X^{(0)} = \bar{X}_{\alpha_0}$.
2) If $X^{(i)}$ is determined then
if $\alpha_{i+1} \in \Delta_0$ then

$$X^{(i+1)} = X^{(i)} \cap X_{\alpha_{i+1}},$$

if $\alpha_{i+1} \in \Delta_1$ then

$$X^{(i+1)} = X^{(i)} \cap \bar{X}_{\alpha_{i+1}}.$$

We note that for all $i \in N$ we have

$$X^{(i+1)} \subseteq X^{(i)} \subseteq X, X^{(i)} \in D.$$

If $\alpha_i \in \Delta_0$ then $\alpha_i \leq x$ for all $x \in X^{(i)}$. If $\alpha_i \in \Delta_1$ then $\alpha_i \not\leq x$ for all $x \in X^{(i)}$.

Since $X^{(i)}$ is infinite set, we can construct a sequence $x_0, x_1, x_2...$ of X for which $x_i \in X^{(i)}$, $x_i \neq x_j$ for all $i \neq j$. Denote X_0 the set of all elements of the sequence. Obviously $|X_0| = \omega_0$.

Let $n \in N$. If $\alpha_n \in \Delta_0$ then $\alpha_n \leq x_m$ for all $m \geq n$. If $\alpha_n \in \Delta_1$ then $\alpha_n \not\leq x_m$ for all $m \geq n$. We see that for any $\alpha \in \Delta_0$ the set $\{x \in X_0 | \, \alpha \not\leq x\}$ is finite, for any $\alpha \in \Delta_1$ the set $\{x \in X_0 | \, \alpha \leq x\}$ is finite.

Since $\Delta_0 \cap \Delta_1 = \emptyset, \Delta_0 \cup \Delta_1 = \Delta$, we have

$$\Delta_0 = \{\alpha \in \Delta | \, \{x \in X_0 | \, \alpha \not\leq x\} \text{ is finite}\},$$

$$\Delta_1 = \{\alpha \in \Delta | \, \{x \in X_0 | \, \alpha \leq x\} \text{ is finite}\}.$$

Let D_0 be an arbitrary non-principal ultrafilter on X_0.

Obviously if $\alpha \in \Delta_0$ then

$$\{x \in X_0 | \, \alpha \leq x\} \in D_0,$$

if $\alpha \in \Delta_1$ then

$$\{x \in X_0 |\, \alpha \nleq x\} \in D_0,$$

i.e

$$\Delta_0 = \{\alpha \in \Delta |\, \{x \in X_0 |\, \alpha \leq x\} \in D_0\},$$

$$\Delta_1 = \{\alpha \in \Delta |\, \{x \in X_0 |\, \alpha \nleq x\} \in D_0\}.$$

Thus we have

$$\lim_D(X) = sup(\Delta_0) = sup\{\alpha \in \Delta |\, \{x \in X_0 |\, \alpha \nleq x\} \text{ is finite}\} =$$

$$= sup\{\alpha \in \Delta |\, \{x \in X_0 |\, \alpha \leq x\} \in D_0\} = \lim_{D_0}(X_0).$$

The theorem is proved.

We generalize the concept of distributivity

$$(a_1 \vee a_2) \wedge (b_1 \vee b_2) = (a_1 \wedge b_1) \vee (a_1 \wedge b_2) \vee (a_2 \wedge b_1) \vee (a_2 \wedge b_2),$$

$$(a_1 \wedge a_2) \vee (b_1 \wedge b_2) = (a_1 \vee b_1) \wedge (a_1 \vee b_2) \wedge (a_2 \vee b_1) \wedge (a_2 \vee b_2)$$

to arbitrary set of elements.

Definition 5. Let $\chi = (M, \leq)$ be a complete lattice. We assume that χ is **a generalized infinite distributive** if

$$inf\{sup(s(i)) |\, i \in I\} =$$

$$= sup\{inf(Rang(f) |\, f : I \to M \text{ is a function, that } f(i) \in s(i), i \in I\},$$

$$sup\{inf(s(i)) |\, i \in I\} =$$

$$= inf\{sup(Rang(f) |\, f : I \to M \text{ is a function, that } f(i) \in s(i), i \in I\}.$$

where I is an arbitrary set of indexes, $s : I \to P(M)$ is an arbitrary function.

Remark 4. If $I = \{0, 1\}, s(0) = \{x\}, s(1) = X$, where $x \in M, X \subseteq M$, then the generalized infinite destributuvity coincides with infinite destributuvity, i.e.

$$inf\{x, sup(X)\} = sup\{inf\{x, y\} | y \in X\},$$

$$sup\{x, inf(X)\} = inf\{sup\{x, y\} | y \in X\}.$$

Theorem 3. Let $\chi = (M, \leq)$ be a generalized infinitely distributive complete lattice. In this case, the operation $C(X)$ satisfies the axioms of closure for a topological space:

1) $C(X_1 \cup X_2) = C(X_1) \cup C(X_2)$,

2) $X \subseteq C(X)$,
3) $C(C(X)) \subseteq C(X)$,
4) $C(\emptyset) = \emptyset$,
where $X, X_1, X_2 \subseteq M$.

Proof.

The properties 1),2),4) were proved in Lemma 2. We will prove the property 3).

Let D be an arbitrary non-principal ultrafilter on $C(X)$ and

$$\lim_D (C(X)) = a,$$

where $a \in M$. We will prove that $a \in C(X)$. For this purpose, it is sufficient to determine a non-principal ultrafilter G on X that

$$a = \lim_G (X).$$

Since D is an ultrafilter on $C(X)$ then $X \in D$ or $C(X) \setminus X \in D$.

If $X \in D$ then we define $G = D|_X$. By Lemma 1, we have

$$\lim_D (C(X)) = \lim_G (X) \in C(X).$$

In this case, the theorem is proved.

We consider the case $C(X) \setminus X \in D$. We define $D_0 = D|_{X_0}$, where $X_0 = C(X) \setminus X$. By Lemma 1, we have

$$\lim_{D_0} (X_0) = \lim_D (C(X)) = a.$$

For any $b \in X_0$, we define a non-principal ultrafilter $F(b)$ on X that

$$\lim_{F(b)} (X) = b.$$

We define

$$R = \{K \subseteq X | \, Z \in D_0, K \in F(b) \, \text{for any} \, b \in Z\}.$$

Since $X \in R$, we have $R \neq \emptyset$. Since $\emptyset \notin F(b)$ for any $b \in X_0$, we have $\emptyset \notin R$.

We will show that R is a non-principal ultrafilter on X.

Let $K_1, ...K_n \in R, n \in N$. There are $Z_1, \ldots, Z_n \in D_0$ that $K_i \in F(b)$ for any $b \in Z_i$, $i = 1, ..., n$. Since $D_0, F(b)$ are ultrafilters, we have $Z_1 \cap ... \cap Z_n \in D_0$, $K_1 \cap ... \cap K_n \in F(b)$ for any $b \in Z_1 \cap ... \cap Z_n$. We have $K_1 \cap ... \cap K_n \in R$.

Let $K \in R, K \subseteq K' \subseteq X$. There is $Z \in D_0$ that $K \in F(b)$ for any $b \in Z$. Since $F(b)$ is ultrafilter, we have $K' \in F(b)$ for any $b \in Z$. We have $K' \in R$.

Let $K \subseteq X$ and $b \in X_0$. We have $K \in F(b)$ or $X \setminus K \in F(b)$. We define $Z = \{b \in X_0 | K \in F(b)\}$. We have $X_0 \setminus Z = \{b \in X_0 | K \notin F(b)\} =$

$\{b \in X_0 | X \setminus K \in F(b)\}$. Since $Z \in D_0$ or $X_0 \setminus Z \in D_0$, we have $K \in R$ or $X \setminus K \in R$.

We have proved that R is a ultrafilter on X.

Let $K \in R$. There is $b \in X_0$ that $K \in F(b)$. Since $F(b)$ is a non-principal ultrafilter, K is a infinite set, i.e. R is a non-principal ultrafilter.

We assume $G = R$.

We will prove

$$\lim_G (X) = a.$$

Let $\Delta = M$ be an approximation base. Let $\alpha \in \Delta, b \in X_0 = C(X) \setminus X$. We define

$$Y_\alpha = \{x \in X_0 | \alpha \leq x\},$$

$$\bar{Y}_\alpha = Y \setminus Y_\alpha = \{x \in X_0 | \alpha \not\leq x\},$$

$$\Delta_0 = \{\alpha \in \Delta | Y_\alpha \in D_0\},$$

$$\Delta_1 = \Delta \setminus \Delta_0 = \{\alpha \in \Delta | \bar{Y}_\alpha \in D_0\},$$

$$X_\alpha = \{x \in X | \alpha \leq x\},$$

$$\bar{X}_\alpha = X \setminus X_\alpha = \{x \in X | \alpha \not\leq x\},$$

$$\Delta_0^{(b)} = \{\alpha \in \Delta | X_\alpha \in F(b)\}, b \in X_0,$$

$$\Delta_1^{(b)} = \Delta \setminus \Delta_0^{(b)} = \{\alpha \in \Delta | \bar{X}_\alpha \in F(b)\}, b \in X_0.$$

We see that

$$sup(\Delta_0) = \lim_{D_0}(C(X) \setminus X) = a,$$

$$sup(\Delta_0^{(b)}) = \lim_{F(b)}(X) = b, b \in X_0.$$

We define

$$\Omega_0 = \{\alpha \in \Delta | X_\alpha \in G\},$$

$$\Omega_1 = \Delta \setminus \Omega_0 = \{\alpha \in \Delta | \bar{X}_\alpha \in G\}.$$

We see that

$$\lim_G (X) = sup(\Omega_0).$$

Let $\alpha \in \Delta_0$. We define

$$\Lambda(\alpha) = \{inf(Rang(f)) | \, f : Y_\alpha \to \Delta, \text{ where } f(b) \in \Delta_0^{(b)}, \ f(b) \leq \alpha \text{ for any } b \in Y_\alpha\},$$

Let $\alpha \in \Delta_0$. We will prove that

1) $\Lambda(\alpha) \subseteq \Delta_0^{(b)}$ for any $b \in Y_\alpha$;
2) $\Lambda(\alpha) \subseteq \Omega_0$;
3) If $\Phi_\alpha^{(b)} = \{\alpha' \in \Delta_0^{(b)} | \alpha' \leq \alpha\}$ then $sup(\Phi_\alpha^{(b)}) = \alpha$ for any $b \in Y_\alpha$;
4) $sup(\Lambda(\alpha)) = \alpha$.

We prove 1). Let $\alpha' \in \Lambda(\alpha)$. There is $f : Y_\alpha \to \Delta$ that $f(b) \in \Delta_0^{(b)}$, $f(b) \leq \alpha$ for any $b \in Y_\alpha$ and $\alpha' = inf(Rang(f))$. Let $b \in Y_\alpha$ is an arbitrary. We have $\alpha' \leq f(b)$. Since $f(b) \in \Delta_0^{(b)}$, we have $X_{f(b)} \in F(b)$. $F(b)$ is an ultrafilter and $X_{\alpha'} \supseteq X_{f(b)}$, i.e. $X_{\alpha'} \in F(b)$. We have $\alpha' \in \Delta_0^{(b)}$, i.e. $\Lambda(\alpha) \subseteq \Delta_0^{(b)}$ for any $b \in Y_\alpha$. Item 1) is proved.

We have also $X_{\alpha'} \in F(b)$ for any $b \in Y_\alpha$. Since $Y_\alpha \in D_0$, we have $X_{\alpha'} \in G$. Hence $\alpha' \in \Omega_0$ and $\Lambda(\alpha) \subseteq \Omega_0$. Item 2) is proved.

Let $b \in Y_\alpha$. By definition $\Delta_0^{(b)}$, we have $sup(\Delta_0^{(b)}) = b$. Since $b \in Y_\alpha$, we have $\alpha \leq b$. If $\beta \in \Delta_0^{(b)}, \beta_0 = inf\{\alpha, \beta\}$ then $\beta_0 \leq \beta$. Since $X_\beta \in F(b)$, $X_{\beta_0} \supseteq X_\beta$ and $F(b)$ is an ultrafilter, we have $X_{\beta_0} \in F(b)$ and $inf\{\alpha, \beta\} = \beta_0 \in \Delta_0^{(b)}$. Hence

$$\{inf\{\alpha, \beta\} | \beta \in \Delta_0^{(b)}\} \subseteq \{\alpha' \in \Delta_0^{(b)} | \alpha' \leq \alpha\}.$$

Since χ is an infinite distributive lattice, it follows (by Remark 4) that

$$sup\{inf\{\alpha, \beta\} | \beta \in \Delta_0^{(b)}\} = inf\{\alpha, sup\{\beta | \, \beta \in \Delta_0^{(b)}\}\} =$$

$$= inf\{\alpha, sup(\Delta_0^{(b)})\} = inf\{\alpha, b\} = \alpha.$$

Hence

$$\alpha = sup\{inf\{\alpha, \beta\} | \beta \in \Delta_0^{(b)}\} \leq sup\{\alpha' \in \Delta_0^{(b)} | \alpha' \leq \alpha\} \leq \alpha,$$

i.e.

$$sup\{\alpha' \in \Delta_0^{(b)} | \alpha' \leq \alpha\} = \alpha.$$

Item 3) is proved.

Since $sup(\Phi_\alpha^{(b)}) = \alpha$ for any $b \in Y_\alpha$, we have (by the generalized infinite destributuvity of χ)

$$\alpha = inf\{sup(\Phi_\alpha^{(b)}) | \, b \in Y_\alpha\} = sup\{inf(Rang(f)) | \, f : Y_\alpha \to \Delta, f(b) \in \Phi_\alpha^{(b)}, b \in Y_\alpha\} =$$

$$= sup\{inf(Rang(f)) | \, f : Y_\alpha \to \Delta, f(b) \in \{\alpha' \in \Delta_0^{(b)} | \alpha' \leq \alpha\} \, \text{for any} \, b \in Y_\alpha\} =$$

$$= sup\{inf(Rang(f)) | \, f : Y_\alpha \to \Delta, f(b) \in \Delta_0^{(b)}, f(b) \leq \alpha \, \text{for any} \, b \in Y_\alpha\} = sup(\Lambda(\alpha)).$$

Thus, we have $sup(\Lambda(\alpha)) = \alpha$. Item 4) is proved.

We define

$$\Lambda_0 = \cup\{\Lambda(\alpha) | \, \alpha \in \Delta_0\}.$$

From 4) it follows that

$$sup(\Lambda_0) = sup(\Delta_0).$$

From 2) it follows that

$$\Lambda_0 \subseteq \Omega_0.$$

We have

$$\lim_G(X) = sup(\Omega_0) \geq sup(\Lambda_0) = sup(\Delta_0) = a,$$

i.e.

$$\lim_G(X) \geq a. \tag{3}$$

We will prove that

$$\Omega_0 \subseteq \Delta_0. \tag{4}$$

Let $\alpha \in \Delta$. We define

$$Z_\alpha = \{b \in X_0 |\, X_\alpha \in F(b)\}.$$

If $Z_\alpha \in D_0$ then $X_\alpha \in G$, i.e. $\alpha \in \Omega_0$. Let $b \in X_0$. If $X_\alpha \in F(b)$ then $\alpha \in \Delta_0^{(b)}$. Since $sup(\Delta_0^{(b)}) = b$, we have $\alpha \leq b$. Thus, we have

$$Z_\alpha = \{b \in X_0 |\, X_\alpha \in F(b)\} \subseteq \{b \in X_0 |\, \alpha \leq b\} = Y_\alpha,$$

i.e. $Z_\alpha \subseteq Y_\alpha$. Since $Z_\alpha \in D_0$, we have $Y_\alpha \in D_0$, i.e. $\alpha \in \Delta_0$.

If $Z_\alpha \notin D_0$ then $X_0 \setminus Z_\alpha \in D_0$. We have

$$X_0 \backslash Z_\alpha = \{b \in X_0 |\, X_\alpha \notin F(b)\} = \{b \in X_0 |\, X \backslash X_\alpha \in F(b)\} = \{b \in X_0 |\, \bar{X}_\alpha \in F(b)\}.$$

Hence $\bar{X}_\alpha \in G$, i.e. $\alpha \in \Delta \setminus \Omega_0$.

Thus, we have the following.

If $Z_\alpha \in D_0$ then $\alpha \in \Omega_0$, $\alpha \in \Delta_0$. If $Z_\alpha \notin D_0$ then $\alpha \in \Delta \setminus \Omega_0$. Thus, we have

$$\alpha \in \Omega_0 \Leftrightarrow Z_\alpha \in D_0.$$

If $\alpha \in \Omega_0$ then $Z_\alpha \in D_0$, i.e. $\alpha \in \Delta_0$.

Thus, we have proved (4), i.e.

$$\Omega_0 \subseteq \Delta_0.$$

We have

$$\lim_G(X) = sup(\Omega_0) \leq sup(\Delta_0) = a.$$

From (3) we have

$$\lim_G(X) = a.$$

The theorem is proved.

Definition 6. Let $\chi = (M, \leq)$ be a part order, $X \subseteq M, x, y \in M$. We define

$$(x, y)|_X = \{z \in X | x \leq z \leq y, x \neq z \neq y\},$$

$$[x, y]|_X = \{z \in X | x \leq z \leq y\},$$

Definition 7. Let $\chi = (M, \leq)$ be a complete lattice, $X \subseteq M$ and $a \in M$.

a is **massive** $\Leftrightarrow$

1) exist $L_1 = (l_\lambda)_{\lambda<\kappa_1}$ that κ_1 is an infinite cardinal, $l_\lambda \in M$, if $\lambda_1 < \lambda_2 < \kappa_1$ then $l_{\lambda_1} < l_{\lambda_2}$, $sup(L_1) = a$.

2) exist $L_2 = (l_\lambda)_{\lambda<\kappa_2}$ that κ_2 is an infinite cardinal, $l_\lambda \in M$, if $\lambda_1 < \lambda_2 < \kappa_2$ then $l_{\lambda_2} < l_{\lambda_1}$, $sup(L_2) = a$.

Lemma 5. Let $\chi = (M, \leq)$ be a complete lattice, $S \subseteq P(M)$. We have that

$$sup(\{inf(Y)|Y \in S\}) = inf(\cap(\{\{y \in M|inf(Y) \leq y\}|Y \in S\})).$$

Proof.

If $x \in M$ then

$$x \geq sup(\{inf(Y)|Y \in S\}) \Leftrightarrow$$

$x \geq inf(Y)$ for any $Y \in S \Leftrightarrow x \in \{y \in M|inf(Y) \leq y\}$ for any $Y \in S \Leftrightarrow$

$$x \in \cap\{\{y \in M|inf(Y) \leq y\}|Y \in S\}.$$

Thus,

$$inf\{x \in M|x \geq sup(\{inf(Y)|Y \in S\})\} = inf(\cap\{\{y \in M|inf(Y) \leq y\}|Y \in S\}),$$

$$sup(\{inf(Y)|Y \in S\}) = inf(\cap\{\{y \in M|inf(Y) \leq y\}|Y \in S\}).$$

Theorem 4. Let $\chi = (M, \leq)$ be a complete lattice and $X \subseteq M$. Let D be an arbitrary non-principal ultrafilter on X and $a \in M$. If $\lim_D(X) = a$ then we have one case of the following:

1) exist $L = (l_\lambda)_{\lambda<\kappa}$ that κ is an infinite cardinal, $l_\lambda \in M$, if $\lambda_1 < \lambda_2 < \kappa$ then $l_{\lambda_1} < l_{\lambda_2}$, $sup(L) = a$.

2) exist non-principal ultrafilter F on $\{x \in X|a \leq x\}$ that for any $Y \in F$ we have $a = inf(X)$.

Proof.

Let $\lim_D(X) = a$.

We define $A = \{inf(Y)|Y \in D\}$. We have that $a = sup(A)$.

We consider the case $a \in A$. In this case there is $X_0 \in D$ that $inf(X_0) = a$. We define $X_\alpha = \{x \in X|a \leq x\}$. We see that $X_0 \subseteq X_\alpha$, i.e. $X_\alpha \in D$.

Let $F = D|_{X_\alpha}$. By Lemma 1 and Proposition 2, we have that

$$\lim_D(X) = \lim_F(X_\alpha) = sup\{inf(Y)|Y \in F\} = a,$$

i.e. for any $Y \in F$ we have $a = inf(Y)$.

Thus, we have the case 2).

We consider the case $a \notin A$, i.e. for any $a' \in A$ we have $a' < a$.

Let $(X_\lambda)_{\lambda<\mu_0}$ be a sequence of D. $\Lambda_0 = (a_\lambda)_{\lambda<\mu_0}$, where μ_0 is a cardinal, $a_\lambda = inf(X_\lambda), \lambda < \mu_0$. We see that $sup(\Lambda_0) = a$.

Let $L_0 = (l_\lambda)_{\lambda<\mu_0}$ be a sequence, where $l_\lambda = sup[a_0, a_\lambda), a_0, a_\lambda \in \Lambda_0$.

We see that $l_{\lambda_0} \leq l_{\lambda_1}$, where $\lambda_0 \leq \lambda_1$ and $sup(L_0) = a$.

We consider the following procedure

a) Suppose that we have defined Λ_n, L_n, μ_n, where Λ_n, L_n are ordered by μ_n. If for any $l \in L_n$ we have $l < a$ then procedure is finished else continue to b).

b) There is $\lambda < \mu_n$ that $l_\lambda = a$. Let λ_0 be a minimum ordinal of the such kind. λ_0 is an infinite ordinal (we will prove it after the procedure). If λ_0 is a limit ordinal then we define

$$\mu_{n+1} = \lambda_0, \Lambda_{n+1} = \{a_\lambda \in \Lambda_n | \lambda < \lambda_0\}, L_{n+1} = \{l_\lambda \in L_n | \lambda < \lambda_0\}.$$

If λ_0 is not a limit ordinal then $\lambda_0 = \lambda'_0 + m$, where λ'_0 is a limit ordinal and $m < \omega$. We define $\mu_{n+1} = \lambda'_0$, Λ_{n+1} is defined as sequence of elements $\{a_{\lambda'_0}, ..., a_{\lambda_0-1}\}$ and elements $\{a_0, ...a_\lambda, ...\}$ from Λ_n, where $0 \leq \lambda < \lambda'_0$. We rename also elements of Λ_{n+1} as $a_0, a_1, ..., a_\lambda, ...$, because we do not want to introduce new notations. Thus we have removed unnecessary elements, and we put the last elements in the first place. We define L_{n+1} on the set Λ_{n+1} similar to the L. We do not introduce new notations also, i.e. $L_{n+1} = (l_\lambda)_{\lambda<\mu_{n+1}}$, where $l_\lambda = sup[a_0, a_\lambda), a_0, a_\lambda \in \Lambda_{n+1}$.

We continue to a).

Now, we prove that λ_0 is an infinite ordinal. We assume the opposite, i.e. λ_0 is finite ordinal. In this case exist finite set $S \subseteq D$ that

$$a = sup\{inf(Y) | Y \in S\}.$$

By Lemma 5, we have that

$$a = sup(\{inf(Y)|Y \in S\}) = inf(\cap(\{\{y \in M | inf(Y) \leq y\} | Y \in S\})) \leq$$

$$\leq inf(\cap(\{\{y \in X | inf(Y) \leq y\} | Y \in S\})).$$

If $Y \in S$ then $Y \in D$,

$$Y \subseteq \{y \in X | inf(Y) \leq y\} \in D,$$

$$\cap(\{\{y \in X | inf(Y) \leq y\} | Y \in S\}) \in D.$$

We see that for any $Z \in D$ we have $inf(Z) \leq a$, i.e.

$$a \leq inf(\cap(\{\{y \in X | inf(Y) \leq y\} | Y \in S\})) \leq a$$

and

$$inf(\cap(\{\{y \in X | inf(Y) \leq y\} | Y \in S\})) = a.$$

Thus, we have the case $a \in A$ that is already considered earlier. We consider the case when $a \notin A$. So, we receive a contradiction, i.e. λ_0 can not be a finite ordinal.

We continue to prove. Thus we obtain

$$\Lambda_0, L_0, \mu_0,$$

$$\Lambda_1, L_1, \mu_1,$$

$$\Lambda_2, L_2, \mu_2,$$

$$...$$

We see that $\mu_0 > \mu_1 > \mu_2 > ...$, i.e. this procedure will end after a finite number of steps. Let

$$\Lambda_{n_0}, L_{n_0}, \mu_{n_0}$$

are the last elements.

We see that μ_{n_0} is a limit ordinal, $sup(L_{n_0}) = a$ and for any $\lambda < \mu_{n_0}$ we have $l_\lambda < a$.

We can select subsequence L of L_{n_0} that L are ordered by cardinal $\kappa = cf(\mu_0)$ and for any $l \in L_{n_0}$ there is $l' \in L$ that $l \leq l'$, i.e.

$$sup(L) = sup(L_{n_0}) = a.$$

We can assume also that all elements of L are different.

Thus, we have the case 1).

The theorem 4 is proved.

Corollary. Let $\chi = (M, \leq)$ be a complete lattice and $X \subseteq M, a \in M$. If D is a non-principal ultrafilter on X that $lim_2(D, X) = lim_3(D, X) = a$ then we have 1) or 2), i.e. a is **massive**.

1) exist $L_1 = (l_\lambda)_{\lambda < \kappa_1}$ that κ_1 is an infinite cardinal, $l_\lambda \in M$, if $\lambda_1 < \lambda_2 < \kappa_1$ then $l_{\lambda_1} < l_{\lambda_2}$, $sup(L_1) = a$.

2) exist $L_2 = (l_\lambda)_{\lambda < \kappa_2}$ that κ_2 is an infinite cardinal, $l_\lambda \in M$, if $\lambda_1 < \lambda_2 < \kappa_2$ then $l_{\lambda_2} < l_{\lambda_1}$, $sup(L_2) = a$.

Proof.

By Theorem 4 we have that for $lim_2(D, X)$ we have 1) or 2) (from condition Theorem 4) and for $Lim_3(D, X) = Lim_2^*(D, X)$ we have 1*) or 2*) (from condition Theorem 4).

We have for possible combinations.

a) $1), 1^*)$

b) $1), 2^*)$

c) $2), 1^*)$

d) $2), 2^*)$

Consider the combination $2), 2^*)$. If 2) then from the proof of the theorem 4 we see that there is $F_0 \in D$ that $inf(F_0) = a$. Similarly, if $2^*)$ then there is $F_1 \in D$ that $sup(F_1) = a$. Thus, we have that $F_0 \cap F_1 = \emptyset$ or $F_0 \cap F_1 = \{a\}$. Since $F_0 \cap F_1 \in D$, then we have $\emptyset \in D$ or $\{a\} \in D$. Since D is a non-principal ultrafilter, then we have obtained a contradiction.

Thus, we have that combination $2), 2^*)$ is impossible.

Combinations a),b),c) imply 1) or 2).

§ 3. Examples

Example 1. Let $\chi = ([0,1], \leq)$ be a lattice on the interval $[0,1]$ with standard interpretation of the relation "$\leq$". Obviously χ is a complete lattice.

Let $X \subseteq [0,1]$, D is an arbitrary non-principal ultrafilter on X. We will show that the point

$$a = \lim_D(X)$$

is a limit point of X in the usual topology, i.e. any open interval of the point a contains points of the set $X \setminus \{a\}$. We assume that $a \neq 0, a \neq 1$. The cases $a = 0, a = 1$ are analyzed in the similar way. We suppose the opposite, i.e. there is an open interval $(b,c) \subset [0,1]$ that $a \in (b,c)$ and $(b,c) \cap X = \{a\}$. We will obtain a contradiction.

Since D is a non-principal ultrafilter, then $X \backslash \{a\} \in D$. Let $X_0 = X \backslash \{a\}$. By Lemma 2

$$\lim_D(X) = \lim_{D_0}(X_0),$$

where $D_0 = D|_{X_0}$. For every $Y \in D_0$ we have $Y \cap (b,c) = \emptyset$, i.e either $lim_2(D_0, X_0) \leq b$ or $lim_2(D_0, X_0) \geq c$. By Proposition 2 we have either $\lim_{D_0}(X_0) \leq b$ or $\lim_{D_0}(X_0) \geq c$. Thus it is $a \neq \lim_D(X)$. This is a contradiction with the assumption that the point a is't a limit point in the usual topology.

Now let the point a is a limit point of the set X in the usual topology, i.e. any open interval containing the point a intersects with $X \setminus \{a\}$. We will show that there is a non-principal ultrafilter D on X that $\lim_D(X) = a$. We assume that $a \neq 0, a \neq 1$. The cases $a = 0, a = 1$ are analyzed in the similar way.

Since any open interval containing the point a has a non-empty intersection with X, then this intersection contains an infinite number of elements. Otherwise it would be possible to pick up an open interval containing the point a and has no intersection with X. Let

$$R = \{X \cap (b,c) | a \in (b,c), (b,c) \subset [0,1]\}.$$

In view of the above remarks any set of R is infinite, the intersection of two sets of R also belongs to R. There exists a non-principal ultrafilter D on X that $R \subseteq D$. By construction D, we have

$$\lim_D(X) = lim_2(D, X) = a.$$

Thus the topology $atop(\chi)$ coincides with the usual topology on $[0, 1]$, which is a compact topology.

Example 2. Let $M = \omega \cup \{\omega\}$, $\chi = (M, \leq)$. Obviously χ is a complete lattice. Close sets in the topology $atop(\chi)$ are finite sets and sets containing ω. Consequently open sets are sets that do not contain ω and sets with a finite supplement. Topology $atop(\chi)$ is a compact. Any open covering must to cover the point ω. The covering must include an open set containing ω and having finite supplement that is covered by a finite number of open sets of the selected covering.

Example 3.

Let M be a non-empty set. Let $E(M)$ is the set of equivalence that $(X, R) \in E(M) \Leftrightarrow$

$$X \subseteq M,\ \text{if}\, x_1, x_2 \in X \,\text{and}\, |[x_1]| > 1, |[x_2]| > 1 \,\text{then}\, [x_1] = [x_2].$$

If $E_0 = (X, R) \in E(M), x \in X, |[x]| > 1$ then we define $Ker(E_0) = [x]$. We see that the equivalence R is uniquely determined by X and $Ker(E_0)$.

Let $\chi = (E(M), \leq)$. If $E_1 = (X_1, R_1), E_2 = (X_2, R_2), E_1, E_2 \in E(M)$ then we define $E_1 \leq E_2 \Leftrightarrow$

1) if $x \in X_1 \cap X_2$ then $[x]_{E_2} \subseteq [x]_{E_1}$,

2) if $x \in X_2 \setminus X_1$ then $[x]_{E_2} \cap X_1 = \emptyset$,

3) if $x \in X_1 \setminus X_2$ then there is $x' \in X_1 \cap X_2$ that $[x]_{E_1} = [x']_{E_1}$.

We have

1) $inf(E(M)) = (M, R)$ that $R = M \times M$,

2) $sup(E(M)) = (M, R)$ that $R = \{(x, x) | x \in M\}$.

If $S \subseteq E(M)$ then

3) $sup(S) = E_0$ that $E_0 = (X, R)$, $Ker(E_0) = \cap\{Ker(E_1)\}|E_1 \in S\}$, $X = Ker(E_0) \cup \bigcup\{Y \setminus Ker(E_1)|E_1 = (Y, R) \in S\}$.

4) $inf(S) = E_0$ that $E_0 = (X, R)$, $Ker(E_0) = \cup\{Ker(E_1)\}|E_1 \in S\}$, $X = Ker(E_0) \cup \bigcap\{Y \setminus Ker(E_1)|E_1 = (Y, R) \in S\}$.

Thus, χ is a complete lattice. By Theorem 1 the topology $atop(\chi)$ is a compact.

Example 4. We will define complete lattice $\chi = (M, \leq_\chi)$ and $X \subseteq M$ that $C(C(X)) \neq C(X)$. χ is't an infinitely distributive.

Let $M^* = P(\Omega)$, where

$$\Omega = \omega^{(0)} \cup \omega^{(1)} \cup \omega^{(2)} \cup ...$$

and

$$\omega^{(m)} = \{0^{(m)}, 1^{(m)}, 2^{(m)}, ...\}, m \in \omega,$$

i.e. $\omega^{(m)}$ is a copy of ω.

We define the relation $\leq^*$ on M^*.

1) $\{n_1^{(m)}\} \leq^* \{n_2^{(m)}\} \Leftrightarrow n_1 \leq n_2$, where $n_1, n_2, m \in \omega$;

2) $\{n^{(m)}\} \leq^* \omega^{(m)}$, where $n, m \in \omega$;

3) $\omega^{(n_1)} \leq^* \omega^{(n_2)} \Leftrightarrow n_1 \leq n_2$, where $n_1, n_2 \in \omega$;

4) We distribute $\leq^*$ to transitivity, i.e. $\{3^{(2)}\} \leq^* \omega^{(2)} \leq^* \omega^{(7)}$ and $\{3^{(2)}\} \leq^* \omega^{(7)}$ but $\{3^{(7)}\} \not\leq^* \omega^{(2)}$, $\{\omega^{(2)}\} \not\leq^* \{3^{(7)}\}, \{3^{(7)}\} \not\leq^* \{3^{(8)}\}, \{3^{(8)}\} \not\leq^* \{3^{(7)}\}$ and so on.

5) We distribute $\leq^*$ to any subsets. Let $X \in M^*$. We define

$$H(X) = \{a \in \Omega | \{a\} \leq^* Y \subseteq X\}.$$

In particular, $H(\emptyset) = \emptyset, H(\Omega) = \Omega$. Let $X_1, X_2 \in M^*$. We define

$$X_1 \leq^* X_2 \Leftrightarrow H(X_1) \subseteq H(X_2).$$

In particular, $\emptyset \leq^* X \leq^* \Omega$ for any $X \in M^*$;

We see that

$$H(\{2^{(1)}\}) = \{0^{(1)}, 1^{(1)}, 2^{(1)}\},$$
$$H(\omega^{(0)}) = \omega^{(0)},$$
$$H(\omega^{(1)}) = \omega^{(0)} \cup \omega^{(1)},$$
$$H(\{2^{(1)}\} \cup \omega^{(1)}) = \omega^{(0)} \cup \omega^{(1)},$$

i.e. $\omega^{(1)} \leq^* \{2^{(1)}\} \cup \omega^{(1)}$ and $\{2^{(1)}\} \cup \omega^{(1)} \leq^* \omega^{(1)}$.

Thus we consider the classes of equivalences.

For any $X_1, X_2 \in M^*$ we have

$$[X_1] = [X_2] \Leftrightarrow H(X_1) = H(X_2).$$

Let $\chi = (M, \leq_\chi)$, where M is a set of equivalence classes of M^* and the relation $\leq_\chi$ correspond to the relation $\leq^*$.

We see that

1) $\leq_\chi$ is a part order on M;

2) if $S \subset M, S \neq \emptyset$ then

$$inf(S) = [\cap\{H(X) | [X] \in S\}],$$

$$sup(S) = [\cup\{H(X) | [X] \in S\}].$$

In particular

$$inf(M) = [\emptyset], sup(M) = [\Omega].$$

Thus, χ is a complete lattice. Let

$$V^{(m)} = \{[\{n^{(m)}\}] | n \leq \omega\}, m \leq \omega,$$

$$V = V^{(0)} \cup V^{(1)} \cup V^{(2)} \cup ...$$

We see that $V \subseteq M$. Let D is an arbitrary non-principal ultrafilter on V. If there are $X \in D$ and $m \in \omega$ that $X \subseteq V^{(m)}$ then $|X| = \omega_0$ and

$$\lim_D(V) = sup(X) = [\omega^{(m)}].$$

Otherwise, for any $X \in D$ there are $n_1, n_2, m_1, m_2 \in \omega$ that $m_1 \neq m_2$ and $[\{n_1^{(m_1)}\}], [\{n_2^{(m_2)}\}] \in X$. If $a \in M$ and $a \leq_\chi [\{n^{(m_1)}\}]$, $a \leq_\chi [\{n^{(m_2)}\}]\}$ then $a = [\emptyset]$. Hence

$$\lim_D(V) = [\emptyset].$$

Thus, we have

$$[\Omega] \notin V \cup \{[\emptyset]\} \cup \{[\omega^{(m)}] | m \leq \omega\} = C(V).$$

Let F is an arbitrary non-principal ultrafilter on

$$W = \{[\omega^{(0)}], [\omega^{(1)}], [\omega^{(2)}], ...\}.$$

We see that $W \subseteq C(V)$ and $W \in F$. Since $[\omega^{(m_1)}] \leq_\chi [\omega^{(m_2)}]$ for any $m_1 \leq m_2$, $m_1, m_2 \in \omega$, we have

$$\lim_F(W) = sup(W) = [\omega^{(0)} \cup \omega^{(1)} \cup \omega^{(2)} \cup ...] = [\Omega].$$

Thus, we have that

$$[\Omega] \in C(W) \subseteq C(C(V)), \Omega \notin C(V),$$

i.e.

$$C(C(V)) \neq C(V).$$

We have

$$inf\{[\omega^{(0)}], sup\{[\{n^{(1)}\}] | n \in \omega\}\} = inf\{[\omega^{(0)}], [\omega^{(1)}]\} = [\omega^{(0)}],$$

$$sup\{inf\{[\omega^{(0)}], [\{n^{(1)}\}]\} | n \in \omega\} = sup\{[\emptyset]\} = [\emptyset],$$

i.e.

$$inf\{[\omega^{(0)}], sup\{[\{n^{(1)}\}] | n \in \omega\}\} \neq sup\{inf\{\omega^{(0)}], [\{n^{(1)}\}]\} | n \in \omega\}.$$

We see, that χ is not a infinite distributive.

Example 5. Let $\chi = (P(\mathbb{N}), \subseteq)$ be the lattice of the subsets of the natural numbers $\mathbb{N}$ by inclusion. Obviously χ is a complete lattice. We consider an approximation base

$$\Delta = \{\{n\} | n \in \mathbb{N}\} \cup \{\emptyset\}.$$

Since $|\Delta| = \omega_0$, according to Theorem 2 the closure of any set $X \subseteq P(\mathbb{N})$ of $atop(\chi)$ can be reduced to the closure of all countable subsets of X.

By Theorem 1 the topology $atop(\chi)$ is a compact.

Example 6. Let $a, b \in \mathbb{R}$ and $a < b$. Let M be the set of all real functions $f : \mathbb{R} \to \mathbb{R}$ that $a \leq f(x) \leq b$. Let $\chi = (M, \leq)$, where " $\leq$ " is a pointwise comparison of functions. Obviously χ is a complete lattice. By Theorem 1 the topology $atop(\chi)$ is a compact.

Example 7. We consider the propositional logic $L = L(A, \Omega, Z, I)$, where $A = \{p_1, p_2, \dots\}$ are propositional variables, $\Omega = \{\neg, \wedge, \vee, \rightarrow\}$ are logical connectives, Z is a set of inference rules (the rule of inference is modus ponens), I is a set of Hilbert axioms.

Let Φ be a set of all formulas of L. Let $\Psi_1, \Psi_2 \subseteq \Phi$. We assume $\Psi_1 \prec \Psi_2 \Leftrightarrow$ for any $\phi \in \Psi_1$ we have $I, \Psi_2 \vdash \phi$.

We denote

$$[\Psi] = \{X \subseteq \Phi | \Psi \prec X \text{ and } X \prec \Psi\}.$$

$[\Psi]$ is a class of equivalence of Ψ.

Let $\chi = (M, \leq)$, where $M = \{[X] | X \subseteq \Phi\}$, if $X_1, X_2 \subseteq \Phi$ then we assume $[X_1] \leq [X_2] \Leftrightarrow X_1 \prec X_2$.

We have $inf(M) = [I]$, if $S \subseteq P(M), S \neq \emptyset$ then

$$sup(S) = [\cup\{\Psi \subseteq \Phi | [\Psi] \in S\}].$$

Thus, χ is a complete upper semilattice with zero.

We have

1) $[I] \leq [X]$ for any $X \subseteq \Phi$, i.e. for any non-principal ultrafilter D on M we have $[I] \leq \lim\limits_{D}(X)$;

2) $[I] = [\emptyset] = [T]$, where T is a set of all formulas $\phi \in \Phi$ that $I \vdash \phi$;

3) $sup(M) = [\Phi] = [\{p_1, \neg p_1\}]$;

4) for any non-principal ultrafilter D on M we have $\lim\limits_{D}(X) \neq [\{p_1, \neg p_1\}]$;

5) if $X \subseteq \Phi$, X is't contradictory and $|X| < \omega_0$ then there exist a non-principal ultrafilter D on M that $\lim\limits_{D}(X) = [X]$.

6) if $X \subseteq \Phi$, X is't contradictory and X is a complete set of formulas (for any $\phi \in \Phi$ we have $X \vdash \phi$ or $X \vdash \neg\phi$) then $[X]$ is't a limit point.

By Theorem 2, the topology $atop(\chi)$ is a compact.

Example 8 (Semilattice of facts). We consider some set Φ of real facts. We consider relation "$\prec$" on Φ. We assume $s_1 \prec s_2 \Leftrightarrow$ when the fact "s_2 implies s_1" belong to Φ. We define classes of equivalence M on subsets of Φ similar with Example 6.

Let $\chi = (M, \leq)$, where $M = \{[X] | X \subseteq \Phi\}$, if $X_1, X_2 \subseteq \Phi$ then we assume $[X_1] \leq [X_2] \Leftrightarrow X_1 \prec X_2$.

We have $inf(M) = [\{"s_1\, implies\, s_1"\}]$, where s_1 is an arbitrary fact. If $S \subseteq P(M)$ and $S \neq \emptyset$ then

$$sup(S) = [\cup\{\Psi \subseteq \Phi | [\Psi] \in S\}].$$

Thus, χ is a complete upper semilattice with zero. By Theorem 2, the topology $atop(\chi)$ is a compact.

§ 5. License agreement

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