EXPANSION OF THE EULER ZIGZAG NUMBERS

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Abstract. This article is based on how to look for a closed-form expression related to the odd zeta function values and explained what meaning of the expansion of the Euler zigzag numbers is.

1. Introduction

Where for s $\in \mathbb{C}$, the Riemann zeta funtion, the Dirichlet lambda and beta function are defined as

$$\zeta(s) = \sum_{m=0}^{\infty} \frac{1}{(m+1)^s} \qquad \Re(s) > 1, \tag{1}$$

$$\lambda(s) = \sum_{m=0}^{\infty} \frac{1}{(2m+1)^s} = \left(1 - \frac{1}{2^s}\right)\zeta(s) \qquad \Re(s) > 1, \tag{2}$$

and

$$\beta(s) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^s} \qquad \Re(s) > 0. \tag{3}$$

The gamma function is related to the Riemann zeta function by [3]

$$\Gamma\left(\frac{s}{2}\right)\pi^{-s/2}\zeta(s) = \Gamma\left(\frac{1-s}{2}\right)\pi^{-(1-s)/2}\zeta(1-s). \tag{4}$$

In addition, we will use the Euler zigzag numbers at the power series of

$$\sec x = \sum_{m=0}^{\infty} \frac{A_{2m}}{(2m)!}x^{2m} \qquad for\ |x| < \frac{\pi}{2}, \tag{5}$$

$$\tan x = \sum_{m=0}^{\infty} \frac{A_{2m+1}}{(2m+1)!}x^{2m+1} \qquad for\ |x| < \frac{\pi}{2}. \tag{6}$$

Then, A'_{2m} would be specified instead of A_{2m} in this article. According to this suggestion, the $\sec x$ is represented by

$$\sec x = \sum_{m=0}^{\infty} \frac{A'_{2m}}{(2m)!}x^{2m} \qquad for\ |x| < \frac{\pi}{2}. \tag{7}$$

2. Basic properties

Lemma 1. *For all $n \in \mathbb{N}$,*

$$\lambda(2n) = \beta(1)\frac{A_{2n-1}}{(2n-1)!}\left(\frac{\pi}{2}\right)^{2n-1}, \tag{8}$$

$$\beta(2n-1) = \beta(1)\frac{A'_{2n-2}}{(2n-2)!}\left(\frac{\pi}{2}\right)^{2n-2}. \tag{9}$$

Lemma 2. *For all $n \in \mathbb{N}$ and $0 < x < \frac{\pi}{2}$,*

$$\ln(\cot x) = 2\sum_{m=1}^{\infty}\frac{\cos((4m-2)x)}{2m-1}. \tag{10}$$

Proof. We consider [2]

$$\ln(\sin x) = -\ln 2 - \sum_{m=1}^{\infty}\frac{\cos(2mx)}{m} \tag{11}$$

,

which was studied by Euler. The (11) is replaced by

$$\ln(\cos x) = -\ln 2 - \sum_{m=1}^{\infty}\frac{(-1)^m}{m}\cos(2mx). \tag{12}$$

To subtract the (11) from the (12) is

$$\ln(\cot x) = 2\sum_{m=1}^{\infty}\frac{\cos((4m-2)x)}{2m-1}.$$

□

Lemma 3. *For all $n \in \mathbb{N}$ and $|x| < \pi$,*

$$\int_0^x \frac{x^n}{\sin x}dx = \left(\frac{\pi}{2}\right)^{n+1}\sum_{m=0}^{\infty}\sum_{l=0}^{n}\frac{(-1)^l A'_{2m}}{(2m+l+1)(2m)!}\binom{n}{l}x^{2m} \tag{13}$$

the binomial coefficient is defined by the next expression

$$\binom{n}{l} = \frac{n!}{(l)!(n-l)!}.$$

Lemma 4. *For all $n \in \mathbb{N}$,* [1]

$$\sum_{m=0}^{\infty}\frac{(-1)^m E_{2m}}{2(2m+2n)!}\left(\frac{\pi}{2}\right)^{2m+2n} = (-1)^{n-1}\sum_{l=1}^{n}\frac{(-1)^{l-1}\beta(2n-2l+2)}{(2l-2)!}\left(\frac{\pi}{2}\right)^{2l-2}, \tag{14}$$

$$\begin{aligned}\sum_{m=0}^{\infty}\frac{(-1)^m E_{2m}}{2(2m+2n+1)!}\left(\frac{\pi}{2}\right)^{2m+2n+1} &= (-1)^n\lambda(2n+1) \\ &+ (-1)^{n-1}\sum_{l=1}^{n}\frac{(-1)^{l-1}\beta(2n-2l+2)}{(2l-1)!}\left(\frac{\pi}{2}\right)^{2l-1}\end{aligned} \tag{15}$$

where E_{2m} is the Euler number.

3. Proof of a theorem

Theorem 1. *For all $n \in \mathbb{N}$,*

$$\sum_{m=0}^{\infty} \frac{A'_{2m}}{(2m+n+1)!}\left(\frac{\pi}{2}\right)^{2m} = \frac{A_n}{n!}\cos\left(\frac{n}{2}\pi\right) + \sum_{l=1}^{n} \frac{A'_l}{(l)!(n-l)!}\sin\left(\frac{l}{2}\pi\right). \tag{16}$$

Proof. Multiplying x^{n-1} and integrating 0 to $\pi/4$ for the both terms of Lemma 2 is

$$\int_0^{\frac{\pi}{4}} x^{n-1}\ln(\cot x)dx = 2\int_0^{\frac{\pi}{4}} \sum_{m=1}^{\infty} \frac{x^{n-1}}{2m-1}\cos((4m-2)x)dx. \tag{17}$$

When we calculate each integral of the (17), it appears to be (18), (19) and (20) which were

$$\beta(2) = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{x}{\sin x}dx, \tag{18}$$

$$\lambda(2n+1) = \frac{(-1)^n}{2(2n)!}\int_0^{\frac{\pi}{2}} \frac{x^{2n}}{\sin x}dx + \sum_{l=0}^{n-1} \frac{(-1)^l\beta(2n-2l)}{(2l+1)!}\left(\frac{\pi}{2}\right)^{2l+1} \tag{19}$$

and

$$\beta(2n+2) = \frac{(-1)^n}{2(2n+1)!}\int_0^{\frac{\pi}{2}} \frac{x^{2n+1}}{\sin x}dx + \sum_{l=0}^{n-1} \frac{(-1)^l\beta(2n-2l)}{(2l+2)!}\left(\frac{\pi}{2}\right)^{2l+2}. \tag{20}$$

Application of the Lemma 3 for each integral term in (18), (19) and (20) yields

$$\lambda(2n+1) = \frac{\beta(2n+1)}{A'_{2n}} \sum_{m=0}^{\infty}\sum_{l=0}^{n} \frac{(-1)^l A'_{2n-2l}A'_{2m}}{(2l+2m+1)(2m)!}\binom{2n}{2l}\left(\frac{\pi}{2}\right)^{2m}, \tag{21}$$

$$\beta(2n) = \frac{\lambda(2n)}{A_{2n-1}} \sum_{m=0}^{\infty}\left(\left(\sum_{l=0}^{n-1} \frac{(-1)^l A_{2n-2l-1}}{2l+2m+1}\binom{2n-1}{2l}\right) - \frac{(-1)^{n-1}}{2m+2n}\right)\frac{A'_{2m}}{(2m)!}\left(\frac{\pi}{2}\right)^{2m}. \tag{22}$$

Application of the Lemma 1 for $\beta(2n+1)$ and $\lambda(2n)$ in each of the (21) and (22) specifies

$$\lambda(2n+1) = \beta(1)\frac{A_{2n}}{(2n)!}\left(\frac{\pi}{2}\right)^{2n}, \tag{23}$$

$$\beta(2n) = \beta(1)\frac{A'_{2n-1}}{(2n-1)!}\left(\frac{\pi}{2}\right)^{2n-1}. \tag{24}$$

Therefore, A_{2n} and A'_{2n-1} are described as

$$A_{2n} = \sum_{m=0}^{\infty}\sum_{l=0}^{n} \frac{(-1)^l A'_{2n-2l}A'_{2m}}{(2l+2m+1)(2m)!}\binom{2n}{2l}\left(\frac{\pi}{2}\right)^{2m}, \tag{25}$$

$$A'_{2n-1} = \sum_{m=0}^{\infty}\left(\left(\sum_{l=0}^{n-1} \frac{(-1)^l A_{2n-2l-1}}{2l+2m+1}\binom{2n-1}{2l}\right) - \frac{(-1)^{n-1}}{2m+2n}\right)\frac{A'_{2m}}{(2m)!}\left(\frac{\pi}{2}\right)^{2m} \tag{26}$$

or

$$A_{2n} = \sum_{m=0}^{\infty} \left(\left(\sum_{l=0}^{2n} \frac{A'_{2n-l}}{2m+l+1} \binom{2n}{l} \cos\left(\frac{l}{2}\pi\right) \right) + \frac{\sin(n\pi)}{2m+2n+1} \right) \frac{A'_{2m}}{(2m)!} \left(\frac{\pi}{2}\right)^{2m}, \tag{27}$$

$$A'_{2n-1} = \sum_{m=0}^{\infty} \left(\left(\sum_{l=0}^{2n-1} \frac{A_{2n-l-1}}{2m+l+1} \binom{2n-1}{l} \cos\left(\frac{l}{2}\pi\right) \right) + \frac{\cos(n\pi)}{2m+2n} \right) \frac{A'_{2m}}{(2m)!} \left(\frac{\pi}{2}\right)^{2m}. \tag{28}$$

Application of the (23) and (24) for $\beta(2n-2l+2)$ in Lemma 4 becomes

$$\sum_{m=0}^{\infty} \frac{A'_{2m}}{(2m+2n)!} \left(\frac{\pi}{2}\right)^{2m} = \sum_{l=1}^{n} \frac{(-1)^{l+n} A'_{2n-2l+1}}{(2l-2)!(2n-2l+1)!}, \tag{29}$$

$$\sum_{m=0}^{\infty} \frac{A'_{2m}}{(2m+2n+1)!} \left(\frac{\pi}{2}\right)^{2m} = \frac{(-1)^n A_{2n}}{(2n)!} + \sum_{l=1}^{n} \frac{(-1)^{l+n} A'_{2n-2l+1}}{(2l-1)!(2n-2l+1)!}. \tag{30}$$

Finally, the (29) and (30) are summarized as

$$\sum_{m=0}^{\infty} \frac{A'_{2m}}{(2m+n+1)!} \left(\frac{\pi}{2}\right)^{2m} = \frac{A_n}{n!} \cos\left(\frac{n}{2}\pi\right) + \sum_{l=1}^{n} \frac{A'_l}{(l)!(n-l)!} \sin\left(\frac{l}{2}\pi\right).$$

□

4. Other explicit fomulas

The value of the Dirichlet lambda function at even positive integers and the Dirichlet beta function at odd positive integers appear by the next expressions

Corollary 1. *For all $n \in \mathbb{N}$,*

$$\frac{1}{n!} = \frac{A'_n}{n!} \cos\left(\frac{n}{2}\pi\right) + \sum_{l=1}^{n} \frac{A_l}{(l)!(n-l)!} \sin\left(\frac{l}{2}\pi\right), \tag{31}$$

Corollary 2. *For all $n \in \mathbb{N}$,*

$$\frac{1}{\Gamma(n)} = \sum_{l=1}^{n} \left(\frac{A_l}{\Gamma(n-l)\Gamma(l+1)} + \frac{A'_{l-1}}{\Gamma(n)\Gamma(l-n+1)} \right) \sin\left(\frac{l}{2}\pi\right), \tag{32}$$

Corollary 3. *For all $n \in \mathbb{N}$,*

$$\frac{1}{(n-1)!} = \frac{A_{n-1}}{(n-1)!} \sin\left(\frac{n}{2}\pi\right) + \sum_{l=0}^{n-1} \frac{A_l}{(l)!(n-l-1)!} \sin\left(\frac{l}{2}\pi\right). \tag{33}$$

Corollary 1 is not defined at A'_0. To define the A'_0, corollary 2 is used by the gamma function. Otherwise, corollary 3 is defined without distinguishing the symbol of A' and A which they specified for (5) and (6).

Conjecture 1. *By the theorem 1, an infinite series*

$$\sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{A'_{2m}}{(2m+n+1)!} \left(\frac{\pi}{2}\right)^{2m} \tag{34}$$

converges to 1.

References

[1] JeonWon Kim. Functional equations related to the dirichlet lambda and beta functions. *arXiv preprint arXiv:1404.5467*, 2014.

[2] Shin-ya Koyama and Nobushige Kurokawa. Euler's integrals and multiple sine functions. *Proceedings of the American Mathematical Society*, pages 1257–1265, 2005.

[3] Eric W Weisstein. Riemann zeta function. 2002.