Adequate quaternionic generalization of complex differentiability

Michael Parfenov*

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To the memory of Lyusya Lyubarskaya

Abstract. The high efficiency of complex analysis is attributable mainly to the ability to represent adequately the Euclidean physical plane essential properties, which have no counterparts on the real axis. In order to provide the similar ability in higher dimensions of space we introduce the general concept of essentially adequate differentiability, which generalizes the key features of the transition from real to complex differentiability. In view of this concept the known Cauchy-Riemann-Fueter equations can be characterized as inessentially adequate. Based on this concept, in addition to the usual complex definition, the quaternionic derivative has to be independent of the method of quaternion division: on the left or on the right. Then we deduce the generalized quaternionic Cauchy-Riemann equations as necessary and sufficient conditions for quaternionic functions to be ℍ-holomorphic. We prove that each ℍ-holomorphic function can be constructed from the ℂ-holomorphic function of the same kind by replacing a complex variable by a quaternionic in an expression for the ℂ-holomorphic function. It follows that the derivatives of all orders of ℍ- holomorphic functions are also ℍ-holomorphic and can be analogously constructed from the corresponding derivatives of ℂ-holomorphic functions. The examples of Liouvillian elementary functions demonstrate the efficiency of the developed theory.

1 Introduction

In accordance with the so-called Meĭlihzon result the admissible set of the quaternion-differentiable functions is restricted to linear functions [1, 2, 3, 4], while the complex analysis gives a large class of the complex-differentiable functions. This also means that we cannot construct any quaternion-differentiable function from a corresponding complex-differentiable function by the direct replacement of a complex variable by a quaternion variable in the expression for the complex function (without change of a functional dependence form), while an analogous procedure is possible (see, e.g., [5], p. 353) by constructing complex-differentiable functions from real-differentiable functions by the direct replacement of a real variable by a complex

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*e-mail: mwparfenov@gmail.com

variable. For example, the complex-differentiable function $\sin(x+iy)$ can be created in this way from the real-differentiable function $\sin x$.

Such a contradiction cannot exist in principle, since each point of any real line is at the same time a point of some plane and space as a whole, and therefore any characterization of differentiability at a point must be the same regardless of whether we think of that point as a point on the real axis or a point in the complex plane, or a point in space. Nevertheless, this contradiction arises within the framework of existing concepts of quaternionic differentiability and does not find a complete solution in accessible materials (see, e.g., [2, 3, 4]) on quaternionic analysis. For example, the prevailing direction of quaternionic analysis [3] constructs the "regular" functions $\psi : \mathbb{H} \to \mathbb{H}$ in an indirect way by means of expressions combining harmonic functions of four real variables and analytic functions of a complex variable. The original Cauchy-Riemann-Fueter equations (see, e.g., [3, 4]), namely,

$$\frac{\partial\psi}{\partial t} + i\frac{\partial\psi}{\partial x} + j\frac{\partial\psi}{\partial y} + k\frac{\partial\psi}{\partial z} = 0 \tag{1.1}$$

for the left-regular quaternionic functions, and

$$\frac{\partial\psi}{\partial t} + \frac{\partial\psi}{\partial x}i + \frac{\partial\psi}{\partial y}j + \frac{\partial\psi}{\partial z}k = 0 \tag{1.2}$$

for the right-regular quaternionic functions (the variable being $q = t + ix + jy + kz$) remain the basic conditions of quaternionic differentiability, but the definitions of the left and right derivatives in the usual sense (as a limit of the left and right difference quotients) are replaced by the definitions using the exterior differential calculus. As noted in [3], such a "definition of 'regular' for a quaternionic function is satisfied by a large class of functions and leads to a development similar to the theory of regular functions of a complex variable". However the problem mentioned above remains on the whole.

The main reason for this contradiction is the prevailing [1, 2, 3] separate consideration of the left and right versions of quaternionic analysis. The separate consideration, as it will be clarified below, is essentially not adequate to properties of 3-dimensional physical space since a representation of an arbitrary rotation of any vector in space by means of quaternions requires the use of both left and right quaternionic multiplication together [6]. The only left or only right version enables us to describe only a part of all rotations in space and cannot be regarded as essentially adequate. We can call them inessentially adequate. Therefore, the essentially adequate quaternionic differentiability theory must be represented by some "construct" of the left and right versions of differentiability together.

It is necessary to say that there are successful results using the left and right versions together [4, 7], however they rather represent "heuristic" formulations than give a consequent theory similar to complex analysis. In particular, they do not solve the above general problem.

The purpose of this paper is to develop a theory of quaternionic differentiability which is essentially adequate to properties of 3-dimensional physical space. This purpose is achieved by introducing the general concept of essentially adequate definitions and conditions of the hypercomplex differentiability. They represent a hypercomplex generalization of key features of the transition from definitions of real differentiability to those of complex differentiability. This concept also contains the requirement of the uniqueness of the derivative value since derivatives of hypercomplex-differentiable functions must represent conservative vector fields in space just as derivatives of complex-differentiable functions in the plane [5, 9].

Based on this concept, we develop the basics of the theory of quaternionic differentiability similar to the theory of complex differentiability. According to this concept, we have to require the equality of the left and right derivatives. This is the necessary step in order to impose the physical reality requirement (uniqueness of a derivative in space) on the mathematical fact that two quaternionic derivatives (left and right) exist. It follows that the essentially adequate conditions of quaternionic differentiability (the generalized Cauchy-Riemann equations) are such that during the check of quaternionic differentiability of any quaternionic function we have to do a definite transition (p. 19) to a 3-dimensional variable in expressions for partial derivatives contained in equations of these conditions. But this does not mean that we deal with triplets in general; such a transition cannot be initially done for quaternionic variables and functions (p. 21). Any quaternionic function of a quaternion variable remains the same quaternionic function regardless of whether we check its quaternionic differentiability or not. This transition can be used in one more case, when after doing all quaternionic calculations we perform the final transition to 3-dimensional physical space to solve some sort of physical problem, if needed.

The developed basics of the essentially adequate quaternionic differentiability give the representation of the full quaternionic derivative as a sum of constituents of the left and right derivatives and enable us to solve the mentioned problem of constructing quaternion-differentiable functions (and their derivatives of all orders) from the complex-differentiable functions (and their derivatives of analogous orders) by the direct replacing of variables.

The sections and subsections of this paper are given as follows:

ity and ℍ-holomorphicity – (13); 4.2 The essentially adequate generalization of Cauchy-Riemann's equations – (14); 4.3 Construction of ℍ-holomorphic functions – (28); 4.4 ℍ-holomorphic derivatives of all orders – (30); 5 Efficiency examples of the presented theory – (34); 6 Conclusions – (44); References – (45).

Examples of elementary functions demonstrate the efficiency of the theory developed, which is confirmed every time, when it is required to create the quaternion-differentiable function from the corresponding complex-differentiable function of the same type.

2 Preliminaries

We assume the reader is familiar enough with the basics of complex numbers and quaternions, as well as complex and quaternionic analysis (see, e.g., [2, 6, 8, 9]). We give the only data which are needed for the sequel.

Objects of study in complex analysis in one independent variable are complex-valued functions $\psi(z) = u(x,y) + v(x,y)i$ of a single complex variable $z = x + yi \in G_2 \in \mathbb{C}$, where x and y are real variables; $u(x,y)$ and $v(x,y)$ are real-valued differentiable (with respect to x and y) functions; G_2 is some connected, open subset called a domain of a function definition (or simply the domain). In the sequel we always understand by a domain a connected, open set of points. We denote it by G_2 in the complex plane $\mathbb{C}$ or by G_4 in the quaternion space $\mathbb{H}$.

The complex derivative of $\psi(z)$ at any point z in its domain is defined by the limit of the difference quotient:

$$\psi'(z) = \lim_{\Delta z \to 0} \frac{\psi(z+\Delta z) - \psi(z)}{\Delta z} = \frac{d\psi(z)}{dz}, \tag{2.1}$$

as the complex increment $\Delta z = \Delta x + \Delta y i$ approaches zero. This is the same as the definition of the derivative for real functions, except that all of the quantities are complex. If the limit (2.1) exists, then the function $\psi(z)$ is called complex-differentiable (briefly, $\mathbb{C}$-differentiable) at the point z. A function $\psi(z)$ is said to be complex-holomorphic (briefly, $\mathbb{C}$-holomorphic) at the point z, if $\psi(z)$ is $\mathbb{C}$-differentiable in some open connected neighborhood of z. If $\psi(z)$ is $\mathbb{C}$-differentiable at every point z in an open set G_2, we say that $\psi(z)$ is $\mathbb{C}$-holomorphic on G_2. The $\mathbb{C}$-holomorphic functions are denoted by $\psi_C(z)$ in the sequel.

The existence of the limit (2.1) is equivalent to independence of the path that Δz follows toward zero. This gives [1, 3, 6] the complex Cauchy-Riemann condition, which can be written as

$$i\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial y} \tag{2.2}$$

where multiplying $\frac{\partial\psi}{\partial x}$ by imaginary unit i reflects the essentially new property of the complex plane, namely, the rotations of vectors in the plane. The differentiability condition (2.2) can be regarded as the essentially adequate condition of complex differentiability since it reflects the essential property of the new dimension of physical space (the complex plane), which have no counterparts in the previous dimension of space (the real axis).

Usually, the requirement (2.2) is represented by two equations, namely, for $u(x, y)$ and $v(x, y)$, the so-called Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{2.3}$$

In the sequel we use the compact notation ∂_s, where s may be any variable, to denote the partial differentiation with respect to this variable. By using this notation, the Cauchy-Riemann equations can be rewritten as

$$\partial_x u = \partial_y v, \quad \partial_y u = -\partial_x v. \tag{2.4}$$

The relationship between real differentiability and complex differentiability is the following. If a complex function $\psi(z) = \psi(x + yi) = u(x, y) + v(x, y)i$ is $\mathbb{C}$-holomorphic, then $u(x, y)$ and $v(x, y)$ have first partial derivatives with respect to x and y (in the sense of real differentiability) and satisfy (the additional complex condition) Cauchy–Riemann's equations.

In the quaternion theory below we use the complex values

$$a = x + yi, \tag{2.5}$$

$$b = z + ui, \tag{2.6}$$

and their conjugates

$$\bar{a} = x - yi, \tag{2.7}$$

$$\bar{b} = z - ui, \tag{2.8}$$

where x, y, z, and u are real numbers. These values define according to the Cayley–Dickson doubling procedure [6] the independent quaternionic variable

$$\begin{aligned} p = {} & x + yi + zj + uk = (x + yi) + (z + ui)j \\ & = a + bj \in \mathbb{H}, \end{aligned} \tag{2.9}$$

where i, j, k are "imaginary" units of the quaternionic algebra with multiplication table

$$i^2 = j^2 = k^2 = -1, ij = -ji = k, jk = -kj = i, ki = -ik = j. \tag{2.10}$$

The Cayley–Dickson doubling procedure is essential to the theory of quaternionic differentiability under consideration. Note that the general scheme (2.9) of "doubling" the complex numbers uses the "imaginary" unit j in $p = a + bj$. The quaternion conjugate of p is defined, as usual [6], by

$$\bar{p} = \bar{a} - bj = x - yi - zj - uk. \tag{2.11}$$

Let $p = (x_1 + y_1 i) + (z_1 + u_1 i) \cdot j = a_1 + b_1 \cdot j$ and $q = (x_2 + y_2 i) + (z_2 + u_2 i) \cdot j = a_2 + b_2 \cdot j$ be two arbitrary quaternions. Then the multiplication rule for quaternions in the Cayley–Dickson doubling form is determined [6] by

$$p \cdot q = (a_1 + b_1 \cdot j) \cdot (a_2 + b_2 \cdot j) = \left(a_1 a_2 - b_1 \overline{b_2}\right) + (a_1 b_2 + \overline{a_2} b_1) \cdot j, \tag{2.12}$$

where by "$\cdot$" is denoted the quaternion multiplication. Putting $a_1 = x_1$ $(y_1 = 0)$, $b_1 = z_1$ $(u_1 = 0)$, $a_2 = \overline{a_2} = x_2$ $(y_2 = 0)$, $b_2 = \overline{b_2} = z_2$ $(u_2 = 0)$ we have two complex numbers $p = x_1 + z_1 j$ and $q = x_2 + z_2 j$; then the multiplication rule for quaternions (2.12) reduces to the multiplication rule for complex numbers:

$$p \cdot q = (x_1 + z_1 \cdot j) \cdot (x_2 + z_2 \cdot j) = (x_1 x_2 - z_1 z_2) + (x_1 z_2 + x_2 z_1) \cdot j$$

where imaginary unit j ($j^2 = -1$) plays a role of the "complex imaginary unit" i.

In the sequel we consider the quaternion-valued (briefly, quaternionic) functions

$$\psi(p) = \psi_1(x,y,z,u) + \psi_2(x,y,z,u)i + \psi_3(x,y,z,u)j + \psi_4(x,y,z,u)k, \tag{2.13}$$

which in accordance with the Cayley–Dickson doubling [6] procedure are represented as

$$\psi(p) = \psi(a,b) = \phi_1(a,b) + \phi_2(a,b) \cdot j, \tag{2.14}$$

where $\psi_1(x,y,z,u)$, $\psi_2(x,y,z,u)$, $\psi_3(x,y,z,u)$, and $\psi_4(x,y,z,u)$ are real-valued functions, and

$$\phi_1(a,b) = \psi_1(a,b) + \psi_2(a,b)i = \psi_1(x,y,z,u) + \psi_2(x,y,z,u)i, \tag{2.15}$$

$$\phi_2(a,b) = \psi_3(a,b) + \psi_4(a,b)i = \psi_3(x,y,z,u) + \psi_4(x,y,z,u)i \tag{2.16}$$

are complex-valued functions. We write briefly $\phi_1(a,b)$ and $\phi_2(a,b)$ bearing in mind the complete notation $\phi_1\left(a,\overline{a},b,\overline{b}\right)$ and $\phi_2\left(a,\overline{a},b,\overline{b}\right)$. As usual [2, 3, 6], the quaternionic conjugate of $\psi(p)$ is determined by

$$\begin{aligned}\overline{\psi}(p) &= \psi_1(x,y,z,u) - \psi_2(x,y,z,u)i - \psi_3(x,y,z,u)j - \psi_4(x,y,z,u)k \\ &= \overline{\phi}_1(a,b) - \phi_2(a,b)j.\end{aligned} \tag{2.17}$$

Quaternionic functions are assumed to be continuous and single-valued everywhere on their definition domains G_4 except, possibly, at certain singularities.

In accordance with definitions of complex analysis [2, 3] we will be concerned with the following differential operators:

$$\partial_a = \tfrac{1}{2}\left(\partial_x - \partial_y \cdot i\right), \tag{2.18}$$

$$\partial_{\overline{a}} = \tfrac{1}{2}\left(\partial_x + \partial_y \cdot i\right), \tag{2.19}$$

$$\partial_b = \tfrac{1}{2}(\partial_z - \partial_u \cdot i), \tag{2.20}$$

$$\partial_{\overline{b}} = \tfrac{1}{2}(\partial_z + \partial_u \cdot i). \tag{2.21}$$

Here the differential operators $\partial_{\overline{a}}$ and $\partial_{\overline{b}}$ represent the so-called Cauchy-Riemann operators in the complex planes $a = x + yi$ and $b = z + ui$, respectively.

The quaternionic generalization [2, 3] of the Cauchy-Riemann operator is denoted by $\overline{\partial}$ and called the Cauchy-Riemann operator too. It and its quaternion conjugate ∂ are represented, as usual, by

$$\overline{\partial} = \partial_x + \partial_y \cdot i + \partial_z \cdot j + \partial_u \cdot k \tag{2.22}$$

$$\partial = \partial_x - \partial_y \cdot i - \partial_z \cdot j - \partial_u \cdot k \tag{2.23}$$

Since $\overline{\partial} = \left(\partial_x + \partial_y \cdot i\right) + \left(\partial_z + \partial_u \cdot i\right) \cdot j$, $\partial = \left(\partial_x - \partial_y \cdot i\right) - \left(\partial_z + \partial_u \cdot i\right) \cdot j$, the quaternion Cauchy-Riemann operator and its quaternion conjugate may be represented in the Cayley–Dickson doubling form as follows:

$$\overline{\partial} = 2\left(\partial_{\overline{a}} + \partial_{\overline{b}} \cdot j\right), \tag{2.24}$$

$$\partial = 2\left(\partial_a - \partial_{\overline{b}} \cdot j\right). \tag{2.25}$$

When it is obvious that the quaternion multiplication is used, we can omit its notation, that is, the dot "·".

A rotation of any vector z in the complex plane through an arbitrary angle φ is represented [5, 8, 9] by means of multiplication of this vector by the complex number $r = \cos\varphi + i\sin\varphi$ (of length 1):

$$z' = rz,$$

where z' is the vector z after the rotation. The commutativity of rotations in the Euclidean plane is adequately represented by the commutative multiplication of complex numbers:

$$z' = r_2 r_1 z = r_1 r_2 z,$$

where r_1 and r_2 are complex numbers corresponding rotations.

A rotation of any 3-dimensional vector v about an arbitrary 3-dimensional vector p_1 of length 1 through an arbitrary angle $2\varphi_1$ is represented [6] by means of multiplication of v on the left by the quaternion $q_1 = \cos\varphi_1 + p_1\sin\varphi_1$ (of length 1) and on the right by the quaternion $q_1^{-1} = \cos\varphi_1 - p_1\sin\varphi_1$:

$$v_1 = q_1 v q_1^{-1} ,$$

where v_1 is the vector v after the rotation, q_1^{-1} is the inverse of the quaternion q_1 such that $q_1 q_1^{-1} = 1$. Clearly, the description of arbitrary rotations in space requires the use of both left and right quaternion multiplication together. Noncommutativity of quaternion multiplication represents adequately noncommutativity of vector rotations in 3-dimensional physical space:

$$q_2(q_1 v q_1^{-1})q_2^{-1} = (q_2 q_1)v(q_2 q_1)^{-1} \neq q_1(q_2 v q_2^{-1})q_1^{-1} = (q_1 q_2)v(q_1 q_2)^{-1},$$

where $q_2 = \cos\varphi_2 + p_2 \sin\varphi_2$ and $q_2^{-1} = \cos\varphi_2 - p_2 \sin\varphi_2$ are quaternions of length 1. We see that a sequence of rotations of any vector v about arbitrary axes p_1 and p_2 through the corresponding arbitrary angles $2\varphi_1$ and $2\varphi_2$ is non-commutative owing to noncommutativity of quaternion multiplication:

$$q_2 q_1 \neq q_1 q_2.$$

The use of the only left or only right version of the quaternion theory is essentially non-adequate to physical properties of 3-dimensional space, because such a use does not describe all arbitrary non-commutative rotations in space.

3 The concept of essentially adequate differentiability

In order to obtain the correct (that is, adequate to properties of physical space) hypercomplex generalization of complex differentiability conditions (2.4), it is necessary to define a general concept (rules) for obtaining the essentially adequate differentiability conditions upon transition from spatial dimension N to dimension $N+1$ (briefly, to a new dimension), where $N = 1, 2$. We formulate this in a general way that allows us to analyze the known hypercomplex generalizations of complex analysis. This concept can be established by the following assertions.

Assertion 3.1 *Essentially adequate differentiability conditions upon the transition to a new dimension must be formulated only on the basis of algebras adequately representing new properties of a new dimension, that is, properties, which have no counterparts in the previous dimensions of space.*

Clearly, the new property of *commutative vector rotations* appears upon transition from the real axis to the Euclidean (physical) plane. This property has no counterpart on the real axis and is adequately represented by the commutative algebra of complex numbers. Therefore, complex differentiability conditions are adequate to physical reality of Euclidean space.

Further on, a new property of *noncommutativity of vector rotations* appears upon transition from the Euclidean physical plane to 3-dimensional Euclidean physical space. This property is adequately represented by the non-commutative algebra of quaternions and has no counterpart in the complex plane, where rotations are commutative.

From this assertion it follows that any generalizations of complex analysis cannot be adequate to 3-dimensional physical space if they are based on algebras with the commutative law of multiplication (see, e.g., S. Rönn's bicomplex analysis in [10], M.S. Marinov's S-regular functions in [2]). It is impossible to expect from such generalizations any results comparable in the "internal perfection and external justification" with results of real and complex theory of differentiability.

It also follows that definitions of quaternionic differentiability only "on the left" and only "on the right" (left-regular and right-regular functions in [2, 3, 4]) cannot be essentially adequate to the 3-dimensional space properties, since the description of the arbitrary vector rotations in space requires the use of both quaternionic multiplications, that is, the left and the right quaternionic versions must be only used together. Thus the statement of the type "For definiteness, we will only consider left-regular functions, which we will call simply 'regular' " (see [3]) cannot be regarded as acceptable. In this sense, all the papers quoted above represent hypercomplex generalizations, which cannot be regarded as essentially adequate.

Assertion 3.2 *The definition of differentiability in higher dimensions of space cannot be "reduced" to the definition of differentiability in the previous lower dimensions. There must be some additional conditions of differentiability, which correspond to the new dimension properties and have no structural (algorithmic) counterparts in the previous lower dimensions*.

This assertion generalizes the known statement of complex analysis [8, 9]: the differentiability in the complex sense cannot be "reduced" (cannot be completely similar) to the differentiability in the real sense since the complex differentiability requires not only the existence of partial derivatives in the real sense (that is, a simple transfer of the corresponding concepts of real analysis) but also the satisfaction the Cauchy–Riemann complex differentiability conditions, which have no counterparts on the real axis and correspond to the new property of commutative rotations of vectors in the physical plane.

Assertion 3.3 *By analogy with real and complex analysis any generalization of differentiability conditions upon transition to the new dimension of space must contain a requirement of the uniqueness of the derivative value. We must also strive to preserve the form (2.1) of a derivative definition upon transition from the complex plane to the new higher dimension of space*.

In complex analysis any holomorphic function $\psi_C(z_0)$ with nonvanishing derivative at a point $z_0 \in \mathbb{C}$ is a conformal (angle-preserving) map at that point. A conformal mapping $\psi_C(z)$ qives a graphical picture of a "linear transformation" (dilation) of an initial complex plane, if we plot images of horizontal and vertical lines under the map $\psi_C(z)$.

This transformation can be "measured" as follows. Firstly, we represent the derivative $\psi'_C(z_0)$ in the known [9] exponential polar form $\psi'_C(z_0) = |\psi'_C(z_0)|e^{i\theta}$ and, secondly, we say that $\psi_C(z_0)$ at the point z_0 has the dilation constant [9] or scale factor [11]

$$r = |\psi'_C(z_0)| > 0, \tag{3.1}$$

and the rotation angle $\theta \in [0,2\pi[$. Thus we associate local dilations of the 2-dimensional complex plane under the map $\psi_C(z)$ with the derivative in the form (2.1), that is, with the limit of the quotient of the line segment "$\Delta\psi(p)$ " in the "dilated" complex plane by the line segment "Δp" in the initial "non-dilated" plane.

This simplest representation of 1- and 2- dimensional local dilations (in the form (2.1)) must be preserved to obtain a correct hypercomplex representation of 3-dimensional local dilations. Indeed, any point of the real axis is also a point of some plane and a point of space. Then the derivative definition at that point must have the same form (2.1) regardless of whether we think of that point as a point on the real axis or a point in the complex plane, or a point in space. Such a representation must have a *unique value* of a derivative (2.1), since it is impossible to imagine that a 3-dimensional local dilation at the same point can have two or more vector values.

On the other hand, the *uniqueness of the derivative value* follows from the fact that the derivative (2.1) of any $\mathbb{C}$-holomorphic function (viewed as a complex potential function) is associated in complex analysis with a complex vector [11] of the corresponding conservative vector field. This vector (a field strength) can have physically the *only unique value*. Therefore, the derivative value *must be unique* regardless of whether we consider it in real or in complex analysis, or in some hypercomplex generalization of complex analysis.

For this reason, if a quaternionic derivative is defined by analogy with formula (2.1) as a limit of the difference quotient, then it must have the same value *regardless of whether we calculate the derivative by using the division on the left or the division on the right*.

It is not superfluous to note that the physical formulation of a problem played an important role initially in the theory of complex-differentiable functions, and the Cauchy-Riemann equations (2.4) were found [8] as early as in 1752 in d'Alembert's doctrine about planar fluid flow.

4 The essentially adequate quaternionic differentiation

First we establish consequences of assertion 3.3 of the essentially adequate (EA) differentiability concept. To get the correct conclusions there is a need to recall the well-known things, which are frequently not taken into consideration in the papers on the generalizations of complex analysis.

The complex division algebra representing operations on vectors in the Euclidean complex 2- dimensional plane is a normed algebra with identity element 1. Since 3-dimensional Euclidean space, say, consists of Euclidean 2-dimensional planes, it follows that a hypercomplex representation of operations on vectors in 3-dimensional space must also be a certain normed algebra with identity element1. We will say a few words about these properties of algebras; for details we refer to [6].

The normability concept characterizes in principle a possibility of "measuring" of a distance between two points in the Euclidean plane and Euclidean space. Such a distance is represented [2, 6, 8, 9] in complex algebra by the absolute value $|a|$ (the norm or length) of the complex vector $a = x + iy$:

$$|a| = \sqrt{a\bar{a}} = \sqrt{x^2 + y^2} = \sqrt{(a, a)},$$

where (a, a) is the so-called scalar product (see, e.g., [6], p. 94).The general expression of normability is defined usually as the norm property

$$|aa'| = |a||a'|.$$

The analogous formulae exist [6] in the 4-dimensional quaternion algebra:

$$|p| = \sqrt{p\bar{p}} = \sqrt{\bar{p}p} = \sqrt{x^2 + y^2 + z^2 + u^2} = \sqrt{(p, p)} = \sqrt{a\bar{a} + b\bar{\bar{b}}}, \ |pp'| = |p||p'|, \quad (4.1)$$

where $p = x + yi + zj + uk$ is an arbitrary quaternion and p' is another arbitrary quaternion.

The possibility of "measuring" of a line segment length such as $|\Delta\psi|$ and $|\Delta z|$ (or $|\Delta p|$), that is, the normability property of an acceptable algebra is the *first requirement*, which enable us in principle to obtain an expression for a spatial dilation constant similar to (3.1). Only in this case, it makes sense to use an expression similar to (2.1) for the definition of a hypercomplex derivative.

The *second requirement* is the possibility of the division operation in an acceptable hypercomplex number system. This enables us to define a hypercomplex derivative as a limiting value of a difference quotient similar to formula (2.1) used in complex analysis.

Now it makes sense to recall [6] the division definition in a hypercomplex number system. A hypercomplex number of dimension n can be written as follows:

$$u = u_1 1 + u_2 i_2 + u_3 i_3 + \ ... \ + u_n i_n, \quad (4.2)$$

where n is a fixed integer, and 1 is an identity element defined by the formula

$$u1 = 1u = u \quad (4.3)$$

for any u; $u_1, u_2, ... \ u_n$ are arbitrary real numbers, and $i_2, i_3, ... \ i_n$ are certain symbols ("imaginary units") with multiplication rule defined by some multiplication table (see, e.g., [6], p. 36). The operations defined in each system of hypercomplex numbers are addition, subtraction, and multiplication. The possibility of division depends on the system.

Let

$$v = v_1 1 + v_2 i_2 + v_3 i_3 + \ ... \ + v_n i_n,$$

be another hypercomplex number, where $v_1, v_2 \ ... \ v_n$ are real numbers such that $v \neq 0$.

A hypercomplex number system is called a division system if for all u and $v \neq 0$ each of equations:

$$vx = u \quad (4.4)$$

and

$$xv = u \qquad (4.5)$$

is uniquely solvable. The solution of equation (4.4) is called the left quotient of u by v, and the solution of equation (4.5) is called the right quotient of u by v. In general, two quotients are different.

The concept of an algebra is more general than that of a hypercomplex system. Any algebra of dimension n consists of elements that are representable in the form

$$u = u_1 i_1 + u_2 i_2 + u_3 i_3 + \ \dots + u_n i_n,$$

and are added, subtracted, multiplied, and divided in the same way as the hypercomplex numbers [6]. Every hypercomplex system may be viewed as an algebra in which the first basis element i_1 (in general, $\neq 1$) is replaced by the identity element 1.

If we "clear away" the terms with $3 \leq k \leq n$ in the expression (4.2) and in the corresponding multiplication table (e.g., in the table (2.10), where the units i_2, i_3, i_4 are denoted, respectively, by i, j, k), then we reduce the hypercomplex numbers of dimension n to the hypercomplex numbers of dimension 2. Starting with $n = 4$, we can write out three hypercomplex numbers of dimensions 4, 3, 2, respectively:

$$u = u_1 1 + u_2 i_2 + u_3 i_3 + u_4 i_4, \qquad (4.6)$$

$$u = u_1 1 + u_2 i_2 + u_3 i_3, \qquad (4.6a)$$

$$u = u_1 1 + u_2 i_2 \ , \qquad (4.6b)$$

where we assume that the latter denotes a complex number.

Instead of a 4-dimensional hypercomplex number (4.6) we consider now an element of an 4-dimensional algebra:

$$u = u_1 i_1 + u_2 i_2 + u_3 i_3 + u_4 i_4, \qquad (4.7)$$

where $i_1 \neq 1$ (the other $i_k \neq 1$, $k = 2, 3, 4$). If there is no identity elements 1 in the expression (4.7), then by starting with (4.7) and "clearing away" any two terms $u_k i_k$ in it, we cannot reduce this expression to the expression (4.6b) of the complex algebra, since the latter has the identity element 1. Hence, each acceptable generalization of the complex algebra corresponding to properties of the physical space and therefore "including" the complex algebra "as a limiting case", must contain the identity element 1. We can regard this as the *third requirement* that must be imposed on the algebra, underlying the EA hypercomplex differentiability.

Thus, we have shown that assertion 3.3 together with the natural third requirement leads to necessity of using of some normed division algebra with the identity element 1 to determine a hypercomplex derivative upon transition from the complex plane to space.

As is well known, (see, e.g., [6], p. 39), any 3-dimensional system of numbers of the form $u_1 1 + u_2 i_2 + u_3 i_3$, with any multiplication table, does not possess a division operation. Hence we need look for a hypercomplex division system in higher dimensions. The next extension (with division) beyond the complex numbers is to the quaternions. This can be explained as follows. According to Hurwitz's theorem, [6] "every normed algebra with an identity is isomorphic to one of following four algebras: the real numbers, the complex numbers, the quaternions, and the Cayley numbers". Hence, the only quaternion algebra can be the nearest algebra underlying the EA hypercomplex differentiability. Assertion 3.1 leads to this conclusion too.

Finally, we can state that the quaternion algebra remains the only algebra that satisfies assertions 3.1 and 3.3 of essentially adequate differentiability conditions. From this it follows that a hypercomplex generalization of complex differentiability must be only realized as a quaternion generalization.

4.1 Principal definitions of $\mathbb{H}$-differentiability and $\mathbb{H}$-holomorphicity

Let $\Delta p = \Delta a + \Delta b j$ be an arbitrary increment of the quaternion variable $p = a + bj$ in the Cayley–Dickson "doubling form" (see (2.9)). A corresponding increment of a quaternion function (see (2.14)) $\psi(p) = \psi(a,b) = \phi_1(a,b) + \phi_2(a,b)j$, at a point $p = a + bj = (a,b)$ can be denoted by $\Delta\psi(p) = \Delta\psi(a,b) = \Delta\phi_1(a,b) + \Delta\phi_2(a,b)j$. Now suppose that a function $\psi(p)$ is defined in domain $G_4 \subseteq \mathbb{H}$ and has in G_4 all first-order partial derivatives of complex functions $\phi_1, \overline{\phi}_1, \phi_2, \overline{\phi}_2$ with respect to complex variables $a, \overline{a}, b, \overline{b}$ in the usual sense, that is, as limiting values of corresponding quotients of the increments $\Delta\phi_1, \overline{\Delta\phi}_1, \Delta\phi_2$ and $\Delta\overline{\phi}_2$ by the increments $\Delta a, \Delta\overline{a}, \Delta b$ and $\overline{\Delta b}$. By a domain G_4 we understand, as usual, a connected, open set of points in the quaternion space $\mathbb{H}$. We define a quaternion-differentiable (briefly, $\mathbb{H}$-differentiable) function in accordance with the above concept of EA differentiability as follows.

Definition 4.1 *A single-valued function $\psi(p) : G_4 \to \mathbb{H}$ is $\mathbb{H}$-differentiable at a point $p \in G_4 \subseteq \mathbb{H}$ if there exists a limiting value (denoted by $\frac{d\psi(p)}{dp}$) of the difference quotient*

$$\frac{\Delta\psi}{\Delta p} \tag{4.8}$$

as $\Delta p \to 0$, and this value is independent of (i) how we let $\Delta p = \Delta a + \Delta b j$ approach zero, and (ii) how we divide $\Delta\psi(p) = \psi(p + \Delta p) - \psi(p)$ by Δp: on the left or on the right. We say also that $\psi(p)$ has a quaternionic derivative $\frac{d\psi(p)}{dp}$ at a point $p \in G_4$ in the mentioned sense.

In its essence, this definition is a "transfer" of complex definition (2.1) with the additional requirement (ii) of the independence of the division way. For the sequel, it is possible to introduce for both requirements (i) and (ii) to be used together in the definition of the quaternionic derivative, more succinctly a single notion of "independence of the way of computation ".

By analogy with complex analysis [9], we make the following definition of the quaternion-holomorphic (briefly, $\mathbb{H}$-holomorphic) functions.

Definition 4.2 *If a quaternionic function $\psi(p)$ is single-valued and $\mathbb{H}$-differentiable in some open connected neighborhood of $p \in \mathbb{H}$, then we say that this function is $\mathbb{H}$-holomorphic at a point p and denote it by $\psi_H(p)$. If $\psi(p)$ is $\mathbb{H}$ -differentiable at every point p in an open connected set $G_4 \subseteq \mathbb{H}$, then we say that $\psi_H(p)$ is $\mathbb{H}$-holomorphic on G_4.*

When speaking of a $\mathbb{H}$-differentiability or a $\mathbb{H}$-holomorphicity in the sequel we will use the general term "$\mathbb{H}$-holomorphicity".

4.2 The essentially adequate generalization of Cauchy-Riemann's equations

Now we show that Definition 4.1 leads to the following

Necessary condition for ψ (p) to be H- holomorphic. Continuing the analogy with real and complex numbers, we consider now two "directions" to approach a limiting point p for $p + \Delta p$ as $\Delta p = \Delta a + \Delta b \cdot j \to 0$: the way A) $\Delta p = \Delta a \to 0$ when $\Delta b \cdot j = 0$, and the way B) $\Delta p = \Delta b \cdot j \to 0$ when $\Delta a = 0$. They must be considered together with division on the left and division on the right in the expression of the difference quotient (4.8).

<u>The division on the left.</u> A) $\Delta \mathrm{p} = \Delta \mathrm{a} \to 0$, when $\Delta \mathrm{b} \cdot \mathrm{j} = 0$.

In this case the difference quotient (4.8) in accordance with (4.4) can be represented in the form

$$\Delta a\left(X_{L1(a)} + X_{L2(a)} \cdot j\right) = \Delta\phi_{1(a)} + \Delta\phi_{2(a)} \cdot j = \Delta\psi_{(a)},$$

where by $\left(X_{L1(a)} + X_{L2(a)} j\right)$ is denoted the solution of this equation for every Δa. For any $\Delta a \neq 0$ it follows that $X_{L1(a)} = \Delta\phi_{1(a)}/\Delta a$ and $X_{L2(a)} = \Delta\phi_{2(a)}/\Delta a$. Now we denote the limiting value of $X_{L1(a)}$ by $'\phi_{1(a)}(p)$ and the limiting value of $X_{L2(a)}$ by $'\phi_{2(a)}(p)$ as $\Delta p = \Delta a \to 0$. We obviously have

$$'\phi_{1(a)}(p) = \partial_a\phi_1, \quad '\phi_{2(a)}(p) = \partial_a\phi_2. \tag{4.9}$$

The partial complex derivatives $\partial_a\phi_1$and $\partial_a\phi_2$ are defined, respectively, as limits of quotients $\Delta\phi_{1(a)}/\Delta a$ and $\Delta\phi_{2(a)}/\Delta a$ as $\Delta p = \Delta a \to 0$, that is, in the same usual way as derivatives in real analysis. We suppose here (and in the sequel) that limits of all quotients, that is, all partial derivatives of functions ϕ_1, $\overline{\phi_1}$, ϕ_2, $\overline{\phi_2}$ with respect to a, $\overline{a}$, b, $\overline{b}$ exist and are independent of how we let Δa and Δb approach zero. Since the "arithmetic" of complex numbers is the same as that of real numbers, we can say that all formulae for computation of complex derivatives must be the same (see, e.g., [9], p. 41) as formulae for real derivatives. Thus using division on the left and the way A) $\Delta p = \Delta a \to 0$ ($\Delta b \cdot j = 0$) in the expression of the difference quotient (4.8), we get the following expression for the left derivative $'\psi_{(a)}(p)$:

$$'\psi_{(a)}(p) = {}'\phi_{1(a)}(p) + {}'\phi_{2(a)}(p)j = \partial_a\phi_1 + \partial_a\phi_2 j, \tag{4.10}$$

where index "(a)" and the left position of the derivative sign " ' " mean, respectively, that the way A) and division on the left are considered. For simplicity we omit the designation " $\cdot$ " of quaternion multiplication in front of " j " bearing in mind in the sequel that multiplication by "j" can be only carried out according to the quaternion multiplication rule.

<u>*The division on the left*</u>. B) $\Delta \mathrm{p} = \Delta \mathrm{bj} \to 0$, when $\Delta \mathrm{a} = 0$.

In this case the difference quotient (4.8) in accordance with (4.4) can be represented in the form

$$\Delta bj \cdot \left(X_{L1(b)} + X_{L2(b)}j\right) = \Delta\phi_{1(b)} + \Delta\phi_{2(b)}j = \Delta\psi_{(b)},$$

where by $\left(X_{L1(b)} + X_{L2(b)}j\right)$ is denoted the solution of this equation for every given Δbj. Using the left distributive law [6] of quaternion multiplication, we obtain

$$\Delta bjX_{L1(b)} + \Delta bjX_{L2(b)}j = \Delta\phi_{1(b)} + \Delta\phi_{2(b)}j.$$

Since the result must be represented in the "doubling form", where the unit " j " is always located after a complex value (see (2.9), (2.14)), we use the known (see, e.g., [6], p. 42) equality $jz = \bar{z}j$, $z \in \mathbb{C}$ as well as the associativity of quaternion multiplication. It follows that

$$-\Delta b\overline{X}_{L2(b)} + \Delta b\overline{X}_{L1(b)}j = \Delta\phi_{1(b)} + \Delta\phi_{2(b)}j.$$

Equating the terms without "j" on the left and the right sides of this equation, and analogically the expressions with "j", we get

$$-\Delta b\overline{X}_{L2(b)} = \Delta\phi_{1(b)}, \qquad \Delta b\overline{X}_{L1(b)} = \Delta\phi_{2(b)}.$$

Denoting by $'\overline{\phi}_{1(b)}(p)$ and by $'\overline{\phi}_{2(b)}(p)$, respectively, the limiting values of $\overline{X}_{L1(b)}$ and $\overline{X}_{L2(b)}$ as $\Delta b \to 0$, we can write

$$'\overline{\phi}_{2(b)}(p) = -\partial_b\phi_1, \qquad '\overline{\phi}_{1(b)}(p) = \partial_b\phi_2,$$

where derivatives are defined in the usual way as the limits of the quotients $\Delta\phi_{1(b)}/\Delta b$ and $\Delta\phi_{2(b)}/\Delta b$ as $\Delta b \to 0$. Finally, the complex conjugation of these expressions gives

$$'\phi_{1(b)}(p) = \overline{(\partial_b\phi_2)} = \partial_{\bar{b}}\overline{\phi}_2, \qquad '\phi_{2(b)}(p) = -\overline{(\partial_b\phi_1)} = -\,\partial_{\bar{b}}\overline{\phi}_1. \tag{4.11}$$

Thus, by using division on the left and the way B) $\Delta p = \Delta bj \to 0$ in the difference quotient (4.8), we get the following expression for the left derivative $'\psi_{(b)}(p)$:

$$'\psi_{(b)}(p) = {}'\phi_{1(b)}(p) + {}'\phi_{2(b)}(p)j = \partial_{\bar{b}}\overline{\phi}_2 - \partial_{\bar{b}}\overline{\phi}_1 j, \tag{4.12}$$

where index "(b)" and the left position of the derivative sign " $'$ " mean, respectively, that the way B) and division on the left are considered.

From the condition (i) of the above definition of quaternionic differentiability it follows that if division on the left is used in the expression (4.8), then it is necessary to satisfy the requirement: $'\psi_{(a)}(p) = {}'\psi_{(b)}(p)$, that is, (see (4.10), (4.12)) the following requirements:

$$'\phi_{1(a)}(p) = {}'\phi_{1(b)}(p), \qquad '\phi_{2(a)}(p) = {}'\phi_{2(b)}(p). \tag{4.13}$$

This gives the necessary equations:

$$\partial_a \phi_1 = \partial_{\overline{b}} \overline{\phi}_2, \qquad \partial_a \phi_2 = -\partial_{\overline{b}} \overline{\phi}_1, \tag{4.14}$$

which we will call the *left quaternionic generalization of the Cauchy-Riemann equations* (2.4).

Now we can state the following general expression for the left quaternionic derivative:

$$'\psi(p) = {}'\phi_1(p) + {}'\phi_2(p)j, \tag{4.15}$$

where in accordance with formulae (4.13), (4.9), and (4.11) we have

$$'\phi_1(p) = {}'\phi_{1(a)}(p) = {}'\phi_{1(b)}(p) = \partial_a \phi_1 = \partial_{\overline{b}} \overline{\phi}_2, \tag{4.16}$$

$$'\phi_2(p) = {}'\phi_{2(a)}(p) = {}'\phi_{2(b)}(p) = \partial_a \phi_2 = -\partial_{\overline{b}} \overline{\phi}_1.$$

In a manner similar as before, we consider now the cases of division on the right in the expression (4.8).

<u>*The division on the right*</u>. A) $\Delta \mathrm{p} = \Delta \mathrm{a} \to 0$, when $\Delta \mathrm{bj} = 0$.

In this case the difference quotient (4.8) in accordance with (4.5) can be represented in the form

$$\left(X_{R1(a)} + X_{R2(a)} j\right) \Delta a = \Delta \phi_{1(a)} + \Delta \phi_{2(a)} j = \Delta \psi_{(a)}.$$

Using the right distributive law [6] of quaternion multiplication, the associative law, and the equality $jz = \overline{z} j$, $z \in \mathbb{C}$, we get the following relations:

$$\phi'_{1(a)}(p) = \partial_a \phi_1, \qquad \phi'_{2(a)}(p) = \partial_{\overline{a}} \phi_2, \tag{4.17}$$

where index "(a)" and the right position of the derivative sign " $'$ " mean, respectively, that the way A) and division on the right are considered. By $\phi'_{1(a)}(p)$ and by $\phi'_{2(a)}(p)$ are denoted, respectively, the limiting values of $X_{R1(a)}$ and $X_{R2(a)}$ as $\Delta a \to 0$.

Finally, by using the way A), we can write the following expression for the right derivative:

$$\psi'_{(a)}(p) = \phi'_{1(a)}(p) + \phi'_{2(a)}(p) j = \partial_a \phi_1 + \partial_{\overline{a}} \phi_2 j. \tag{4.18}$$

<u>*The division on the right*</u>. B) $\Delta p = \Delta bj \to 0$, when $\Delta a = 0$.

In this case the difference quotient (4.8) in accordance with (4.5) can be represented in the form

$$\left(X_{R1(b)} + X_{R2(b)} j\right) \Delta bj = \Delta \phi_{1(b)} + \Delta \phi_{2(b)} j = \Delta \psi_{(b)}.$$

Denoting by $\phi'_{1(b)}(p)$ and by $\phi'_{2(b)}(p)$, respectively, the limiting values of $X_{R1(b)}$ and $X_{R2(b)}$ as $\Delta b \to 0$, we have

$$\phi'_{1(b)}(p) = \partial_b \phi_2, \qquad \phi'_{2(b)}(p) = -\partial_{\overline{b}} \phi_1, \tag{4.19}$$

where partial derivatives are defined in the usual way as limits of the quotients $\Delta\phi_{2(b)}/\Delta b$ and $\Delta\phi_{1(b)}/\Delta\overline{b}$ as Δb, $\Delta\overline{b} \to 0$.

Thus, using division on the right and the way B) $\Delta p = \Delta bj \to 0$ $(\Delta a = 0)$ in the difference quotient (4.8), we get the following expression for the right derivative $\psi'_{(b)}(p)$:

$$\psi'_{(b)}(p) = \phi'_{1(b)}(p) + \phi'_{2(b)}(p)j = \partial_b\phi_2 - \partial_{\overline{b}}\phi_1 j. \tag{4.20}$$

From the condition (i) of Definition 4.1 it follows that if the division on the right in the expression (4.8) is used, then the requirement $\psi'_{(a)}(p) = \psi'_{(b)}(p)$ must be satisfied. If we bear in mind formulae (4.18), (4.20), then from this requirement, we get the conditions:

$$\phi'_{1(a)}(p) = \phi'_{1(b)}(p), \qquad \phi'_{2(a)}(p) = \phi'_{2(b)}(p), \tag{4.21}$$

which in accordance with (4.17) and (4.19) lead to the following necessary equations:

$$\partial_a\phi_1 = \partial_b\phi_2, \qquad \partial_{\overline{a}}\phi_2 = -\,\partial_{\overline{b}}\phi_1. \tag{4.22}$$

We will call equations (4.22) *the right quaternionic generalization of the Cauchy-Riemann equations* (2.4).

Now we can state the general expression for the right quaternionic derivative:

$$\psi'(p) = \phi'_1(p) + \phi'_2(p)j, \tag{4.23}$$

where in accordance with formulae (4.21), (4.17), and (4.19) we have

$$\phi'_1(p) = \phi'_{1(a)}(p) = \phi'_{1(b)}(p) = \partial_a\phi_1 = \partial_b\phi_2 \tag{4.24}$$

$$\phi'_2(p) = \phi'_{2(a)}(p) = \phi'_{2(b)}(p) = \partial_{\overline{a}}\phi_2 = -\partial_{\overline{b}}\phi_1.$$

Expressions (4.15) and (4.23) for the left and right quaternionic derivative (just as the equations of the left and right quaternion generalization of the Cauchy-Riemann equations) are obtained as the result of satisfying the requirement (i) of Definition 4.1. Now our intention is to satisfy the requirement (ii) of that definition. To do this we *have to require* the equality of the left (4.15) and right (4.23) quaternionic derivative:

$$'\psi(p) \equiv \psi'(p),$$

that is,

$$'\phi_1(p) + {}'\phi_2(p)j \equiv \phi'_1(p) + \phi'_2(p)j, \tag{4.25}$$

where (and in the sequel) the symbol "≡" means that we require an additional equality. This means that in addition to differentiability conditions (4.14) and (4.22) we must also consider the following essential requirements:

$$'\phi_1(p) \equiv \phi'_1(p), \quad {}'\phi_2(p) \equiv \phi'_2(p).$$

Using (4.16) and (4.24), we can write the last conditions as

$$'\phi_1(p) = \partial_a\phi_1 = \partial_{\overline{b}}\overline{\phi}_2 \equiv \phi'_1(p) = \partial_a\phi_1 = \partial_b\phi_2, \tag{4.26}$$

$$'\phi_2(p) = \partial_a\phi_2 = -\,\partial_{\overline{b}}\overline{\phi}_1 \equiv \phi'_2(p) = \partial_{\overline{a}}\phi_2 = -\,\partial_{\overline{b}}\phi_1. \tag{4.27}$$

Since the partial derivative $\partial_a \phi_1$ is contained in the expressions for ${}'\phi_1(p)$ and $\phi_1'(p)$, it follows that the condition (4.26) is satisfied, so to say, "automatically", if left and right differentiability conditions (4.14) and (4.22) are satisfied. In order to satisfy the condition (4.27) we need only to require the equality of partial derivatives $\partial_a \phi_2$ and $\partial_{\overline{a}} \phi_2$, that is, $\partial_a \phi_2 \equiv \partial_{\overline{a}} \phi_2$, from which it follows that the equality $\partial_{\overline{b}} \overline{\phi}_1 = \partial_{\overline{b}} \phi_1$ holds too if conditions (4.14) and (4.22) are satisfied.

Formally, the requirement $\partial_a \phi_2 \equiv \partial_{\overline{a}} \phi_2$ can be satisfied in only two ways:

$$\partial_a \equiv \partial_{\overline{a}} \left(\equiv \tfrac{1}{2} \partial_x\right) \tag{4.28}$$

and

$$a \equiv \overline{a} \; (\equiv x), \tag{4.29}$$

where expressions in parentheses are obtained from the formulae (2.5), (2.7), (2.18), and (2.19). We will mostly use the simple sign of equality "$=$" instead of "$\equiv$".

The first of these requirements is imposed on the differential operators ∂_a and $\partial_{\overline{a}}$; the second is only imposed on the variables a and $\overline{a}$ contained in expressions of "already obtained" partial derivatives in (4.14) and (4.22). Note that the requirement (4.29) cannot be initially imposed on a quaternionic variable and a quaternionic function, only on "computed" partial derivatives. Since the partial derivatives with respect to a and $\overline{a}$ are "already computed" when formulating the *left* (4.14) and *right* (4.22) generalized Cauchy-Riemann's equations (an application of differential operators has been done), it is impossible to modify differential operators in these equations (in (4.27) too), that is, use (4.28). We can only use the condition (4.29), when formulating the *complete quaternionic generalization* of Cauchy-Riemann's equations and further when using it for checking holomorphicity of functions. As regards the condition (4.28), it can be only interpreted as an additional differential requirement that unlike the condition (4.29) can be used further to clarify the expressions for the complete quaternionic derivatives.

Thus, we have established that the requirement $a = \overline{a} = x$ imposed on the variables in expressions for "computed" partial derivatives in (4.14) and (4.22) is the EA condition of quaternionic differentiability (holomorphicity), according to the requirement (ii) of Definition 4.1.

Note that since the equality $\partial_a \phi_2 = \partial_{\overline{a}} \phi_2$ holds upon application of $a = \overline{a} = x$, it follows from (4.27) that the equality $\partial_{\overline{b}} \overline{\phi}_1 = \partial_{\overline{b}} \phi_1$ holds too, and we can state that the equality

$$\phi_1(p) = \overline{\phi_1}(p) = \psi_1(x, y, z, u) \tag{4.30}$$

follows from the requirement $a = \overline{a} = x$ for derivatives of $\mathbb{H}$-holomorphic functions.

Summarizing the results of the left (4.14) and right (4.22) quaternion generalizations of the Cauchy-Riemann equations and the condition (4.29), we can write the following *general system of EA conditions of quaternionic differentiability*:

$$1)\ \partial_a \phi_1 = \partial_{\overline{b}} \overline{\phi}_2, \qquad 2)\ \partial_a \phi_2 = -\partial_{\overline{b}} \overline{\phi}_1$$

$$\text{(after doing } a \equiv \overline{a} \equiv x\text{)} \tag{4.31}$$

$$3)\ \partial_a \phi_1 = \partial_b \phi_2, \qquad 4)\ \partial_{\overline{a}} \phi_2 = -\partial_{\overline{b}} \phi_1.$$

We will call this system the *complete EA quaternionic generalization of the Cauchy-Riemann equations (GCR-equations).* It may be remarked, by the way, that the system (4.31) has been submitted previously in [12] by the author. We see that the conditions of quaternionic holomorphicity (4.31) are defined so that *in order to check the holomorphicity of a function, we need to carry out the transition $a = \overline{a} = x$ in expressions for its derivatives (and only in them).* However, any quaternion function of a quaternion variable remains the same quaternion function regardless of whether we check its quaternionic holomorphicity or not.

It makes sense to speak of "*conceptual requirements*" of the EA differentiability theory under consideration bearing in mind the necessity of satisfying the requirements (4.28) and (4.29). The requirement (4.28) of this concept exists, so to say, "in parallel" with the condition (4.29) but independently of the formulation of GCR-equations (4.31). Thus the system (4.31) remains the same when checking not only the quaternionic differentiability of any function but also the quaternionic differentiability of its derivatives. The condition (4.28) can give equalities, which don't belong to the system (4.31). Equations (4.31-1) and (4.31-2) correspond to the left quaternion division in the expression (4.8), and equations (4.31-3) and (4.31-4) to the right quaternion division. It is easy to check by direct computation that among power functions p^n, where n is integer, the only functions of degrees $n = 0$ and $n = 1$ satisfy equations (4.31-1) and (4.31-2) as well as equations (4.31-3) and (4.31-4) by themselves, that is, without the condition $a = \overline{a}$. This corresponds to the so-called Meĭlihzon result [1, 2, 3] that states that only linear functions are solutions of the left or right equations of the same type as (4.31-1,2), or (4.31-3,4). In this special case the partial derivatives in (4.31) are independent of variables a, $\overline{a}$, b, $\overline{b}$. Conversely, we will show below that power functions $p^n$of degrees $n \geq 2$ are solutions of the EA system (4.31), that is, of the system of the left and right equations together with the condition $a = \overline{a} = x$.

The condition $a = \overline{a} = x$ is essential to the quaternionic differentiability theory under consideration. It implements assertions 3.2 and 3.3 of the above concept of EA conditions of differentiability. This additional condition is associated with a new property of quaternionic analysis having no counterparts in complex analysis, namely, the existence of two different results of division. Therefore, the system (4.31) can't be "reduced" to the system of Cauchy-Riemann's complex equations.

It is of interest to compare equations (4.31) with the Cauchy-Riemann-Fueter equations (1.1) and (1.2). For this we can represent equations (1.1) and (1.2) as equations $\overline{\partial}\psi = 0$ and $\psi\overline{\partial} = 0$ [2, 3], using the formal multiplication (2.12) of the Cauchy-Riemann operator $\overline{\partial}$ (see (2.24)) by a quaternion function $\psi(p) = \phi_1 + \phi_2 j$ in the Cayley–Dickson doubling form. Multiplying $\overline{\partial}$ on the left by $\psi(p)$ we obtain

$$\overline{\partial}\psi = 2(\partial_{\overline{a}} + \partial_{\overline{b}} j) \cdot (\phi_1 + \phi_2 j) = 2(\partial_{\overline{a}}\phi_1 - \partial_{\overline{b}}\overline{\phi}_2) + 2(\partial_{\overline{a}}\phi_2 + \partial_{\overline{b}}\overline{\phi}_1)j = 0,$$

whence follows the system of the left-regularity equations equivalent to (1.1):

$$\partial_{\overline{a}}\phi_1 = \partial_{\overline{b}}\overline{\phi}_2, \qquad \partial_{\overline{a}}\phi_2 = -\,\partial_{\overline{b}}\overline{\phi}_1. \tag{4.32}$$

Similarly, we get the system of the right-regularity equations equivalent to (1.2):

$$\partial_{\overline{a}}\phi_1 = \partial_b\phi_2, \qquad \partial_a\phi_2 = -\,\partial_{\overline{b}}\phi_1. \tag{4.33}$$

The systems (4.32) and (4.33) cannot be regarded as essentially adequate. They do not satisfy assertions 3.2 and 3.3 of the above concept of EA differentiability. Note that there is no essential difference between these systems considered together and the system (4.31) taken without the condition $a = \overline{a}$. All these systems can be, in principle, "reduced" to Cauchy–Riemann's equations of complex analysis due to the absence of an essentially new requirement (similar to $a = \overline{a}$) reflecting the essential difference between the complex plane and space.

Thus, the Cauchy-Riemann-Fueter equations (1.1) and (1.2) can be regarded only as inessentially adequate to properties of space.

Taking into account (4.15), (4.23), (4.25) - (4.27), we obtain the following expression for the EA quaternionic derivative after doing the condition $a = \overline{a}$:

$$\frac{\partial\psi}{\partial p} := k(\partial_p\phi_1 + \partial_p\phi_2 j), \tag{4.34}$$

where k is a constant factor associated with the obvious linearity of equations (4.31); $\partial_p\phi_1$ and $\partial_p\phi_2$ are determined by relations

$$\partial_p\phi_1 := \partial_a\phi_1 = \partial_{\overline{b}}\overline{\phi}_2 = \partial_b\phi_2, \tag{4.35}$$

$$\partial_p\phi_2 := \partial_a\phi_2 = -\,\partial_{\overline{b}}\overline{\phi}_1 = \partial_{\overline{a}}\phi_2 = -\,\partial_{\overline{b}}\phi_1,$$

that must be valid after doing the condition $a = \overline{a}$ if $\psi(p) = \phi_1 + \phi_2 j$ is $\mathbb{H}$-differentiable ($\mathbb{H}$-holomorphic) at a point p. *We see that the expression (4.34) follows from Definition 4.1 and gives the derivative that is "independent of the way of computation".*

When considering complex variables a and b in expressions for functions ϕ_1 and ϕ_2 we speak (by analogy with [10]) of a C^2-representation. As a rule, the C^2-representation leads to the shortest calculations. If we consider real variables x, y, z, u in expressions for functions ϕ_1 and ϕ_2, then we speak of an R^4-representation. As noted earlier, the equality $\phi_1 = \overline{\phi}_1 = \psi_1(x, y, z, u)$ follows from the condition $a = \overline{a} = x$ whenever the function $\psi(p) = \psi(a, b) = \phi_1(a, b) + \phi_2(a, b) \cdot j$ satisfies equations (4.31). It follows that in the R^4-representation the equalities $y = 0$, $\psi_2(x, y, z, u) = 0$ hold too if $a = \overline{a} = x$ holds. Substituting (2.18) and (2.19) into (4.28), we can get the requirement (4.28) in the R^4-representation: $\partial_y = 0$.

If the conditions $a = \overline{a}$ and $\phi_1 = \overline{\phi}_1$ or, respectively, $y = 0$ and $\psi_2(x, y, z, u) = 0$ are fulfilled first of all, then we have immediately 3-dimensional hypercomplex expressions (triplets), namely, $p = x + zj + uk$ and $\psi(p) = \psi_1(x, z, u) + \psi_3(x, z, u)j + \psi_4(x, z, u)k$, for which the operation of division and hence Definitions 4.1 and 4.2 are impossible. Therefore, it is important to recall that *the requirement (4.29) , that is, $a = \overline{a} = x$ cannot be initially imposed on a quaternionic variable or a quaternionic function. It can be only executed in expressions after computation of partial derivatives of the functions ϕ_1 and ϕ_2 to be used in system (4.31).* In other words, it is possible to use the only following sequence of actions.

Computation rule in the C^2-representation. Firstly, we compute the partial derivatives of the functions ϕ_1, ϕ_2, $\overline{\phi}_1$ and $\overline{\phi}_2$ with respect to the variables a, $\overline{a}$, b, or $\overline{b}$ contained in the system (4.31); secondly, we put $a = \overline{a} = x$ in the computed expressions of partial derivatives; and thirdly, we check whether equations (4.31) hold.

The same sequence of actions but when calculating partial derivatives with respect to x, y, z, u and performing the condition $y = 0$, must be carried out if we check whether equations (4.31) hold in the R^4-representation. This representation of equations (4.31) can be readily obtained by substituting (2.15), (2.16), (2.18), (2.20) and their conjugates into (4.31), however, we shall not dwell on this here.

To denote the correct sequence of actions when we apply the requirement $a = \overline{a} = x$, we introduce a special notation. Let $f(a, b, \overline{a}, \overline{b})$ be any function; then the notation $(f(a, b, \overline{a}, \overline{b})|$ (as well as $[f(a, b, \overline{a}, \overline{b})|$ or $\{f(a, b, \overline{a}, \overline{b})|$ for "complicated" expressions), briefly, brackets $(..|$, where instead of the end parenthesis we use the vertical bar, will show that we have put $a = \overline{a} = x$ in the expression in brackets, that is, in $f(a, b, \overline{a}, \overline{b})$. Using this notation we can rewrite equations (4.31) as follows:

$$1)\ \ (\partial_a \phi_1| = (\partial_{\overline{b}} \overline{\phi}_2|, \qquad 2)\ \ (\partial_a \phi_2| = -\,(\partial_{\overline{b}} \overline{\phi}_1|, \qquad (4.36)$$
$$3)\ \ (\partial_a \phi_1| = (\partial_b \phi_2|, \qquad 4)\ \ (\partial_{\overline{a}} \phi_2| = -\,(\partial_{\overline{b}} \phi_1|.$$

We will call this system just as the system (4.31), the *complete EA quaternionic generalization of the Cauchy-Riemann equations (GCR-equations).*

Further, the expressions (4.34) and (4.35) for the quaternionic derivative can be rewritten as follows:

$$\frac{\partial\psi}{\partial p} := k[(\partial_p\phi_1| + (\partial_p\phi_2|j], \tag{4.37}$$

where $(\partial_p\phi_1|$ and $(\partial_p\phi_2|$ are determined by expressions

$$(\partial_p\phi_1| := (\partial_a\phi_1| = (\partial_{\overline{b}}\overline{\phi}_2| = (\partial_b\phi_2|, \tag{4.38}$$

$$(\partial_p\phi_2| := (\partial_a\phi_2| = -\,(\partial_{\overline{b}}\overline{\phi}_1| = (\partial_{\overline{a}}\phi_2| = -\,(\partial_{\overline{b}}\phi_1|,$$

and by $\frac{\partial\psi}{\partial p}$ is denoted the derivative after performing the conceptual condition (4.29). In particular, it follows that

$$\frac{\partial\psi}{\partial p} := k[(\partial_a\phi_1| + (\partial_a\phi_2|j]. \tag{4.39}$$

This expression is the quaternion analogue of the complex derivative in the usual notation [9]:

$$\frac{\partial\psi}{\partial z} = \partial_x u(x,y) + i\partial_x v(x,y).$$

We have shown that the system of equations (4.31) (or (4.36)) is the necessary condition for a quaternionic function $\psi(p)$ to be $\mathbb{H}$-holomorphic in $G_4 \subseteq \mathbb{H}$. Now our intention is to show that this system is also the sufficient condition.

Sufficient condition for ψ (p) to be $\mathbb{H}$-holomorphic. To show this we suppose that a quaternionic function $\psi(p,\overline{p}) = \phi_1(a,\overline{a},b,\overline{b}) + \phi_2(a,\overline{a},b,\overline{b})\cdot j$ is single-valued and continuous at all points $p \in G_4 \subseteq \mathbb{H}$ and that it has in G_4 the continuous first-order partial derivatives of functions ϕ_1 and ϕ_2 with respect to variables $a,\overline{a},b,\overline{b}$. Then we can (see, e.g., [9]) write

$$\Delta\phi_1 = \phi_1(a+\Delta a, b+\Delta b, \overline{a}+\Delta\overline{a}, \overline{b}+\Delta\overline{b}) - \phi_1(a,b,\overline{a},\overline{b})$$
$$= (\partial_a\phi_1)\Delta a + (\partial_b\phi_1)\Delta b + (\partial_{\overline{a}}\phi_1)\Delta\overline{a} + (\partial_{\overline{b}}\phi_1)\Delta\overline{b} + o_1(|\Delta p|),$$
$$\Delta\phi_2 = \phi_2(a+\Delta a, b+\Delta b, \overline{a}+\Delta\overline{a}, \overline{b}+\Delta\overline{b}) - \phi_2(a,b,\overline{a},\overline{b})$$
$$= (\partial_a\phi_2)\Delta a + (\partial_b\phi_2)\Delta b + (\partial_{\overline{a}}\phi_2)\Delta\overline{a} + (\partial_{\overline{b}}\phi_2)\Delta\overline{b} + o_2(|\Delta p|),$$

where $o_1(|\Delta p|)$ and $o_2(|\Delta p|)$ converge to zero faster than $|\Delta p| = |\Delta a + \Delta b j| = |\Delta\overline{p}|$.

Thus altogether,

$$\Delta\psi(p) = \Delta\phi_1 + \Delta\phi_2\cdot j = (\partial_a\phi_1)\Delta a + (\partial_b\phi_1)\Delta b + (\partial_{\overline{a}}\phi_1)\Delta\overline{a} + (\partial_{\overline{b}}\phi_1)\Delta\overline{b}$$
$$+(\partial_a\phi_2)\Delta a j + (\partial_b\phi_2)\Delta b j + (\partial_{\overline{a}}\phi_2)\Delta\overline{a} j + (\partial_{\overline{b}}\phi_2)\Delta\overline{b} j + o(|\Delta p|),$$

where $o(|\Delta p|) = o_1(|\Delta p|) + o_2(|\Delta p|)j$ converges to zero faster than $|\Delta p|$, that is, $\frac{o(|\Delta p|)}{|\Delta p|} \to 0$ as $|\Delta p| \to 0$. This expression represents the total infinitesimal increment of the function $\psi(p,\overline{p}) = \phi_1(a,\overline{a},b,\overline{b}) + \phi_2(a,\overline{a},b,\overline{b})\cdot j$ owing to infinitesimal increments of all its arguments. Rearranging the terms, we obtain

$$\Delta\psi(p) = \{(\partial_a\phi_1)\Delta a + (\partial_a\phi_2)\Delta a j + (\partial_{\overline{b}}\phi_2)\Delta\overline{b} j + (\partial_b\phi_1)\Delta b\}$$
$$+\{(\partial_{\overline{a}}\phi_1)\Delta\overline{a} + (\partial_{\overline{a}}\phi_2)\Delta\overline{a} j + (\partial_b\phi_2)\Delta b j + (\partial_{\overline{b}}\phi_1)\Delta\overline{b}\} + o(|\Delta p|). \quad (4.40)$$

Now our intention is to show by means of transformations of this expression that if the functions $\phi_1(a,\overline{a},b,\overline{b})$ and $\phi_2(a,\overline{a},b,\overline{b})$ satisfy GCR-equations (4.36), then $\frac{\partial\psi}{\partial p}$ exists (Definition 4.1) and coincides up to a constant factor k with the one of expressions (4.37), in particular, with (4.39). We must use the operations of taking limits when $\Delta a, \Delta\overline{a}, \Delta b, \Delta\overline{b}$ tend to zero together with the additional condition $a = \overline{a} = x$. It is possible to perform these operations step by step until the process is fully completed. We assume that it is possible to replace a certain term in (4.40), say, "X" by another term "Y" if both of them are eventually equal by using these operations, equations (4.36), and the conceptual requirements (4.28) and (4.29).

For now it is important to compare the derivatives $\partial_a\phi_1$ and $\partial_{\overline{a}}\phi_1$. Formally, it follows from the conceptual requirement (4.28) that these derivatives are equal (in principle, "when $a = \overline{a}$"). We cannot use the introduced notation $(..|$ directly in this case, because the straightforward computation in accordance with the above computation rule in the C^2-representation is not obligatory to lead to the equality of derivatives $\partial_a\phi_1$and $\partial_{\overline{a}}\phi_1$. This is so because such an equality doesn't belong to the system of equations (4.36). Therefore, such an equality, based on the general concept of the theory under consideration, can only be regarded as the additional requirement imposed on the operators ∂_a and $\partial_{\overline{a}}$. We can use the simple notation in accordance with (4.28) as follows:

$$\partial_a\phi_1 \equiv \partial_{\overline{a}}\phi_1 \left(\equiv \tfrac{1}{2}\,\partial_x\phi_1\right). \quad (4.41)$$

Then we can formally state that the expression

$$[(\partial_{\overline{a}}\phi_1)d\overline{a}| = [(\partial_a\phi_1)da|$$

is valid. In this case we can, as noted earlier, replace the fifth term $(\partial_{\overline{a}}\phi_1)\Delta\overline{a}$ in the expression (4.40) by the term $(\partial_a\phi_1)\Delta a$. After this partial replacement we can use further the computation rule in the C^2-representation and the notation $(..|$.

Now we consider the third term in (4.40), namely, $(\partial_{\overline{b}}\phi_2)\Delta\overline{b} j$. We want to show that the relation

$$(\partial_b\phi_2)\Delta b = (\partial_{\overline{b}}\phi_2)\Delta\overline{b} \quad (4.42)$$

is valid in the limit when $\Delta b, \Delta\overline{b}$ tend to zero and if equations (4.36) hold. We can assume in our proof that this relation holds approximately for sufficiently small values $\Delta b, \Delta\overline{b}$ and show further how it can be reduced to the precise equality in the limit $\Delta b, \Delta\overline{b} \to 0$. From (4.26) it follows that the relation

$$\partial_{\overline{b}}\overline{\phi}_2 = \partial_b\phi_2 \tag{4.43}$$

is valid. Note that we do not even need to require $a = \overline{a} = x$ in this case. Substituting this relation into (4.42), we obtain the following expression:

$$(\partial_{\overline{b}}\overline{\phi}_2)\Delta b = (\partial_{\overline{b}}\phi_2)\Delta\overline{b}.$$

It is now easy to see that this expression can be formally reduced to

$$\frac{\partial\overline{\phi}_2}{\Delta\overline{b}} = \frac{\partial\phi_2}{\Delta b},$$

and hence to the expression

$$\frac{\partial\overline{\phi}_2}{(\partial\overline{b}+\varepsilon_1\Delta\overline{b})} = \frac{\partial\phi_2}{(\partial b+\varepsilon_2\Delta b)},$$

where $\varepsilon_1 \to 0$ as $\Delta\overline{b} \to 0$ and $\varepsilon_2 \to 0$ as $\Delta b \to 0$, that is, $\varepsilon_1\Delta\overline{b} \to 0$ more rapidly than $\partial\overline{b} \to 0$ as $\Delta\overline{b} \to 0$ and $\varepsilon_2\Delta b \to 0$ more rapidly than $\partial b \to 0$ as $\Delta b \to 0$. Taking the limits as $\Delta b, \Delta\overline{b} \to 0$ in the last expression we get the true relation (4.43) from (4.42). Therefore (4.42) is valid in the above sense. Thus, the replacement of the third term $(\partial_{\overline{b}}\phi_2)\Delta\overline{b}j$ by the term $(\partial_b\phi_2)\Delta bj$ in (4.40) is possible.

Making the noted replacements of the third and fifth terms in (4.40), we obtain

$$\Delta\psi(p) = \{(\partial_a\phi_1|\Delta a + (\partial_a\phi_2)\Delta aj + (\partial_b\phi_2)\Delta bj + (\partial_b\phi_1)\Delta b\} +$$
$$\{(\partial_a\phi_1|\Delta a + (\partial_{\overline{a}}\phi_2)\Delta\overline{a}j + (\partial_b\phi_2)\Delta bj + (\partial_{\overline{b}}\phi_1)\Delta\overline{b}\} + o(|\Delta p|),$$

where $o(|\Delta p|) = o_1(|\Delta p|) + o_2(|\Delta p|)j$ converges to zero faster than $|\Delta p|$.
Using the multiplicative commutativity of complex numbers, the associativity of quaternion multiplication as well as formulae $cj = j\overline{c}$ for $c \in \mathbb{C}$ (see, e.g., [3, 6]) and $j^2 = -1$, we get

$$\Delta\psi(p) = \{\Delta a(\partial_a\phi_1| + \Delta a(\partial_a\phi_2)j + \Delta bj(\partial_{\overline{b}}\overline{\phi}_2) - \Delta bj(\partial_{\overline{b}}\overline{\phi}_1)j\} +$$
$$\{(\partial_a\phi_1|\Delta a + (\partial_{\overline{a}}\phi_2)j\Delta a + (\partial_b\phi_2)\Delta bj - (\partial_{\overline{b}}\phi_1)j\Delta bj\} + o(|\Delta p|).$$

Further, taking into account the left and right distributive laws (see [6], p. 38) of quaternion multiplication, we get the following expression:

$$\Delta\psi(p) = \{\Delta a[(\partial_a\phi_1| + (\partial_a\phi_2)j] + \Delta bj[(\partial_{\overline{b}}\overline{\phi}_2) - (\partial_{\overline{b}}\overline{\phi}_1)j]\} + \tag{4.44}$$
$$\{[(\partial_a\phi_1| + (\partial_{\overline{a}}\phi_2)j]\Delta a + [(\partial_b\phi_2) - (\partial_{\overline{b}}\phi_1)j]\Delta bj\} + o(|\Delta p|).$$

Setting $a = \overline{a} = x$ only in expressions for derivatives in (4.44) and using equations (4.36-1,2) in the first braces (the third and fourth terms) as well as equations (4.36-3,4) in the second braces (the third and fourth terms), we obtain

$$\Delta\psi(p) = \{\Delta a[(\partial_a\phi_1| + (\partial_a\phi_2|j] + \Delta bj[(\partial_a\phi_1| + (\partial_a\phi_2|j]\} +$$
$$\{[(\partial_a\phi_1| + (\partial_{\overline{a}}\phi_2|j]\Delta a + [(\partial_a\phi_1| + (\partial_{\overline{a}}\phi_2|j]\Delta bj\} + o(|\Delta p|).$$

The application of the left and right distributive laws to this expression yields

$$\Delta\psi(p) = \Delta\phi_1 + \Delta\phi_2 j = (\Delta a + \Delta bj) \cdot [(\partial_a\phi_1| + (\partial_a\phi_2|j] + [(\partial_a\phi_1| + (\partial_{\overline{a}}\phi_2|j] \cdot (\Delta a + \Delta bj) + o(|\Delta p|), \quad (4.45)$$

where $o(|\Delta p|) = o_1(|\Delta p|) + o_2(|\Delta p|)j$ converges to zero faster than $|\Delta p|$.

It is not difficult to see that the first and second terms in (4.45) are, respectively, the "left" and "right" (total) infinitesimal changes in the value of function $\psi(p)$ due to the infinitesimal change $\Delta p = \Delta a + \Delta bj$. The left increment in (4.45) includes the left derivative $'\psi(p) = (\partial_a\phi_1| + (\partial_a\phi_2|j$ defined by (4.15) and the right increment includes the right derivative $\psi'(p) = (\partial_a\phi_1| + (\partial_{\overline{a}}\phi_2|j$ defined by (4.23). Since the increments Δa, Δb, and $\Delta p = \Delta a + \Delta bj$ are arbitrary, it follows that both derivatives are independent of how we let $\Delta p = \Delta a + \Delta bj$ approach zero. *Thus the condition (i) of Definition 4.1 is satisfied.*

In accordance with (4.38) the equality $(\partial_a\phi_2| = (\partial_{\overline{a}}\phi_2|$ holds, then the left and right quaternion derivatives in (4.45) are equal and hence the *condition (ii) of Definition 4.1 is satisfied too*. Both derivatives coincide up to a constant factor with the derivative defined by (4.39).

Thus, we have shown that GCR-equations (4.36) (or (4.31)) are not only necessary but also sufficient conditions for the function $\psi(p)$ to be $\mathbb{H}$-holomorphic in G_4 if we assume that the continuous first-order partial derivatives of functions ϕ_1 and ϕ_2 with respect to variables $a, b, \overline{a}, \overline{b}$ exist at points $p \in G_4 \in \mathbb{H}$. This allows us to introduce the following definition of a $\mathbb{H}$-holomorphic function in complete analogy to complex analysis.

Definition 4.3 *A single-valued quaternion function $\psi(p) = \psi(a, b) = \phi_1(a, b) + \phi_2(a, b) \cdot j$, where $\phi_1(a, b)$ and $\phi_2(a, b)$ have continuous first-order partial derivatives with respect to $a, \overline{a}, b$, and $\overline{b}$ in some open connected neighborhood G_4 of a point $p = a + bj \in G_4 \in \mathbb{H}$, is $\mathbb{H}$-holomorphic at that point if and only if the functions $\phi_1(a, b)$ and $\phi_2(a, b)$ satisfy equations (4.36) in G_4.*

From (4.45), in the limit as Δa, Δb, and hence $\Delta p = \Delta a + \Delta bj \to 0$, we get the following expression for the total differential of a $\mathbb{H}$-holomorphic function $\psi(p, \overline{p}) = \phi_1(a, \overline{a}, b, \overline{b}) + \phi_2(a, \overline{a}, b, \overline{b}) \cdot j$:

$$d\psi(p) = dp \cdot [(\partial_a\phi_1| + (\partial_a\phi_2|j] + [(\partial_a\phi_1| + (\partial_{\overline{a}}\phi_2|j] \cdot dp, \quad (4.46)$$

where $dp = da + dbj$. Using the argumentation as above, it is not difficult to show that

$$dp \cdot [(\partial_a\phi_1| + (\partial_a\phi_2|j] = [(\partial_a\phi_1| + (\partial_{\overline{a}}\phi_2|j] \cdot dp.$$

We shall not dwell on this here. Note that for simplicity we do not use here the notation $d(p|$ for the final transition $a = \overline{a} = x$. Taking into account this equality and $(\partial_a\phi_2| = (\partial_{\overline{a}}\phi_2|$ we can rewrite (4.46) as follows:

$$d\psi(p) = 2[(\partial_a\phi_1| + (\partial_a\phi_2|j] \cdot dp. \quad (4.47)$$

It makes sense to compare this expression with the expression for the total differential (see, e.g., [2, 8, 9]) of the complex function $\psi(a)$:

$$d\psi(a) = (\partial_a \psi)da + (\partial_{\overline{a}}\psi)d\overline{a}, \tag{4.48}$$

where by $a = x + yi$ is denoted a complex variable; ∂_a and $\partial_{\overline{a}}$ are differential operators defined by (2.18) and (2.19). If $\psi(a)$ is $\mathbb{C}$-holomorphic (analytic) [2, 8, 9], then

$$\partial_{\overline{a}}\psi = 0\ , \tag{4.49}$$

and (4.48) becomes

$$d\psi(a) = (\partial_a \psi)da \tag{4.50}$$

The expression (4.47) for the total differential of a $\mathbb{H}$-holomorphic function is the EA generalization of the expression (4.50) for the total differential of a $\mathbb{C}$-holomorphic function. Note that expressions for the total differentials in both cases of holomorphicity are similar in the sense that the expression (4.47) is independent of the conjugate quaternion variable $\overline{p}$ just as the expression (4.50) is independent of the conjugate complex variable $\overline{a}$ [2, 9]. Taking into account the above formulae too, we can see that the presented theory of quaternionic differentiability gives expressions for the $\mathbb{H}$-holomorphic functions similar to expressions for the $\mathbb{C}$-holomorphic functions. A more detailed study of these matters is beyond the scope of the present paper.

Comparing the formulae (4.47) and (4.50), we can establish the following expression for the first-order quaternionic derivative after doing the transition $a = \overline{a} = x$:

$$\frac{\partial\psi}{\partial p} = 2[(\partial_a \phi_1| + (\partial_a \phi_2|j] = 2(\partial_a \phi_1| + 2(\partial_a \phi_2|j, \tag{4.51}$$

which allows us to specify the constant factor k in the expressions (4.34), (4.37), and (4.39), namely $k = 2$.

The expression (4.51) can represent, so to say, a certain final "materialization" of the 4-dimensional quaternionic derivative in 3-dimensional physical space (triplets with basis $1, j, k$) upon the transition $a = \overline{a} = x$ if we want to use the presented theory in practical applications. In principle, we can compute in this way the 3-dimensional "dilations" associated with some physical conservative vector field represented by a quaternionic function. Since calculus of triplets is restricted, it seems certain that we need first perform all quaternionic computations, and then the final transition to obtain the 3-dimensional physical result. Thus, it is important to know the initial expression for the full quaternionic derivative, that is, before we do the transition $a = \overline{a} = x$, especially when examining the second and higher order derivatives.

We have obtained the final expression (4.51) as a sum of contributions of the final left and right derivatives of the function $\psi(p)$. If we want to find the general expression for the computation of the quaternionic derivative before we do the transition $a = \overline{a} = x$ (in this case we denote the derivative by $\psi(p)'$, not to be confused with the notation $'\psi(p)$ and $\psi'(p)$), then we need have a sum of contributions of the left and right derivatives before we do the transition $a = \overline{a} = x$. To achieve this we write first the required expression as

$$\psi(p)' = \phi_1^{(\prime)} + \phi_2^{(\prime)} j, \tag{4.52}$$

where $\phi_1^{(\prime)}$ and $\phi_2^{(\prime)}$ are constituents of the derivative $\psi(p)'$ in the Cayley–Dickson doubling form before doing the transition $a = \overline{a} = x$. In the sequel we will merely use the notation with operator $\frac{\partial}{\partial p} = \partial_p$ or brackets $[, (..|$ for quaternionic derivatives and their constituents *after* doing the transition $a = \overline{a} = x$, e. g., $\frac{\partial \psi(p)}{\partial p}, [\psi(p)'|, \left(\phi_1^{(\prime)}\right|, (\partial_a \phi_2|$, as well as the notation without them for quaternionic derivatives and their constituents *before* doing the transition $a = \overline{a} = x$, e. g., $\psi(p)', \phi_2^{(\prime)}, \partial_a \phi_2$. If the function $\psi(p) = \phi_1(a,b) + \phi_2(a,b) \cdot j$ is $\mathbb{H}$-holomorphic, then in accordance with (4.51) we have

$$\left(\phi_1^{(\prime)}\right| = 2(\partial_a \phi_1|, \tag{4.53}$$

$$\left(\phi_2^{(\prime)}\right| = 2(\partial_a \phi_2|. \tag{4.54}$$

To obtain a sum of contributions of the left and right derivatives we consider first the expression (4.54). Since in accordance with (4.38) we have $(\partial_a \phi_2| = (\partial_{\overline{a}} \phi_2|$, it is possible to rewrite (4.54) as a sum

$$\left(\phi_2^{(\prime)}\right| = 2(\partial_a \phi_2| = (\partial_a \phi_2| + (\partial_{\overline{a}} \phi_2|,$$

whence

$$\phi_2^{(\prime)} = \partial_a \phi_2 + \partial_{\overline{a}} \phi_2. \tag{4.55}$$

The derivative $\partial_a \phi_2$ belongs to the "left" equation (4.36-2), the derivative $\partial_{\overline{a}} \phi_2$ belongs to the "right" equation (4.36-4), hence we have in (4.55) the sum of the left and right contributions to the constituent $\phi_2^{(\prime)}$ of the complete derivative.

To clarify (4.53) we can use the additional information about constructing the derivative following from the conceptual requirement (4.41). We can rewrite (4.53) as

$$\left(\phi_1^{(\prime)}\right| = 2(\partial_a \phi_1| = 2[\tfrac{1}{2}(\partial_x \phi_1|] = (\partial_x \phi_1|.$$

Using further the operator identity $\partial_x = \partial_a + \partial_{\overline{a}}$ based on (2.18) and (2.19), we get

$$\left(\phi_1^{(\prime)}\right| = (\partial_x \phi_1| = (\partial_a \phi_1 + \partial_{\overline{a}} \phi_1|$$

whence

$$\phi_1^{(\prime)} = \partial_a \phi_1 + \partial_{\overline{a}} \phi_1. \tag{4.56}$$

Finally, combining (4.52), (4.55), and (4.56), we get the following expression for the quaternionic derivative of the $\mathbb{H}$-holomorphic function $\psi_H(p)$ in the C[2]-representation:

$$\psi_H(p)' = \phi_1^{(\prime)} + \phi_2^{(\prime)} j, \tag{4.57}$$

where

$$\phi_1^{(\prime)} = \partial_a \phi_1 + \partial_{\bar{a}} \phi_1, \qquad \phi_2^{(\prime)} = \partial_a \phi_2 + \partial_{\bar{a}} \phi_2,$$

4.3 Construction of $\mathbb{H}$-holomorphic functions

We consider now a theorem that will play an important role in the sequel.

Theorem 4.4 (*the extension of complex holomorphicity to quaternionic*). *Let a complex function $\psi_C(\xi): G_2 \to \mathbb{C}$ be $\mathbb{C}$-holomorphic everywhere in a connected open set $G_2 \subseteq \mathbb{C}$, except, possibly, at certain singularities. Then a $\mathbb{H}$-holomorphic function $\psi_H(p)$ of the same kind as $\psi_C(\xi)$ can be constructed (without change of a kind of function) from $\psi_C(\xi)$ by replacing a complex variable $\xi \in G_2$ in an expression for $\psi_C(\xi)$ by a quaternionic variable $p \in G_4 \subseteq \mathbb{H}$, where G_4 is defined (except, possibly, at certain singularities) by the relation $G_4 \supset G_2$ in the sense that G_2 exactly follows from G_4 upon transition from p to ξ.*

Proof. To prove this theorem we need consider the transformation of equations (4.31) when making a transition to the complex case. The transition to the 3-dimensional case $a = \bar{a} = x$ is already implied for derivatives in equations (4.31). We have the only two next possibilities to complete the transition from $p = a + bj = a + (z + ui)j$ to the complex case: by putting 1) $b = \bar{b} = z$, $(u = 0)$ or 2) $b = -\bar{b} = ui$, $(z = 0)$. Consider the first way and then show that the second way gives the same result.

For operators ∂_a, $\partial_{\bar{a}}$, ∂_b, $\partial_{\bar{b}}$ the transition conditions $y = 0$, $u = 0$ (respectively $\partial_y = \partial_u = 0$) yield in accordance with (2.18), (2.19), (2.20), (2.21) the following transition formulae:

$$\partial_a = \partial_{\bar{a}} = \tfrac{1}{2}\partial_x , \quad \partial_b = \partial_{\bar{b}} = \tfrac{1}{2}\partial_z. \tag{4.58}$$

We see that upon transition to the complex case we factually rule out the dimensions with imaginary units i and k. Then a quaternionic function $\psi(p) = \psi_1(x,y,z,u) + \psi_2(x,y,z,u)i + \psi_3(x,y,z,u)j + \psi_4(x,y,z,u)k = \phi_1(a,b) + \phi_2(a,b) \cdot j$, where $p = x + yi + zj + uk = a + bj \in G_4 \subseteq \mathbb{H}$, becomes $\psi(\xi) = \psi_1(x,z) + \psi_3(x,z,)j$, where $\xi = x + zj \in G_2 \subseteq \mathbb{C}$; and $G_4 \supset G_2$ *in the sense that* G_2 *exactly follows from* G_4 upon transition to the complex case.

Thus we get for the quaternionic function $\psi(p)$ the following transition formulae:

$$\phi_1(a,b) = \overline{\phi}_1(a,b) = \psi_1(x,z), \;\; \phi_2(a,b) = \overline{\phi}_2(a,b) = \psi_3(x,z). \tag{4.59}$$

It is also easy to see that the first formula follows from equations (4.31-2) and (4.31-4) when $\partial_a = \partial_{\overline{a}}$, and the second from equations (4.31-1) and (4.31-3) when $\partial_b = \partial_{\overline{b}}$ in accordance with (4.58).

Substituting the transition formulae (4.58) and (4.59) into the system of quaternionic holomorphicity equations (4.31), we transform this system to the following system:

$$1)\ \tfrac{1}{2}\partial_x\psi_1 = \tfrac{1}{2}\partial_z\psi_3, \qquad 2)\ \tfrac{1}{2}\partial_x\psi_3 = -\tfrac{1}{2}\partial_z\psi_1,$$

$$3)\ \tfrac{1}{2}\partial_x\psi_1 = \tfrac{1}{2}\partial_z\psi_3, \qquad 4)\ \tfrac{1}{2}\partial_x\psi_3 = -\tfrac{1}{2}\partial_z\psi_1.$$

This system represents the Cauchy-Riemann complex holomorphicity equations for the functions $\psi_C(\xi) = \psi_1(x,z) + \psi_3(x,z,)j$ in the complex plane $\xi = x + zj$, that is, with "imaginary unit" j ($j^2 = -1$):

$$\partial_x\psi_1 = \partial_z\psi_3, \qquad \partial_x\psi_3 = -\partial_z\psi_1. \tag{4.60}$$

Any given kind of a function ψ remains unchanged (ψ preserves the same form) when we make a replacement of p by ξ, and, correspondingly, replacements (designated further by "→") $a, \overline{a} \to x$, $b, \overline{b} \to z$, $\phi_1(a,b)$, $\overline{\phi}_1(a,b) \to \psi_1(x,z)$, and $\phi_2(a,b), \overline{\phi}_2(a,b) \to \psi_3(x,z)$. For example, the function $\psi(p) = p^3$ becomes $\psi(\xi) = \xi^3$ *without change of a kind of function*.

Thus, the replacement $p \to \xi$ without change of a kind of function ψ transforms the ℍ-holomorphicity equations (4.31) into ℂ-holomorphicity equations (4.60). Each ℍ-holomorphic function becomes ℂ-holomorphic. Since all ℂ-holomorphic functions satisfy the Cauchy-Riemann equations we can state that each ℂ-holomorphic function follows *always* from the corresponding ℍ-holomorphic function when we replace a quaternion argument $p = x + yi + zj + uk$ by a complex argument $\xi = x + zj$ in an expression for a quaternionic function *without change of a kind of function.*

In principle, the transition $p \to \xi$ is invertible; then any ℍ-holomorphic function $\psi_H(p)$ can be created from a ℂ-holomorphic function of the same kind by replacing ξ by p in a general expression $\psi_C(\xi)$ for a ℂ-holomorphic function. Indeed, if a function ψ(p), obtained when replacing a complex variable ξ by a quaternion variable p in an expression for $\psi_C(\xi)$, were non-ℍ-holomorphic, then this would mean that a ℂ-holomorphic function can follow from a non-ℍ-holomorphic function when reverse replacing $p \to \xi$ that contradicts the fact that *each* ℂ-holomorphic function follows *always* from the corresponding ℍ-holomorphic function if a kind of function ψ remains unchanged (otherwise the invariance of a kind of function when replacing p by ξ would be broken). This contradiction proves our theorem for case 1) when $p = x + yi + zj + uk$ becomes $\xi = x + zj$.

Let us now see what happens in the case 2) when we assume that the transition conditions are $a = \overline{a} = x$, $(y = 0)$ and $b = -\overline{b} = ui$, $(z = 0)$. In this case $p = x + yi + zj + uk$ becomes $\xi = x + uk$. For operators ∂_a, $\partial_{\overline{a}}$, ∂_b, $\partial_{\overline{b}}$ the transition conditions $y = 0$, $z = 0$ (respectively $\partial_y = \partial_z = 0$) give in accordance with (2.18), (2.19), (2.20), (2.21) the following transition formulae:

$$\partial_a = \partial_{\overline{a}} = \tfrac{1}{2}\partial_x, \qquad \partial_b = -\partial_{\overline{b}} = -i\tfrac{1}{2}\partial_u. \tag{4.61}$$

Using these formulae we get from equations (4.31-2), (4.31-4) and (4.31-1), (4.31-3) the transition formulae, respectively, for the function ϕ_1 and ϕ_2:

$$\phi_1(a,b) = \overline{\phi}_1(a,b) = \psi_1(x,u), \qquad \phi_2(a,b) = -\overline{\phi}_2(a,b) = i\psi_4(x,u). \tag{4.62}$$

In other words, the function $\psi(\mathrm{p}) = \phi_1(a,b) + \phi_2(a,b)j = \psi_1(x,y,z,u) + \psi_2(x,y,z,u)i + \psi_3(x,y,z,u)j + \psi_4(x,y,z,u)k$ becomes $\psi(\xi) = \psi_1(x,u) + \psi_4(x,u)k$.

Substituting the transition formulae (4.61) and (4.62) into the system of quaternion holomorphicity equations (4.31) we transform it to the following system:

$$\partial_x\psi_1 = \partial_u\psi_4, \qquad \partial_x\psi_4 = -\partial_u\psi_1,$$

which represents the Cauchy-Riemann equations for complex functions $\psi_C(\xi) = \psi_1(x,u) + \psi_4(x,u)k$ in the complex plane $\xi = x + uk$ with imaginary unit k $(k^2 = -1)$. We see that in the second possible case of the transition to the complex plane each ℂ-holomorphic function follows allways from the corresponding ℍ-holomorphic function of the same kind. Then by using the argumentation as in the case 1) we prove our theorem for case 2). This completes the proof of the theorem in whole. □

In the sequel, we can without loss of generality consider the only replacement of $\xi = x + zj$ by $p = a + bj$, and vice versa.

4.4 ℍ-holomorphic derivatives of all orders

Theorem 4.4 enables us to establish the ℍ-holomorphicity of derivatives of all orders, computed by using the expression (4.57). We begin by the following theorem.

Theorem 4.5 *Let a continuous quaternion function* $\psi_H(p) = \phi_1(a,b) + \phi_2(a,b)j$, *where* $\phi_1(a,b)$ *and* $\phi_2(a,b)$ *are differentiable with respect to* a, $\overline{a}$, b *and* $\overline{b}$, *be ℍ-holomorphic everywhere in its domain of definition* $G_4 \subseteq \mathbb{H}$. *Then its quaternion derivative, defined by*

$$\psi_H(p)' = \phi_1^{(\prime)} + \phi_2^{(\prime)}j\ ,$$

where

$$\phi_1^{(\prime)} = \partial_a\phi_1(a,b) + \partial_{\overline{a}}\phi_1(a,b), \qquad \phi_2^{(\prime)} = \partial_a\phi_2(a,b) + \partial_{\overline{a}}\phi_2(a,b),$$

,

is also ℍ-holomorphic in G_4, except, possibly, at certain singularities. If a quaternion function $\psi(p)$ is once ℍ-differentiable in G_4, then it possesses derivatives of all orders in G_4, each one ℍ-holomorphic.

Proof. Let a complex-valued function $\psi_C(\xi) = \psi_1(x,z) + \psi_3(x,z,)j$ be ℂ-holomorphic in an open connected domain $G_2 \subseteq \mathbb{C}$, $\xi = x + zj \in G_2$. The complex derivative of $\psi_C(\xi)$ at a point ξ is defined [5, 9] by

$$\psi_C(\xi)' = \partial_x\psi_1 + j\partial_x\psi_3 = \left(\partial_\xi\psi_1 + \partial_{\overline{\xi}}\psi_1\right) + \left(\partial_\xi\psi_3 + \partial_{\overline{\xi}}\psi_3\right)j, \tag{4.63}$$

where ∂_ξ and $\partial_{\overline{\xi}}$ are operators in the complex plane $\xi = x + zj$, defined by $\partial_\xi = \frac{1}{2}(\partial_x - \partial_z \cdot i)$, $\partial_{\overline{\xi}} = \frac{1}{2}(\partial_x + \partial_z \cdot i)$, whence $\partial_\xi + \partial_{\overline{\xi}} = \partial_x$.

It is easy to see that we can formally state the following identity for operators:

$$\partial_x = \partial_\xi + \partial_{\overline{\xi}} = \partial_a + \partial_{\overline{a}}\,, \tag{4.64}$$

where $\partial_a = \frac{1}{2}\left(\partial_x - \partial_y \cdot i\right)$ and $\partial_{\overline{a}} = \frac{1}{2}\left(\partial_x + \partial_y \cdot i\right)$ correspond to (2.18) and (2.19).
It is possible because the differential operators ∂_z and ∂_y disappear upon summation, respectively, of operators ∂_ξ, $\partial_{\overline{\xi}}$ and ∂_a, $\partial_{\overline{a}}$.

By using the identity (4.64) the expression (4.63) can be rewritten as follows:

$$\psi_C(\xi)' = (\partial_a\psi_1 + \partial_{\overline{a}}\psi_1) + (\partial_a\psi_3 + \partial_{\overline{a}}\psi_3)j. \tag{4.65}$$

The transition from the ℂ-holomorphic function $\psi_C(\xi) = \psi_1(x,z) + \psi_3(x,z,)j$ to the quaternion case can be carried out in accordance with Theorem 4.4 by replacing ξ by p in the expression for $\psi_C(\xi)$, that is, by means of transition $\psi_C(\xi) \to \psi_H(p)$ without change of an initial kind of function ψ_C. This is equivalent to the replacements $\xi \to p$, $\psi_1(x,z) \to \phi_1(a,b)$, and $\psi_3(x,z,) \to \phi_2(a,b)$, where $\psi_1(x,z)$ and $\psi_3(x,z,)$ are defined by the initial complex function $\psi_C(\xi) = \psi_1(x,z) + \psi_3(x,z,)j$ and $\phi_1(a,b)$, $\phi_2(a,b)$ are defined as constituents of the Cayley–Dickson doubling form $\psi_H(p) = \phi_1(a,b) + \phi_2(a,b)j$ of the function $\psi_H(p)$, obtained by the replacement $\xi \to p$ in $\psi_C(\xi)$. Making these replacements in (4.65) we get

$$\psi_H(p)' = [\partial_a\phi_1(a,b) + \partial_{\overline{a}}\phi_1(a,b)] + [\partial_a\phi_2(a,b) + \partial_{\overline{a}}\phi_2(a,b)]j, \tag{4.66}$$

that is, the above expression (4.57) for a derivative of a ℍ-holomorphic function $\psi_H(p)$ obtained in another way.

We see that the differential algorithm (4.65) for finding the complex derivative is the same as the differential algorithm (4.66) for finding the quaternionic derivative: $\partial_a + \partial_{\overline{a}}$. Since upon transition from (4.65) to (4.66), the differential algorithm for finding the derivative and a kind of function ψ, to which this algorithm is applied, remain unchanged, we can state that the dependence of the complex derivative on ξ, viewed as a function $\varphi(\xi)$ $(= \psi_C(\xi)')$, is the same as the dependence of the quaternionic derivative on p, that is, $\varphi(p)(=\psi_H(p)')$. In other words, the replacement $\xi \to p$ in the expression (4.65) for the complex derivative gives the expression (4.66) for the quaternionic derivative without change of a kind of a derivative function.

Then, taking into consideration the fact that the first derivative (4.63) of the $\mathbb{C}$-holomorphic function is also $\mathbb{C}$-holomorphic [2, 9], and using Theorem 4.4 we have proved that the first quaternionic derivative $\psi_H(p)'$ of the $\mathbb{H}$-holomorphic function is also $\mathbb{H}$-holomorphic in $G_4 \subseteq \mathbb{H}$ such that $G_4 \supset G_2$ in the sense that G_2 exactly follows from G_4 upon transition to the complex case.

As usual, a second derivative of $\psi(p)$ is defined by differentiation of a first derivative, a third derivative by differentiation of a second derivative, and so on. We denote a derivative with respect to ξ or p by a prime, so that the second derivative is written as $\psi(\xi)''$ or $\psi(p)''$, a third as $\psi(\xi)'''$ or $\psi(p)'''$. Introducing the designation $\partial_{a,\overline{a}} = \partial_a + \partial_{\overline{a}} = \partial_x$ we can briefly rewrite the formulae (4.65) and (4.66) as follows:

$$\psi_C(\xi)' = \partial_{a,\overline{a}}\psi_1 + \partial_{a,\overline{a}}\psi_3 j = \psi_1' + \psi_3' j,$$

$$\psi_H(p)' = \partial_{a,\overline{a}}\phi_1 + \partial_{a,\overline{a}}\phi_2 j = \phi_1' + \phi_2' j.$$

Further, we can write for the second and third derivatives the following expressions:

$$\psi_C(\xi)'' = \partial_{a,\overline{a}}\psi_1' + \partial_{a,\overline{a}}\psi_3' j = \partial_{a,\overline{a}}\partial_{a,\overline{a}}\psi_1 + \partial_{a,\overline{a}}\partial_{a,\overline{a}}\psi_3 j = \psi_1'' + \psi_3'' j, \quad (4.67)$$

$$\psi_H(p)'' = \partial_{a,\overline{a}}\phi_1' + \partial_{a,\overline{a}}\phi_2' j = \partial_{a,\overline{a}}\partial_{a,\overline{a}}\phi_1 + \partial_{a,\overline{a}}\partial_{a,\overline{a}}\phi_2 j = \phi_1'' + \phi_2'' j, \quad (4.68)$$

$$\psi_C(\xi)''' = \partial_{a,\overline{a}}\psi_1'' + \partial_{a,\overline{a}}\psi_3'' j = \partial_{a,\overline{a}}\partial_{a,\overline{a}}\partial_{a,\overline{a}}\psi_1 + \partial_{a,\overline{a}}\partial_{a,\overline{a}}\partial_{a,\overline{a}}\psi_3 j = \psi_1''' + \psi_3''' j \quad (4.69)$$

$$\psi_H(p)''' = \partial_{a,\overline{a}}\phi_1'' + \partial_{a,\overline{a}}\phi_2'' j = \partial_{a,\overline{a}}\partial_{a,\overline{a}}\partial_{a,\overline{a}}\phi_1 + \partial_{a,\overline{a}}\partial_{a,\overline{a}}\partial_{a,\overline{a}}\phi_2 j = \phi_1''' + \phi_2''' j. \quad (4.70)$$

We assume that all functions occurring in expressions for derivatives possess the sufficient differentiability in the corresponding regions (G_1, G_2 or G_4), except, possibly, at certain singularities.

It is easy to see that the replacement $\xi \to p$, $\psi_1(x,z) \to \phi_1(a,b)$, $\psi_3(x,z,) \to \phi_2(a,b)$ in the expression (4.67) for the second complex derivative gives the expression (4.68) for the second quaternionic derivative. Here the differential algorithm $\partial_{a,\overline{a}}\partial_{a,\overline{a}}$ upon the transition to the quaternion case remains unchanged. Similarly, the differential algorithm $\partial_{a,\overline{a}}\partial_{a,\overline{a}}\partial_{a,\overline{a}}$ upon the transition from the complex expression (4.69) to the quaternion expression (4.70) for the third derivative remains unchanged. Since a kind of function ψ $(= \psi_C = \psi_H)$ upon the transition $\xi \to p$ remains also unchanged, we can conclude that the replacement $\xi \to p$, $\psi_1(x,z) \to \phi_1(a,b)$, $\psi_3(x,z,) \to \phi_2(a,b)$ in the expressions for the second and third complex derivatives carries out the transition to the expressions for the second and third quaternionic derivatives *without change of a kind of derivatives functions*. It is evident that the same is true for all higher derivatives. Then, taking into consideration the well-known fact [2, 9] that ℂ-holomorphic functions have ℂ-holomorphic derivatives of all orders, and using Theorem 4.4, we have proved that ℍ-holomorphic functions have ℍ-holomorphic derivatives of all orders.

This completes the proof of the theorem. □

From the proof of Theorem 4.5, we can easily see that, by using (4.66) all ℍ-holomorphic derivatives of ℍ-holomorphic functions follow from ℂ-holomorphic derivatives of ℂ-holomorphic functions when replacing a complex argument ξ by a quaternion argument p *without changing of a kind of derivatives functions*. Then we can formulate the following

Corollary 4.6 *All expressions for derivatives of a ℍ-holomorphic function $\psi_H(p)$ of the same kind as a ℂ-holomorphic function $\psi_C(\xi)$ have the same forms as the expressions for corresponding derivatives of a function $\psi_C(\xi)$.*

For example, if the derivative of the ℂ-holomorphic function $\psi_C(\xi) = \xi^n$, where n is an integer, has the expression $\psi_C(\xi)' = n\xi^{n-1}$, then the derivative of the ℍ-holomorphic function $\psi_H(p) = p^n$, that is, of the same kind of function, when computing by (4.66), must have the similar expression $\psi_H(p)' = np^{n-1}$.

The Cauchy-Riemann equations, in particular (4.60), give four (see, e.g., [9]) different variants of expressions for computing the derivatives of ℂ-holomorphic functions, for example, one of them $\psi_C(\xi)' = \partial_x\psi_1 - j\partial_z\psi_1$. It is of interest to establish in addition to (4.66) other equivalent expressions for computing the ℍ-holomorphic derivatives of all orders. In order to make this we state the following equations for ℍ-holomorphic functions without proof:

$$\partial_a\phi_2(a,b) = -\,\partial_{\overline{b}}\phi_1(a,b), \qquad \partial_{\overline{a}}\phi_2(a,b) = -\,\partial_{\overline{b}}\overline{\phi}_1(a,b), \tag{4.71}$$

which unlike the ℍ-holomorphicity equations (4.31) or (4.36) *are valid without any requirements imposed on variables in derivatives contained in them*. These equations can be shown to be true for ℍ-holomorphic functions in general, but since the proof is rather long and long-winded, we will not present it here, but merely state them. Using (4.71) in (4.66) we obtain the following equivalent expressions for derivatives of ℍ-holomorphic functions:

$$\psi_H(p)' = \partial_a\phi_1(a,b) + \partial_{\overline{a}}\phi_1(a,b) + \Big(-\partial_{\overline{b}}\phi_1(a,b) + \partial_{\overline{a}}\phi_2(a,b)\Big)j, \tag{4.72}$$

$$\psi_H(p)' = \partial_a\phi_1(a,b) + \partial_{\overline{a}}\phi_1(a,b) + \Big(\partial_a\phi_2(a,b) - \partial_{\overline{b}}\overline{\phi}_1(a,b)\Big)j,$$

$$\psi_H(p)' = \partial_a\phi_1(a,b) + \partial_{\overline{a}}\phi_1(a,b) - \Big(\partial_{\overline{b}}\phi_1(a,b) + \partial_{\overline{b}}\overline{\phi}_1(a,b)\Big)j.$$

Other equivalent expressions for derivatives of ℍ-holomorphic functions can be obtained by using the equality

$$\partial_a\overline{\phi}_1(a,b) = \partial_{\overline{a}}\phi_1(a,b).$$

This equality follows from equations (4.71) when differentiating them with respect to $\overline{a}$ and a, and using the commutativity of mixed partial derivatives. Using this equality in expressions (4.66) and (4.72) we get the following additional expressions for derivatives of $\mathbb{H}$-holomorphic functions:

$$\psi_H(p)' = \left(\partial_a\phi_1(a,b) + \partial_a\overline{\phi}_1(a,b)\right) + \left(\partial_a\phi_2(a,b) + \partial_{\overline{a}}\phi_2(a,b)\right)j, \qquad (4.73)$$
$$\psi_H(p)' = \left(\partial_a\phi_1(a,b) + \partial_a\overline{\phi}_1(a,b)\right) + \left(-\partial_{\overline{b}}\phi_1(a,b) + \partial_{\overline{a}}\phi_2(a,b)\right)j,$$
$$\psi_H(p)' = \left(\partial_a\phi_1(a,b) + \partial_a\overline{\phi}_1(a,b)\right) + \left(\partial_a\phi_2(a,b) - \partial_{\overline{b}}\overline{\phi}_1(a,b)\right)j,$$
$$\psi_H(p)' = \left(\partial_a\phi_1(a,b) + \partial_a\overline{\phi}_1(a,b)\right) - \left(\partial_{\overline{b}}\phi_1(a,b) + \partial_{\overline{b}}\overline{\phi}_1(a,b)\right)j.$$

Altogether, expressions (4.66), (4.72), and (4.73) represent the quaternionic generalizations of the known expressions for complex derivatives [9] based on Cauchy-Riemann's equations. Equations (4.71) and (4.43) *play the same role as Cauchy-Riemann's equations*, when obtaining the different expressions for derivatives of holomorphic functions. We shall not dwell on this here. Each of these expressions may be used in order to formulate the general expression for the derivatives of all orders of $\mathbb{H}$-holomorphic functions. For now it suffices to formulate the general expression in the Cayley–Dickson doubling form based on the formula (4.66):

$$\psi(p)^{(k)} = \phi_1^{(k)} + \phi_2^{(k)}j = \left[\partial_a\phi_1^{(k-1)} + \partial_{\overline{a}}\phi_1^{(k-1)}\right] + \left[\partial_a\phi_2^{(k-1)} + \partial_{\overline{a}}\phi_2^{(k-1)}\right]j, \qquad (4.74)$$

where $\psi(p)^{(k)}$ is the k'th derivative of $\psi(p)$; $\phi_1^{(k-1)}$ and $\phi_2^{(k-1)}$ are the constituents of the $(k-1)'$th derivative of $\psi(p)$, represented in the Cayley–Dickson doubling form: $\psi(p)^{(k-1)} = \phi_1^{(k-1)} + \phi_2^{(k-1)}j$; $k \geq 1$; $\phi_1^{(0)} = \phi_1(a,b)$ and $\phi_2^{(0)} = \phi_2(a,b)$ for $k = 1$.

5 Efficiency examples of the presented theory

It is clear that "complicated" holomorphic functions are practically representable in terms of algebraic operations and compositions, applied to "elementary" holomorphic functions. The usual elementary functions are well known [2, 4, 9, 11], for example, the functions p^n (with $n = 0, \pm 1, \pm 2, \ldots$), e^z, $\cos z$, $\sin z$. We use here, however, a more generalized concept of elementary functions: the so-called Liouvillian elementary functions. A function of one (complex) variable is said to be the Liouvillian elementary function if it has an explicit representation in terms of a finite number of algebraic operations (functions), logarithms, and exponentials [13]. Taking into account the linearity of equations (4.31) and the fact that the power functions are basic elements of algebraic functions, it suffices only to consider in details the following

Liouvillian elementary functions: the power function, exponential and logarithmic functions, in order to demonstrate the rightness of the presented theory of $\mathbb{H}$-differentiability.

Example 5.1 As the first example, we consider the power function $\psi(p) = \phi_1 + \phi_2 \cdot j = p^4$. The straightforward computation in accordance with the multiplication rule (2.12) yields the following expressions for components ϕ_1 and ϕ_2 of this function in the Cayley–Dickson doubling form:

$$\phi_1 = a^4 - \left(3a^2 + 2a\overline{a} + \overline{a}^2\right)b\overline{b} + b^2\overline{b}^2, \phi_2 = \left(a^3 + a^2\overline{a} + a\overline{a}^2 + \overline{a}^3\right)b - 2(a+\overline{a})b^2\overline{b}$$

as well as their conjugates

$$\overline{\phi}_1 = \overline{a}^4 - \left(3\overline{a}^2 + 2a\overline{a} + a^2\right)b\overline{b} + b^2\overline{b}^2, \overline{\phi}_2 = \left(\overline{a}^3 + \overline{a}^2 a + \overline{a}a^2 + a^3\right)\overline{b} - 2(a+\overline{a})\overline{b}^2 b.$$

Since the function $\psi(\xi) = \xi^4$ is $\mathbb{C}$-holomorphic in the complex plane, the function $\psi(p) = p^4$ must be $\mathbb{H}$-holomorphic in the quaternion space in accordance with Theorem 4.4. In order to verify the $\mathbb{H}$-holomorphicity of this function we initially compute the partial derivatives: $\partial_a\phi_1 = 4a^3 - (6a + 2\overline{a})b\overline{b}$, $\partial_b\phi_2 = \partial_{\overline{b}}\overline{\phi}_2 = \left(a^3 + a^2\overline{a} + a\overline{a}^2 + \overline{a}^3\right) - 4(a+\overline{a})b\overline{b}$, as well as $\partial_a\phi_2 = -\partial_{\overline{b}}\phi_1 = \left(3a^2 + 2a\overline{a} + \overline{a}^2\right)b - 2b^2\overline{b}$, $\partial_{\overline{a}}\phi_2 = -\partial_{\overline{b}}\overline{\phi}_1 = \left(a^2 + 2a\overline{a} + 3\overline{a}^2\right)b - -2b^2\overline{b}$. We see that the equations (4.71) and (4.43) hold.

The final transition $a = \overline{a} = x$ in the computed derivatives yields the following equalities:

$$1)\ (\partial_a\phi_1| = (\partial_{\overline{b}}\overline{\phi}_2| = 4x^3 - 8xb\overline{b};\quad 2)\ (\partial_a\phi_2| = -(\partial_{\overline{b}}\overline{\phi}_1| = 6x^2b - 2b^2\overline{b};$$

$$3)\ (\partial_a\phi_1| = (\partial_b\phi_2| = 4x^3 - 8xb\overline{b};\quad 4)\ (\partial_{\overline{a}}\phi_2| = -(\partial_{\overline{b}}\phi_1| = 6x^2b - 2b^2\overline{b}.$$

Thus the $\mathbb{H}$-holomorphicity equations (4.36) hold, and the function $\psi(p) = p^4$ is $\mathbb{H}$-holomorphic everywhere in $\mathbb{H}$.

The first derivative. Computing additionally the derivative $\partial_{\overline{a}}\phi_1 = -2(a+\overline{a})b\overline{b}$, we obtain in accordance with (4.74) the following expression for the first derivative of $\psi(p) = p^4$:

$$(p^4)^{(1)} = \phi_1^{(1)} + \phi_2^{(1)}j = (\partial_a\phi_1 + \partial_{\overline{a}}\phi_1) + (\partial_a\phi_2 + \partial_{\overline{a}}\phi_2)j = \left[4a^3 - (6a + 2\overline{a})b\overline{b} - 2(a+\overline{a})b\overline{b}\right] + \left[\left(3a^2 + 2a\overline{a} + \overline{a}^2\right)b - 2b^2\overline{b} + \left(a^2 + 2a\overline{a} + 3\overline{a}^2\right)b - 2b^2\overline{b}\right]j$$

$$= 4\left[a^3 - (2a + \overline{a})b\overline{b}\right] + 4\left[\left(a^2 + a\overline{a} + \overline{a}^2\right)b - b^2\overline{b}\right]j = 4p^3,$$

where $p^3 = \left[a^3 - (2a+\overline{a})b\overline{b}\right] + \left[\left(a^2 + a\overline{a} + \overline{a}^2\right)b - b^2\overline{b}\right]j$. It is not difficult to obtain the last formula for p^3 by means of the direct computation $p^3 = (a + bj)^3$ using the multiplication formula (2.12) in the Cayley–Dickson doubling form. According to Corollary 4.6, the first derivative $(p^4)^{(1)} = 4p^3$ has the same form as the one in complex (and real) analysis: $(\xi^4)^{(1)} = 4\xi^3$. The final transition $a = \overline{a} = x$ yields the following expression for the first derivative:

$$\frac{\partial(p^4)}{\partial p} = [(p^4)^{(1)}| = (\phi_1^{(1)}| + (\phi_2^{(1)}|j = (4p^3| = 4\left[x^3 - 3xb\overline{b}\right] + 4\left[3x^2b - b^2\overline{b}\right]j,$$

which is "independent of the way of computation" (see Definition 4.1). Therefore, this 3-dimensional expression can represent in principle the corresponding conservative vector field in space just as the first derivative of the $\mathbb{C}$-holomorphic function $\psi(\xi) = \xi^4$ can represent in principle some conservative vector field in the plane [5, 9]. In all examples below we bear in mind this meaning of the final transitions to the 3-dimensional expressions.

From the expression for $(p^4)^{(1)}$ it follows that the first derivative has the components $\phi_1^{(1)} = 4\left[a^3 - (2a + \overline{a})b\overline{b}\right]$ and $\phi_2^{(1)} = 4\left[\left(a^2 + a\overline{a} + \overline{a}^2\right)b - b^2\overline{b}\right]$. Their conjugates are as follows: $\overline{\phi_1^{(1)}} = 4\left[\overline{a}^3 - (2\overline{a} + a)b\overline{b}\right]$ and $\overline{\phi_2^{(1)}} = 4\left[\,\left(\overline{a}^2 + a\overline{a} + a^2\right)\overline{b} - \overline{b}^2 b\right]$.

We get the following partial derivatives: $\partial_a\phi_1^{(1)} = 4\left(3a^2 - 2b\overline{b}\right)$; $\partial_b\phi_2^{(1)} = \partial_{\overline{b}}\overline{\phi_2^{(1)}} = 4\left[\left(a^2 + a\overline{a} + \overline{a}^2\right) - 2b\overline{b}\right]$; and $\partial_a\phi_2^{(1)} = -\,\partial_{\overline{b}}\phi_1^{(1)} = 4(2a + \overline{a})b$; $\partial_{\overline{a}}\phi_2^{(1)} = -\,\partial_{\overline{b}}\overline{\phi_1^{(1)}} = 4(a + 2\overline{a})b$. After performing the transition $a = \overline{a} = x$ we see that the $\mathbb{H}$-holomorphicity equations (4.36) hold:

$$1)\ (\partial_a\phi_1^{(1)}| = (\partial_{\overline{b}}\overline{\phi_2^{(1)}}| = 4\left(3x^2 - 2b\overline{b}\right);\quad 2)\ (\partial_a\phi_2^{(1)}| = -\,(\partial_{\overline{b}}\overline{\phi_1^{(1)}}| = 12xb;$$

$$3)\ (\partial_a\phi_1^{(1)}| = (\partial_b\phi_2^{(1)}| = 4\left(3x^2 - 2b\overline{b}\right);\quad 4)\ (\partial_{\overline{a}}\phi_2^{(1)}| = -\,(\partial_{\overline{b}}\phi_1^{(1)}| = 12xb.$$

Thus the first derivative $(p^4)^{(1)} = \phi_1^{(1)} + \phi_2^{(1)}j$ of the $\mathbb{H}$-holomorphic function $\psi(p) = p^4$ is $\mathbb{H}$-holomorphic everywhere in $\mathbb{H}$, according to Theorem 4.5. We see that equations (4.71) and (4.43) hold too.

<u>*The second derivative*</u>. Using the computed partial derivatives of $\phi_1^{(1)}$ and $\phi_2^{(1)}$ as well as the derivative $\partial_{\overline{a}}\phi_1^{(1)} = -4b\overline{b}$, we find from (4.74) the following expression for the second derivative of $\psi(p) = p^4$:

$$(p^4)^{(2)} = \phi_1^{(2)} + \phi_2^{(2)}j = \left(\partial_a\phi_1^{(1)} + \partial_{\overline{a}}\phi_1^{(1)}\right) + \left(\partial_a\phi_2^{(1)} + \partial_{\overline{a}}\phi_2^{(1)}\right)j =$$

$$\left[4\left(3a^2 - 2b\overline{b}\right) - 4b\overline{b}\right] + \left[4(2a + \overline{a})b + 4(a + 2\overline{a})b\right]j = 4\cdot 3p^2 = 12p^2,$$

where $p^2 = (a + bj)^2 = \left(a^2 - b\overline{b}\right) + b(a + \overline{a})j$, $\phi_1^{(2)} = 12\left(a^2 - b\overline{b}\right)$, and $\phi_2^{(2)} = 12(a + \overline{a})b$. According to Corollary 4.6, the second derivative $(p^4)^{(2)} = 12p^2$ has the same form as the one in real and complex analysis: $(\xi^4)^{(2)} = 12\xi^2$. The final transition $a = \overline{a} = x$ yields the following expression for the second derivative:

$$\frac{\partial^2(p^4)}{\partial p^2} = [(p^4)^{(2)}| = (\phi_1^{(2)}| + (\phi_2^{(2)}|j = 12(p^2| = 12\left[\left(x^2 - b\overline{b}\right) + 2xbj\right].$$

To check the $\mathbb{H}$-holomorphicity of the second derivative $(p^4)^{(2)}$ we compute the partial derivatives of functions $\phi_1^{(2)}$ and $\phi_2^{(2)}$: $\partial_a\phi_1^{(2)} = 24a$; $\partial_b\phi_2^{(2)} = \partial_{\overline{b}}\overline{\phi_2^{(2)}} = 12(a + \overline{a})$; and

$\partial_a\phi_2^{(2)} = -\,\partial_{\overline{b}}\phi_1^{(2)} = 12b$; $\partial_{\overline{a}}\phi_2^{(2)} = -\,\partial_{\overline{b}}\overline{\phi_1^{(2)}} = 12b$. After performing the transition $a = \overline{a} = x$ we see that the ℍ-holomorphicity equations (4.36) hold:

$$1)\ (\partial_a\phi_1^{(2)}| = (\partial_{\overline{b}}\overline{\phi_2^{(2)}}| = 24x;\quad 2)\ (\partial_a\phi_2^{(2)}| = -\,(\partial_{\overline{b}}\overline{\phi_1^{(2)}}| = 12b;$$

$$3)\ (\partial_a\phi_1^{(2)}| = (\partial_b\phi_2^{(2)}| = 24x;\quad 4)\ (\partial_{\overline{a}}\phi_2^{(2)}| = -\,(\partial_{\overline{b}}\phi_1^{(2)}| = 12b.$$

Thus the second derivative $(p^4)^{(2)}$ of $\psi(p) = p^4$ is ℍ-holomorphic everywhere in ℍ, according to Theorem 4.5. Equations (4.71) and (4.43) hold too.

The third derivative. Using the computed partial derivatives of $\phi_1^{(2)}$ and $\phi_2^{(2)}$ as well as the derivative $\partial_{\overline{a}}\phi_1^{(2)} = 0$ we find from (4.74) the following expression for the third derivative of the function $\psi(p) = p^4$:

$$(p^4)^{(3)} = \phi_1^{(3)} + \phi_2^{(3)}j = \left(\partial_a\phi_1^{(2)} + \partial_{\overline{a}}\phi_1^{(2)}\right) + \left(\partial_a\phi_2^{(2)} + \partial_{\overline{a}}\phi_2^{(2)}\right)j =$$

$$= 24a + 24bj = 24p$$

,whence $\phi_1^{(3)} = 24a$ and $\phi_2^{(3)} = 24b$. According to Corollary 4.6, the third derivative: $(p^4)^{(3)} = 24p$ has the same form as the one in real and complex analysis: $(\xi^4)^{(3)} = 24\xi$.

Computing the partial derivatives of $\phi_1^{(3)}$ and $\phi_2^{(3)}$ yields $\partial_a\phi_1^{(3)} = 24$; $\partial_b\phi_2^{(3)} = \partial_{\overline{b}}\overline{\phi_2^{(3)}} = 24$; and $\partial_a\phi_2^{(3)} = -\,\partial_{\overline{b}}\phi_1^{(3)} = 0$; $\partial_{\overline{a}}\phi_2^{(3)} = -\,\partial_{\overline{b}}\overline{\phi_1^{(3)}} = 0$. We see that the ℍ-holomorphicity equations (4.36) hold in this case without requiring the transition $a = \overline{a} = x$:

$$1)\ (\partial_a\phi_1^{(3)}| = (\partial_{\overline{b}}\overline{\phi_2^{(3)}}| = 24;\quad 2)\ (\partial_a\phi_2^{(3)}| = -\,(\partial_{\overline{b}}\overline{\phi_1^{(3)}}| = 0;$$

$$3)\ (\partial_a\phi_1^{(3)}| = (\partial_b\phi_2^{(3)}| = 24;\quad 4)\ (\partial_{\overline{a}}\phi_2^{(3)}| = -\,(\partial_{\overline{b}}\phi_1^{(3)}| = 0.$$

Thus the third derivative $(p^4)^{(3)} = \phi_1^{(3)} + \phi_2^{(2)}j$ of the ℍ-holomorphic function $\psi(p) = p^4$ is ℍ-holomorphic everywhere in ℍ too, according to Theorem 4.5. Equations (4.71) and (4.43) hold too.

The final transition $a = \overline{a} = x$ yields the following expression for the third derivative:

$$\frac{\partial^3\psi}{\partial p^3} = [(p^4)^{(3)}| = (\phi_1^{(3)}| + (\phi_2^{(3)}|j = 24x + 24bj = 24(p|.$$

The fourth derivative. We find that

$$(p^4)^{(4)} = \phi_1^{(4)} + \phi_2^{(4)}j = \left(\partial_a\phi_1^{(3)} + \partial_{\overline{a}}\phi_1^{(3)}\right) + \left(\partial_a\phi_2^{(3)} + \partial_{\overline{a}}\phi_2^{(3)}\right)j = 24 + 0j = 24,$$

whence $\phi_1^{(4)} = 24$, $\phi_2^{(4)} = 0$.

It is not difficult to see that in this case the ℍ-holomorphicity equations (4.36) hold too:

$$1)\ (\partial_a\phi_1^{(4)}| = (\partial_{\overline{b}}\overline{\phi_2^{(4)}}| = 0;\qquad 2)\ (\partial_a\phi_2^{(4)}| = -\,(\partial_{\overline{b}}\overline{\phi_1^{(4)}}| = 0;$$

$$3)\ (\partial_a\phi_1^{(4)}| = (\partial_b\phi_2^{(4)}| = 0;\qquad 4)\ (\partial_{\overline{a}}\phi_2^{(4)}| = -\,(\partial_{\overline{b}}\phi_1^{(4)}| = 0.$$

According to Corollary 4.6, the fourth derivative: $(p^4)^{(4)} = 24$ has the same form as the one in real and complex analysis: $(\xi^4)^{(4)} = 24$. The final transition $a = \overline{a} = x$ yields

$$\frac{\partial^4(p^4)}{\partial p^4} = [(p^4)^{(4)}| = (\phi_1^{(4)}| + (\phi_2^{(4)}|j = 24.$$

Equations (4.71) and (4.43) hold too.

Example 5.2 Consider the power function $\psi(p) = p^{-1}$. Using the formulae (4.1) and (2.11), we get the following expression for this function in the Cayley–Dickson doubling form:

$$\psi(p) = \phi_1 + \phi_2 \cdot j = p^{-1} = \frac{1}{p} = \frac{\overline{p}}{|p|^2} = \frac{\overline{a}}{(a\overline{a}+b\overline{b})} - \frac{b}{(a\overline{a}+b\overline{b})}j,$$

whence $\phi_1 = \frac{\overline{a}}{(a\overline{a}+b\overline{b})}$ and $\phi_2 = -\frac{b}{(a\overline{a}+b\overline{b})}$ as well as $\overline{\phi}_1 = \frac{a}{(a\overline{a}+b\overline{b})}$ and $\overline{\phi}_2 = -\frac{\overline{b}}{(a\overline{a}+b\overline{b})}$.

In accordance with Theorem 4.4, since the complex function $\psi(\xi) = \xi^{-1}$ is $\mathbb{C}$-holomorphic at points $\xi \in \mathbb{C} \setminus \{0\}$, the quaternion function $\psi(p) = p^{-1}$ must be $\mathbb{H}$-holomorphic at points $p \in \mathbb{H} \setminus \{0\}$. To check the $\mathbb{H}$-holomorphicity of the function $\psi(p) = p^{-1}$ we compute the partial derivatives of functions ϕ_1 and ϕ_2: $\partial_a\phi_1 = -\frac{\overline{a}^2}{(a\overline{a}+b\overline{b})^2}$; $\partial_b\phi_2 = \partial_{\overline{b}}\overline{\phi}_2 = -\frac{a\overline{a}}{(a\overline{a}+b\overline{b})^2}$; and $\partial_a\phi_2 = -\partial_{\overline{b}}\phi_1 = \frac{\overline{a}b}{(a\overline{a}+b\overline{b})^2}$; $\partial_{\overline{a}}\phi_2 = -\partial_{\overline{b}}\overline{\phi}_1 = \frac{ab}{(a\overline{a}+b\overline{b})^2}$. After performing the transition $a = \overline{a} = x$ we see that the $\mathbb{H}$-holomorphicity equations (4.36) hold:

1) $(\partial_a\phi_1| = (\partial_{\overline{b}}\overline{\phi}_2| = -\frac{x^2}{(x^2+b\overline{b})^2}$; 2) $(\partial_a\phi_2| = -(\partial_{\overline{b}}\overline{\phi}_1| = \frac{xb}{(x^2+b\overline{b})^2}$;

3) $(\partial_a\phi_1| = (\partial_b\phi_2| = -\frac{x^2}{(x^2+b\overline{b})^2}$; 4) $(\partial_{\overline{a}}\phi_2| = -(\partial_{\overline{b}}\phi_1| = \frac{xb}{(x^2+b\overline{b})^2}$.

Thus we have shown that the function $\psi(p) = \phi_1 + \phi_2 \cdot j = p^{-1}$ is $\mathbb{H}$-holomorphic at points $p \in \mathbb{H} \setminus \{0\}$. Equations (4.71) and (4.43) hold too.

The first derivative. Using the expressions for $\partial_a\phi_1, \partial_a\phi_2, \partial_{\overline{a}}\phi_2$ and $\partial_{\overline{a}}\phi_1 = \frac{b\overline{b}}{(a\overline{a}+b\overline{b})^2}$ in (4.74), we get the following expression for the first derivative of $\psi(p) = p^{-1}$:

$$(p^{-1})^{(1)} = \phi_1^{(1)} + \phi_2^{(1)}j = (\partial_a\phi_1 + \partial_{\overline{a}}\phi_1) + (\partial_a\phi_2 + \partial_{\overline{a}}\phi_2)j =$$

$$= \left[-\frac{\overline{a}^2}{(a\overline{a}+b\overline{b})^2} + \frac{b\overline{b}}{(a\overline{a}+b\overline{b})^2}\right] + \left[\frac{\overline{a}b}{(a\overline{a}+b\overline{b})^2} + \frac{ab}{(a\overline{a}+b\overline{b})^2}\right]j$$

$$= -\frac{\left[\overline{a}^2 - b\overline{b} - b(a+\overline{a})j\right]}{(a\overline{a}+b\overline{b})^2} = -\frac{\overline{p}^2}{(a\overline{a}+b\overline{b})^2} = -\frac{\overline{p}^2}{|p|^4},$$

where the expression $\overline{p}^2 = (\overline{a} - bj) \cdot (\overline{a} - bj) = \overline{a}^2 - b\overline{b} - b(a + \overline{a})j$ is evident. Taking into account that $p^{-2} = p^{-1}p^{-1} = \frac{\overline{p}^2}{|p|^4}$, we get the following expression for the first derivative of the function $\psi(p) = p^{-1}$:

$$(p^{-1})^{(1)} = \phi_1^{(1)} + \phi_2^{(1)} j = -p^{-2}\ ,$$

where $\phi_1^{(1)} = \frac{b\overline{b}-\overline{a}^2}{(a\overline{a}+b\overline{b})^2}$ and $\phi_2^{(1)} = \frac{b(a+\overline{a})}{(a\overline{a}+b\overline{b})^2}$. The complex conjugation of $\phi_1^{(1)}$ and $\phi_2^{(1)}$ yields $\overline{\phi_1^{(1)}} = \frac{b\overline{b}-a^2}{(a\overline{a}+b\overline{b})^2}$ and $\overline{\phi_2^{(1)}} = \frac{\overline{b}(\overline{a}+a)}{(a\overline{a}+b\overline{b})^2}$. According to Corollary 4.6, the first derivative: $(p^{-1})^{(1)} = -p^{-2}$ has the same form as the one in real and complex analysis: $(\xi^{-1})^{(1)} = -\xi^{-2}$.

To verify the $\mathbb{H}$-holomorphicity of the first derivative $(p^{-1})^{(1)}$ we compute the partial derivatives of functions $\phi_1^{(1)}$ and $\phi_2^{(1)}$: $\partial_a\phi_1^{(1)} = \frac{2\overline{a}(\overline{a}^2-b\overline{b})}{(a\overline{a}+b\overline{b})^3}$; $\partial_b\phi_2^{(1)} = \partial_{\overline{b}}\overline{\phi_2^{(1)}} = \frac{(a+\overline{a})(a\overline{a}-b\overline{b})}{(a\overline{a}+b\overline{b})^3}$; and $\partial_a\phi_2^{(1)} = -\partial_{\overline{b}}\phi_1^{(1)} = \frac{b(b\overline{b}-a\overline{a}-2\overline{a}^2)}{(a\overline{a}+b\overline{b})^3}$; $\partial_{\overline{a}}\phi_2^{(1)} = -\partial_{\overline{b}}\overline{\phi_1^{(1)}} = \frac{b(b\overline{b}-a\overline{a}-2a^2)}{(a\overline{a}+b\overline{b})^3}$. Equations (4.71) and (4.43) hold.

After performing the transition $a = \overline{a} = x$ we see that the $\mathbb{H}$-holomorphicity equations (4.36) hold for the first derivative of $\psi(p) = p^{-1}$:

$$1)\ (\partial_a\phi_1^{(1)}| = (\partial_{\overline{b}}\overline{\phi_2^{(1)}}| = \frac{2x(x^2-b\overline{b})}{(x^2+b\overline{b})^3};\quad 2)\ (\partial_a\phi_2^{(1)}| = -(\partial_{\overline{b}}\overline{\phi_1^{(1)}}| = \frac{b(b\overline{b}-3x^2)}{(x^2+b\overline{b})^3};$$

$$3)\ (\partial_a\phi_1^{(1)}| = (\partial_b\phi_2^{(1)}| = \frac{2x(x^2-b\overline{b})}{(x^2+b\overline{b})^3};\quad 4)\ (\partial_{\overline{a}}\phi_2^{(1)}| = -(\partial_{\overline{b}}\phi_1^{(1)}| = \frac{b(b\overline{b}-3x^2)}{(x^2+b\overline{b})^3}.$$

Thus the first derivative of the $\mathbb{H}$-holomorphic function $\psi(p) = p^{-1}$ is $\mathbb{H}$-holomorphic at the points $p \in \mathbb{H} \setminus \{0\}$, according to Theorem 4.5.

At last, the transition $a = \overline{a} = x$ gives the final 3-dimensional expression for the first derivative of $\psi(p) = p^{-1}$:

$$\frac{\partial(p^{-1})}{\partial p} = [(p^{-1})^{(1)}| = (\phi_1^{(1)}| + (\phi_2^{(1)}|j = \frac{b\overline{b}-x^2}{(x^2+b\overline{b})^2} + \frac{2xb}{(x^2+b\overline{b})^2}j = -(p^{-2}|.$$

The second derivative. Using the formula (4.74) and derivatives $\partial_a\phi_1^{(1)}$, $\partial_a\phi_2^{(1)}$, $\partial_{\overline{a}}\phi_2^{(1)}$, $\partial_{\overline{a}}\phi_1^{(1)} = -\frac{2b\overline{b}(a+\overline{a})}{(a\overline{a}+b\overline{b})^3}$, we can write the following expression for the second derivative of the function $\psi(p) = p^{-1}$:

$$(p^{-1})^{(2)} = \phi_1^{(2)} + \phi_2^{(2)} j = \left(\partial_a\phi_1^{(1)} + \partial_{\overline{a}}\phi_1^{(1)}\right) + \left(\partial_a\phi_2^{(1)} + \partial_{\overline{a}}\phi_2^{(1)}\right)j$$

$$= \left[\frac{2\overline{a}(\overline{a}^2-b\overline{b})}{(a\overline{a}+b\overline{b})^3} - \frac{2b\overline{b}(a+\overline{a})}{(a\overline{a}+b\overline{b})^3}\right] + \left[\frac{b(b\overline{b}-a\overline{a}-2\overline{a}^2)}{(a\overline{a}+b\overline{b})^3} + \frac{b(b\overline{b}-a\overline{a}-2a^2)}{(a\overline{a}+b\overline{b})^3}\right]j$$

$$= \frac{2\left[\overline{a}^3-2\overline{a}b\overline{b}-ab\overline{b}\right]+2\left[b^2\overline{b}-\overline{a}^2b-ab(\overline{a}+a)\right]j}{(a\overline{a}+b\overline{b})^3},$$

whence $\phi_1^{(2)} = \frac{2\left[\overline{a}^3-(2\overline{a}+a)b\overline{b}\right]}{(a\overline{a}+b\overline{b})^3}$, $\phi_2^{(2)} = \frac{2\left[b^2\overline{b}-\left(a^2+a\overline{a}+\overline{a}^2\right)b\right]}{(a\overline{a}+b\overline{b})^3}$.

Taking into account $\overline{p}^2 = (\overline{a}-bj)(\overline{a}-bj) = \overline{a}^2 - b\overline{b} - b(a+\overline{a})j$; $\overline{p}^3 = \overline{p}^2\overline{p} =$

$[\overline{a}^2 - b\overline{b} - b(a+\overline{a})j](\overline{a} - bj) = [\overline{a}^3 - 2\overline{a}b\overline{b} - ab\overline{b}] + [b^2\overline{b} - \overline{a}^2 b - ab(\overline{a}+a)]j;\ p^{-3} = p^{-1}p^{-1}p^{-1} = \frac{\overline{p}^3}{|p|^6} = \frac{\overline{p}^3}{(a\overline{a}+b\overline{b})^3}$, we obtain the following expression for the second derivative of the function $\psi(p) = p^{-1}$:

$$(p^{-1})^{(2)} = 2\frac{\overline{p}^3}{(a\overline{a}+b\overline{b})^3} = 2p^{-3}.$$

The obtained expression has the same form as the one in complex (and real) analysis: $(\xi^{-1})^{(2)} = 2\xi^{-3}$. The transition $a = \overline{a} = x$ leads to the final 3-dimensional expression for the second derivative of $\psi(p) = p^{-1}$:

$$\frac{\partial^2(p^{-1})}{\partial p^2} = [(p^{-1})^{(2)}| = 2(p^{-3}| = \frac{2(\overline{p}^3|}{(x^2+b\overline{b})^3} = 2\frac{(x^3-3xb\overline{b})+(b^2\overline{b}-3x^2b)j}{(x^2+b\overline{b})^3}.$$

To verify the $\mathbb{H}$-holomorphicity of the second derivative $(p^{-1})^{(2)}$ we compute the partial derivatives of components $\phi_1^{(2)}$ and $\phi_2^{(2)}$: $\partial_a\phi_1^{(2)} = \frac{2[-3\overline{a}^4+2\overline{a}(3\overline{a}+a)b\overline{b}-b^2\overline{b}^2]}{(a\overline{a}+b\overline{b})^4}$; $\partial_b\phi_2^{(2)} = \partial_{\overline{b}}\overline{\phi_2^{(2)}} = \frac{2[-a\overline{a}(a^2+a\overline{a}+\overline{a}^2)+2(a^2+2a\overline{a}+\overline{a}^2)b\overline{b}-b^2\overline{b}^2]}{(a\overline{a}+b\overline{b})^4}$; $\partial_a\phi_2^{(2)} = -\partial_{\overline{b}}\phi_1^{(2)} = \frac{2\overline{a}(a^2+2a\overline{a}+3\overline{a}^2)b}{(a\overline{a}+b\overline{b})^4} -$

$$-\frac{4(a+2\overline{a})b^2\overline{b}}{(a\overline{a}+b\overline{b})^4}\ ;\ \partial_{\overline{a}}\phi_2^{(2)} = -\partial_{\overline{b}}\overline{\phi_1^{(2)}} = \frac{2[a\overline{a}(\overline{a}+2a)b+3a^3b-2(\overline{a}+2a)b^2\overline{b}]}{(a\overline{a}+b\overline{b})^4}.$$

After performing the transition $a = \overline{a} = x$ we see that the $\mathbb{H}$-holomorphicity equations (4.36) hold for the second derivative of $\psi(p) = p^{-1}$:

1) $(\partial_a\phi_1^{(2)}| = (\partial_{\overline{b}}\overline{\phi_2^{(2)}}| = \frac{2[-3x^4+8x^2b\overline{b}-b^2\overline{b}^2]}{(x^2+b\overline{b})^4}$; 2) $(\partial_a\phi_2^{(2)}| = -(\partial_{\overline{b}}\overline{\phi_1^{(2)}}| = \frac{12xb(x^2-b\overline{b})}{(x^2+b\overline{b})^4}$

3) $(\partial_a\phi_1^{(2)}| = (\partial_b\phi_2^{(2)}| = \frac{2[-3x^4+8x^2b\overline{b}-b^2\overline{b}^2]}{(x^2+b\overline{b})^4}$; 4) $(\partial_{\overline{a}}\phi_2^{(2)}| = -(\partial_{\overline{b}}\phi_1^{(2)}| = \frac{12xb(x^2-b\overline{b})}{(x^2+b\overline{b})^4}$.

Thus the second derivative of the $\mathbb{H}$-holomorphic function $\psi(p) = p^{-1}$ is $\mathbb{H}$-holomorphic at points $p \in \mathbb{H} \setminus \{0\}$ too, according to Theorem 4.5. Equations (4.71) and (4.43) hold too.

Higher derivatives. As the reader can verify, Theorem 4.5 and Corollary 4.6 remain valid for the higher derivatives of $\psi(p) = p^{-1}$.

Example 5.3 Consider the quaternionic exponential function $\psi(p) = e^p = \phi_1 + \phi_2 \cdot j$, where e is the base of the natural logarithm. Since this function can be obtained from the principal branch of the $\mathbb{C}$-holomorphic function [9, 10, 11] $\psi(z) = e^z, z \in \mathbb{C}$ by the direct replacement of z by p, it follows that the quaternionic exponential function $\psi(p) = \phi_1 + \phi_2 \cdot j = e^p$ must be $\mathbb{H}$-holomorphic in $\mathbb{H}$, according to Theorem 4.4. To verify the $\mathbb{H}$-holomorphicity of the exponential function we need first to represent this function in the Cayley–Dickson doubling form and then to obtain the functions ϕ_1 and ϕ_2.

First of all, we represent the quaternion variable $p = x + yi + zj + uk$ as a sum of real and imaginary parts. So we obtain the quaternion representation in the form $p = x + vr$, where $v = \sqrt{y^2 + z^2 + u^2}$ is a real value, $r = \frac{yi+zj+uk}{\sqrt{y^2+z^2+u^2}}$ is a purely imaginary unit quaternion, so its square is -1. Since $r^2 = -1$ as well as x and v are real values, the quaternion formula $p = x + vr$ is algebraically equivalent to the complex formula $z = x + yi$. Then the complex expression $e^z = e^{(x+yi)} = e^x e^{yi} = e^x(\cos y + i \sin y)$, where trigonometric functions cosine and sine, and Euler's formula [2,9] $e^{yi} = \cos y + i \sin y$ are used, can be extended to the quaternion case as follows:

$$\psi(p) = \phi_1 + \phi_2 \cdot j = e^p = e^{(x+vr)} = e^x e^{vr} = e^x(\cos v + r\sin v)$$
$$= e^x\left(\cos v + \frac{yi\sin v}{v}\right) + e^x \frac{(z+ui)\sin v}{v} \cdot j\,,$$

whence $\phi_1 = e^x\left(\cos v + \frac{yi\sin v}{v}\right)$, $\phi_2 = e^x \frac{(z+ui)\sin v}{v}$. In this way a "kind of function" remains unchanged, according to Theorem 4.4.

Using expressions $x = \frac{a+\overline{a}}{2}$, $y = \frac{a-\overline{a}}{2i}$, $z = \frac{b+\overline{b}}{2}$, $u = \frac{b-\overline{b}}{2i}$, following from (2.5) - (2.8), we rewrite finally the expressions for ϕ_1 and ϕ_2 as functions of a, $\overline{a}$, b, $\overline{b}$:

$$\phi_1 = 2\beta\cos v + \frac{\beta(a-\overline{a})\sin v}{v},\ \ \phi_2 = \frac{2\beta b\sin v}{v}\ \ ,$$

where $\beta = \frac{e^{\frac{a+\overline{a}}{2}}}{2}$, $|p| = \sqrt{x^2 + y^2 + z^2 + u^2} = \sqrt{a\overline{a} + b\overline{b}}$, and $v = \frac{\sqrt{[4(a\overline{a}+b\overline{b})-(a+\overline{a})^2]}}{2} = \frac{\sqrt{[4|p|^2-(a+\overline{a})^2]}}{2}$ are real values. Thus the expression for the exponential function takes the form

$$\psi(p) = \phi_1 + \phi_2 \cdot j = e^p = 2\beta\cos v + \frac{\beta(a-\overline{a})\sin v}{v} + \frac{2\beta b\sin v}{v} \cdot j.$$

It is easy to verify that $(\phi_1| = (\overline{\phi}_1|$ is valid here just as in all the examples.

To check the $\mathbb{H}$-holomorphicity of the exponential function $\psi(p) = e^p = \phi_1 + \phi_2 \cdot j$ we first compute the partial derivatives of functions ϕ_1 and ϕ_2: $\partial_a\phi_1 = \beta\left[\cos v + \frac{(a-\overline{a}+1)\sin v}{v} - \frac{(a-\overline{a})^2(v\cos v-\sin v)}{4v^3}\right]$; $\partial_b\phi_2 = \partial_{\overline{b}}\overline{\phi}_2 = \beta\left[\frac{2\sin v}{v} + \frac{b\overline{b}(v\cos v-\sin v)}{v^3}\right]$; and $\partial_a\phi_2 = -\,\partial_{\overline{b}}\phi_1 = \beta b\left[\frac{\sin v}{v} - \frac{(a-\overline{a})(v\cos v-\sin v)}{2v^3}\right]$; $\partial_{\overline{a}}\phi_2 = -\,\partial_{\overline{b}}\overline{\phi}_1 = \beta b\left[\frac{\sin v}{v} + \frac{(a-\overline{a})(v\cos v-\sin v)}{2v^3}\right]$.

Performing the transition $a = \overline{a} = x$ and taking into consideration that $b\overline{b} = |b|^2$, we have $v = |b|$ and $\beta = \frac{e^x}{2}$. Then, it follows that the $\mathbb{H}$-holomorphicity equations (4.36) hold:

1) $(\partial_a\phi_1| = (\partial_{\overline{b}}\overline{\phi}_2| = \frac{e^x(\cos|b|+|b|^{-1}\sin|b|)}{2}$; 2) $(\partial_a\phi_2| = -\,(\partial_{\overline{b}}\overline{\phi}_1| = \frac{e^x b|b|^{-1}\sin|b|}{2}$;

3) $(\partial_a\phi_1| = (\partial_b\phi_2| = \frac{e^x(\cos|b|+|b|^{-1}\sin|b|)}{2}$; 4) $(\partial_{\overline{a}}\phi_2| = -\,(\partial_{\overline{b}}\phi_1| = \frac{e^x b|b|^{-1}\sin|b|}{2}$.

Thus, according to Theorem 4.4, the quaternion exponential function $\psi(p) = e^p$ is ℍ-holomorphic everywhere in ℍ except, possibly, at certain singularities if they exist. Equations (4.71) and (4.43) hold too.

The first derivative. Substituting the expressions for derivatives $\partial_a \phi_1, \partial_a \phi_2, \partial_{\overline{a}} \phi_2$ as well as $\partial_{\overline{a}} \phi_1 = \beta \left[\frac{(v\cos v - \sin v)}{v} + \frac{(a-\overline{a})^2 (v \cos v - \sin v)}{4v^3}\right]$ into (4.74), we get, after some algebra, the expression for the first derivative of the function $\psi(p) = e^p$:

$$(e^p)^{(1)} = \phi_1^{(1)} + \phi_2^{(1)} j = (\partial_a \phi_1 + \partial_{\overline{a}} \phi_1) + (\partial_a \phi_2 + \partial_{\overline{a}} \phi_2) \cdot j$$

$$= \beta \cos v + \frac{\beta(a-\overline{a}+1)\sin v}{v} - \frac{\beta(a-\overline{a})^2(v\cos v - \sin v)}{4v^3} + \frac{\beta(v\cos v - \sin v)}{v} + \frac{\beta(a-\overline{a})^2(v\cos v - \sin v)}{4v^3} +$$

$$+ \left[\frac{\beta b \sin v}{v} - \frac{\beta b(a-\overline{a})(v\cos v - \sin v)}{2v^3} + \frac{\beta b \sin v}{v} + \frac{\beta b(a-\overline{a})(v\cos v - \sin v)}{2v^3}\right] \cdot j$$

$$= 2\beta \cos v + \frac{\beta(a-\overline{a})\sin v}{v} + \frac{2\beta b \sin v}{v} \cdot j = e^p,$$

where the above expression for e^p is used. According Corollary 4.6, this formula has the same form as the one in complex (and real) analysis: $(e^z)^{(1)} = e^z$. The transition $a = \overline{a} = x$ leads to the final 3-dimensional expression for the first derivative of $(p) = e^p$:

$$\frac{\partial(e^p)}{\partial p} = \left[(e^p)^{(1)}\right| = (e^p|.$$

Higher derivatives. Since the function e^p is its own derivative, it follows that all higher derivatives of the quaternionic exponential function, according to Theorem 4.5, are ℍ-holomorphic everywhere in ℍ except, possibly, at certain singularities. According to Corollary 4.6, each k'th derivative has the same form $(e^p)^{(k)} = e^p, k = 1, 2, \ldots$ as the one in complex (and real) analysis: $(e^z)^{(k)} = e^z, k = 1, 2, \ldots$.

Example 5.4 Consider the quaternion natural logarithmic function $\psi(p) = \ln p$. The initial complex logarithmic function is given by $(z) = \ln z = \ln|z| + i \arg z = \ln|z| + i \arccos\frac{z+\overline{z}}{2|z|}$, where arg z is defined up to an additive multiple of 2π. We consider the principal branch (see, e.g., [9, 11]) of $\arg z$. Arguing as in Example 5.3, we can write the initial expression for the quaternion natural logarithmic function as $\psi(p) = \ln p = \ln|p| + r \cdot \mathrm{Arccos}\frac{p+\overline{p}}{2|p|}$, where $|p|$ and r are defined as in Example 5.3. Using (2.5) - (2.8), we finally obtain the expression for the quaternion logarithmic function as a function of variables $a, \overline{a}$, b, $\overline{b}$:

$$\psi(p) = \ln p = \phi_1 + \phi_2 \cdot j = \ln|p| + \frac{(a-\overline{a})\mathrm{Arccos}\frac{a+\overline{a}}{2|p|}}{2v} + \frac{b \cdot \mathrm{Arccos}\frac{a+\overline{a}}{2|p|}}{v} \cdot j\ ,$$

where v is the same as in Example 5.3: $v = \frac{\sqrt{[4|p|^2 - (a+\overline{a})^2]}}{2} = \sqrt{a\overline{a} + b\overline{b} - \frac{(a+\overline{a})^2}{4}}$, whence

$$\phi_1 = \ln|p| + \frac{(a-\bar{a})\text{Arccos}\frac{a+\bar{a}}{2|p|}}{2v}; \quad \phi_2 = \frac{b\cdot\text{Arccos}\frac{a+\bar{a}}{2|p|}}{v}, p, v \neq 0.$$

The computation results can be represented as follows. The partial derivatives are $\partial_a\phi_1 = +\frac{\bar{a}}{2|p|^2} + \frac{\theta}{2v} - \frac{(a-\bar{a})[2|p|^2-(a+\bar{a})\bar{a}]}{2|p|^2[4|p|^2-(a+\bar{a})^2]} + \frac{(a-\bar{a})^2\theta}{(2v)^3}$, $\partial_b\phi_2 = \partial_{\bar{b}}\overline{\phi}_2 = \theta\left(\frac{1}{v} - \frac{|b|^2}{2v^3}\right) + \frac{(a+\bar{a})|b|^2}{|p|^2[4|p|^2-(a+\bar{a})^2]}$, $\partial_a\phi_2 = -\partial_{\bar{b}}\phi_1 = -\frac{b[2|p|^2-(a+\bar{a})\bar{a}]}{|p|^2[4|p|^2-(a+\bar{a})^2]} + \frac{(a-\bar{a})b\theta}{4v^3}$, $\partial_{\bar{a}}\phi_2 = -\partial_{\bar{b}}\overline{\phi}_1 = -\frac{b[2|p|^2-(a+\bar{a})a]}{|p|^2[4|p|^2-(a+\bar{a})^2]} - \frac{(a-\bar{a})b\theta}{4v^3}$, where $\theta = \text{Arccos}\frac{a+\bar{a}}{2|p|}$. For $a, b \neq 0$, after performing the transition $a = \bar{a} = x$ we have relations $|p| = |p_3| = \sqrt{x^2+z^2+u^2} = \sqrt{x^2+|b|^2} \neq 0$, $4|p|^2 - (a+\bar{a})^2 = 4|p_3|^2 - 4x^2 = 4|b|^2 \neq 0$, $v = \sqrt{b\bar{b}} = |b| \neq 0$, $\theta = \text{Arccos}\frac{x}{|p_3|}$. Then we see that $\mathbb{H}$-holomorphicity equations (4.36) hold:

$$1)\ (\partial_a\phi_1| = (\partial_{\bar{b}}\overline{\phi}_2| = \frac{x}{2|p_3|^2} + \frac{\text{Arccos}\frac{x}{|p_3|}}{2|b|}; \quad 2)\ (\partial_a\phi_2| = -(\partial_{\bar{b}}\overline{\phi}_1| = -\frac{b}{2|p_3|^2};$$

$$3)\ (\partial_a\phi_1| = (\partial_b\phi_2| = \frac{x}{2|p_3|^2} + \frac{\text{Arccos}\frac{x}{|p_3|}}{2|b|}; \quad 4)\ (\partial_{\bar{a}}\phi_2| = -(\partial_{\bar{b}}\phi_1| = -\frac{b}{2|p_3|^2}.$$

Hence, the quaternion (natural) logarithmic function $\psi(p) = \ln p$ is $\mathbb{H}$-holomorphic (according to Theorem 4.4) everywhere in $\mathbb{H} \setminus \{0\}$. Equations (4.71) and (4.43) hold too.

<u>*The first derivative*</u>. Taking into consideration the fact that $\partial_{\bar{a}}\phi_1 = \frac{a}{2|p|^2} - \frac{\theta}{2v} - \frac{(a-\bar{a})^2\theta}{(2v)^3} - \frac{(a-\bar{a})[2|p|^2-(a+\bar{a})a]}{2|p|^2[4|p|^2-(a+\bar{a})^2]}$ we get the following expression for the first derivative of the function $\psi(p) = \ln p$:

$$(\ln p)^{(1)} = \phi_1^{(1)} + \phi_2^{(1)}\cdot j = (\partial_a\phi_1 + \partial_{\bar{a}}\phi_1) + (\partial_a\phi_2 + \partial_{\bar{a}}\phi_2)\cdot j =$$
$$= \frac{(a+\bar{a})}{2|p|^2} - \frac{(a-\bar{a})[4|p|^2-(a+\bar{a})^2]}{2|p|^2[4|p|^2-(a+\bar{a})^2]} - \frac{b[4|p|^2-(a+\bar{a})^2]}{|p|^2[4|p|^2-(a+\bar{a})^2]}\cdot j = \frac{\bar{a}}{|p|^2} - \frac{b}{|p|^2}\cdot j = \frac{\bar{p}}{|p|^2} = \frac{1}{p}.$$

According Corollary 4.6, this formula has the same form as the one in complex (and real) analysis: $(\ln z)^{(1)} = \frac{1}{z}$. Since the function $\frac{1}{p}$, according to Example 5.2 above, is $\mathbb{H}$-holomorphic, the first derivative of the function $\ln p$ is $\mathbb{H}$-holomorphic too. The transition $a = \bar{a} = x$ leads to the final 3-dimensional expression for the first derivative of $\psi(p) = \ln p$:

$$\frac{\partial(\ln p)}{\partial p} = [(\ln p)^{(1)}| = \frac{1}{(p|}.$$

<u>*Higher derivatives*</u>. Since the first derivative of the function $\ln p$ is $\frac{1}{p}$, the computation of the higher derivatives of this function repeats the results of Example 5.2. Thus the quaternion logarithmic function $\psi(p) = \ln p$ possesses the $\mathbb{H}$-holomorphic derivatives of all orders everywhere in $\mathbb{H} \setminus \{0\}$. Equations (4.71) and (4.43) hold too. According to Corollary 4.6, each k'th derivative, $k = 2,3$ ..., has the same form as the one in the complex (and real) analysis.

As a result, we conclude that the presented theory is totally confirmed by the considered examples of Liouvillian elementary functions.

It is not difficult to see that the function $\phi_2(a,b)$ satisfying equation (4.43) as the constituent of any ℍ-holomorphic function $\psi_H(p) = \phi_1(a,b) + \phi_2(a,b) \cdot j$ can only include those forms of dependence on variables a and $\overline{a}$ that are invariant under complex conjugation. Since the quaternionic derivatives of ℍ-holomorphic functions are ℍ-holomorphic too, the same is valid for functions $\phi_2^{(1)}$, $\phi_2^{(2)}$, $\phi_2^{(3)}$ and so on. From examples discussed above it follows that such forms are $a\overline{a}$ and $(a\overline{a})_m = a^m + a^{m-1} \cdot \overline{a} + a^{m-2} \cdot \overline{a}^2 + \ldots + a^2 \cdot \overline{a}^{m-2} + a \cdot \overline{a}^{m-1} + \overline{a}^m$, where m is some positive integer and by "$\cdot$" is denoted the complex multiplication.

If we consider, for example, the first derivative of the function $\psi(p) = p^4$, then we have $\phi_2^{(1)} = (\partial_a \phi_2 + \partial_{\overline{a}} \phi_2) = 4\left[\left(a^2 + a\overline{a} + \overline{a}^2\right)b - b^2\overline{b}\right] = 4\left[(a\overline{a})_2 b - b^2\overline{b}\right]$ with the "invariant" (symmetric in variables a and $\overline{a}$) form $(a\overline{a})_2$. The next examples, the functions $\psi(p) = p^{-1}$ and $\psi(p) = e^p$, have, respectively, $\phi_2^{(1)} = \frac{b(a+\overline{a})}{(a\overline{a}+b\overline{b})^2} = \frac{b(a\overline{a})_1}{(a\overline{a}+b\overline{b})^2}$ and $\phi_2^{(1)} = \frac{2\beta b \sin v}{v} = e^{\frac{a+\overline{a}}{2}} \cdot \frac{b \sin\sqrt{a\overline{a}+b\overline{b}-\frac{(a+\overline{a})^2}{4}}}{\sqrt{a\overline{a}+b\overline{b}-\frac{(a+\overline{a})^2}{4}}}$ with the symmetric forms $a\overline{a}$ and $(a\overline{a})_1 = a + \overline{a}$. In addition, we consider one more example of the function $\psi(p) = p^6$. Omitting some algebra we can see that this function possesses $\phi_2 = (a\overline{a})_5 b - 2[2(a\overline{a})_3 + a\overline{a}(a\overline{a})_1]b^2\overline{b} + 3(a\overline{a})_1 b^3\overline{b}^2$ and $\phi_2^{(1)} = 6(a\overline{a})_4 b - 2[9(a\overline{a})_2 + 3a\overline{a}]b^2\overline{b} + 6b^3\overline{b}^2$. In this case we have the symmetric invariant forms $a\overline{a}$, $(a\overline{a})_1$, $(a\overline{a})_3$, $(a\overline{a})_5$ and $a\overline{a}$, $(a\overline{a})_2$, $(a\overline{a})_4$, respectively.

We can state that the constituent $\phi_2^{(1)}$ of the complete quaternion derivative $\psi(p)^{(1)} = \phi_1^{(1)} + \phi_2^{(1)} j$ puts together its "left and right fragments", namely, the derivative $(\partial_a \phi_2)$, belonging to the left quaternionic derivative (4.15) and the derivative $(\partial_{\overline{a}} \phi_2)$, belonging to the right quaternionic derivative (4.23). The analogous statement is valid, of course, for the constituent ϕ_2 of an initial ℍ-holomorphic function. On the other hand, note that the functions ϕ_1, $\phi_1^{(1)}$, $\phi_1^{(2)}$, … are symmetric in variables b and $\overline{b}$ and may be interpreted as a sum of the corresponding parts of the left and right quaternionic derivatives too.

6 Conclusions

A corner stone of our work has been the independence of the quaternionic derivative (as a limit of a difference quotient) of the "way of its computation", that is, the independence not only of the limiting path but also of the way of quaternion division: on the left or on the right.

Such a requirement based on the general concept of essentially adequate differentiability extends the complex derivative definition to the definition in space in the natural way. To develop the theory of quaternionic differentiability similar to a complex one, we have generalized the basic notions of the differentiation theory based on the derivative definition as a limit of a difference quotient to the quaternion case step by step.

The main stumbling block to develop the quaternion differentiability theory – the impossibility to obtain a wide class of differentiable functions by the separate consideration of the left and right versions due to their incomplete adequacy to properties of space – is eliminated, since the complete essentially adequate quaternionic derivative puts together the partial derivatives contained in the left and right quaternionic derivatives.

According to theorems proved, the expressions for $\mathbb{H}$-holomorphic functions and their derivatives are similar to the corresponding expressions for $\mathbb{C}$-holomorphic functions and their derivatives. Each of $\mathbb{H}$-holomorphic functions (and its derivatives) can be now created from the corresponding $\mathbb{C}$- holomorphic function (and its derivatives) by the direct replacing of variables. In principle, this solves the known problem of Fueter's polynomials, for which the reduction to the complex case gives the incorrect power functions $(-iz)^n$ instead of z^n [2].

The complete quaternionic derivative connecting the left and right differentiability versions contains the symmetric in variables a and $\overline{a}$ as well as b and $\overline{b}$ forms, representing undoubtedly the harmony of space symmetry.

A study of many other matters is beyond the scope of the present paper. Primarily, there is a need to expound a proof of equations (4.71) as well as explore the essentially adequate quaternionic generalizations of antiholomorphic functions, Laplace's equation and harmonic functions, scalar potential function for the fluid flow in space and for other applications that, hopefully, will be represented in subsequent papers.

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