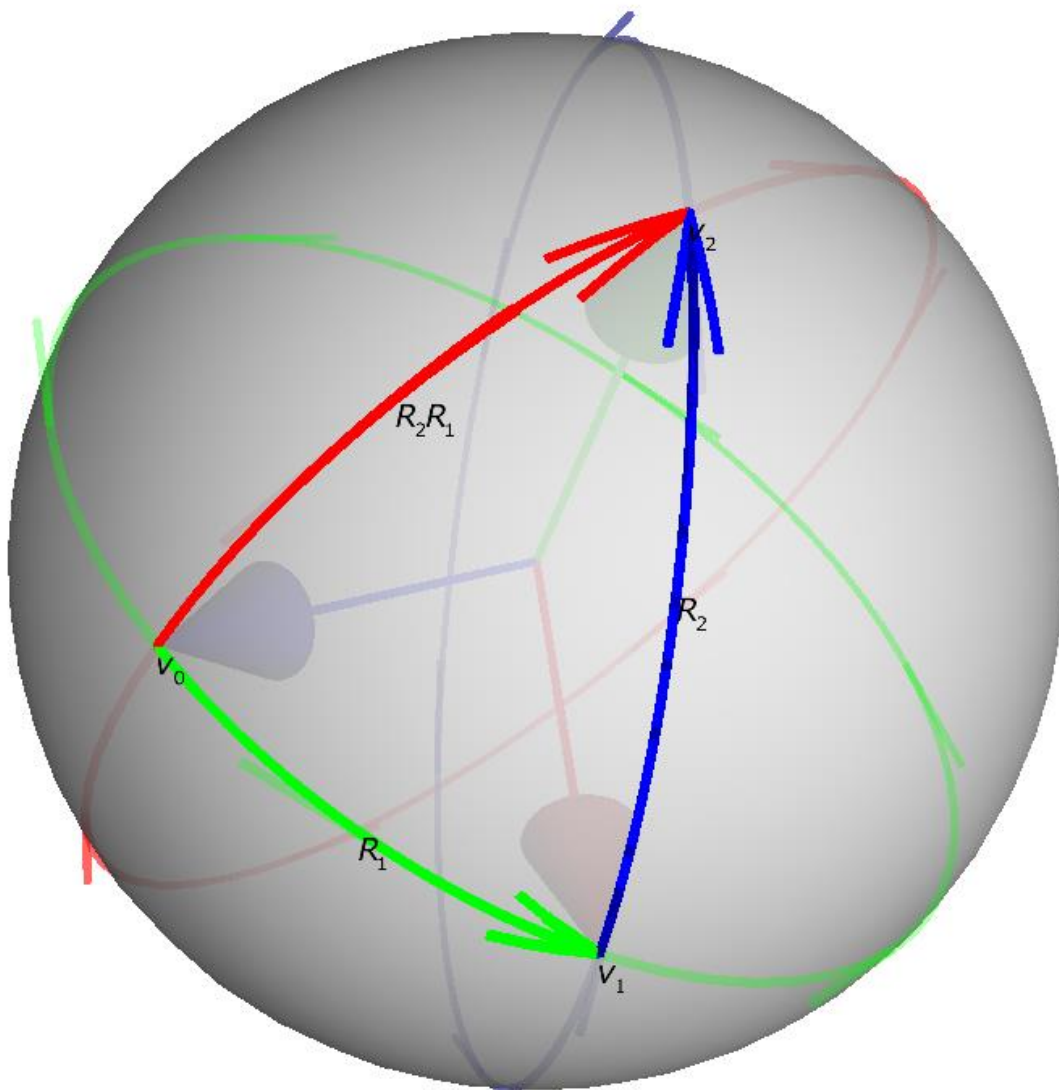


Miroslav Josipović

# Multiplication of vectors and structure of 3D Euclidean space



„So God made the people speak many different languages ...“

## Virus

„The **World Health Organization** has announced a world-wide epidemic of the **Coordinate Virus** in mathematics and physics courses at all grade levels. Students infected with the virus exhibit compulsive **vector avoidance** behavior, unable to conceive of a vector except as a list of numbers, and seizing every opportunity to replace vectors by coordinates. At least two thirds of physics graduate students are severely infected by the virus, and half of those may be permanently damaged so they will never recover. The most promising treatment is a strong dose of **Geometric Algebra**“ . (Hestenes)

## Cat

„When the spiritual teacher and his disciples began their evening meditation, the cat who lived in the monastery made such noise that it distracted them. So the teacher ordered that the cat be tied up during the evening practice. Years later, when the teacher died, the cat continued to be tied up during the meditation session. And when the cat eventually died, another cat was brought to the monastery and tied up. Centuries later, learned descendants of the spiritual teacher wrote scholarly treatises about the religious significance of tying up a cat for meditation practice.“ (Zen story)

## Empty your cup

„A university professor went to visit a famous Zen master. While the master quietly served tea, the professor talked about Zen. The master poured the visitor's cup to the brim, and then kept pouring. The professor watched the overflowing cup until he could no longer restrain himself. - *It's overfull! No more will go in!* - the professor blurted. - *You are like this cup,*- the master replied, - *How can I show you Zen unless you first empty your cup?*“ (Zen story)

## Division algebra

“Geometric algebra is, in fact, the largest possible associative division algebra that integrates all algebraic systems (algebra of complex numbers, vector algebra, matrix algebra, quaternion algebra, etc.) into a coherent mathematical language that augments the powerful geometric intuition of the human mind with the precision of an algebraic system.”

(Sabbata: *Geometric algebra and applications in physics* [28])

## Preface

The aim of this paper is to introduce the interested reader to the world of geometric algebra. Why?

Alright, imagine the Neelix and Vulcan (from the starship Voyager) conversation. The goal is to sell a new product to the Vulcan (Tuvok). This can be achieved so that Neelix quickly intrigue the Vulcan, giving him as little information as possible, and the ultimate goal is that Vulcan, after using it, be surprised by the quality of the product and recommend it to the others. Let's start.

Neelix: "Mr Vulcan, would you like to rotate objects without matrices, in any dimension?"

Vulcan: "Mr Neelix, do you offering me quaternions?"

Neelix: "No, they only work in 3D, I have something much better. In addition you will be able to do spinors, too."

Vulcan: "Spinors? Come on, mr Neelix, you're not going to say that I will be able to work with complex numbers, too?"

Neelix: "Yes, mr Vulcan, the whole complex analysis, generalized to higher dimensions. And you will be able to get rid of tensors."

Vulcan: "Excuse me, what? I'm a physicist, it will not pass ..."

Neelix: "It will, you do not need the coordinates. And you will be able to do the special theory of relativity and quantum mechanics using the same tool. And all integral theorems that you know, including the complex area, become a single theorem."

Vulcan: "Come on ... nice idea ... I work a lot with the Lie algebras and groups ..."

Neelix: "In the package ..."

Vulcan: "Are you kidding me, mr Neelix? Ok, let's say that I believe you, how much would that product cost me?"

Neelix: "Pennyworth, mr Vulcan, You must multiply vectors differently."

Vulcan: "That's all? All of this you offer me for such a small price? What's trap?"

Neelix: "There is no one. But true, you will have to spend some time to learn to use the new tool".

Vulcan: "Time? Just do not have ... And why would I ever forgo coordinates? You know, I am quite adept at juggling indices, I have my career ..."

Neelix: "Do physical processes you are studying depend on the coordinate systems you choose?"

Vulcan: "I hope not."

Neelix: "There. Does a rotation by matrices provides you a clear geometric meaning when you do it?"

Vulcan: "No. I have to work hard to find it out."

Neelix: "Now you will not have to, it will be available to you at each step."

Vulcan: "Mr. Neelix, I'm curious, where did you get this new tool?"

Neelix: "Well, mr Vulcan, it is an old tool from Earth, 19th century, I think, invented by humans Grassmann and Clifford."

Vulcan: "What? How is that I'm not aware of it? Isn't it strange?"

Neelix: "Well, I think that human Gibbs and his followers had a hand in it. Allegedly, human Hestenes was trying to tell the other humans about it, but they did not listen to him. You will agree, mr Vulcan, that humans are really funny sometimes."

Vulcan: "Mr Neelix, this is a rare occasion when I have to agree with you."

Vulcan buys and lives long and prosper. And, of course, recommends the new tool to the captain ...

This text is not intended as a textbook, it is more motivationally directed, to see „what's up“. It is intended mainly to young people. Also, intention here was to use simple examples and reader is referred to the independent problem solving. The active reading of the text is recommended, with paper and pencil in hand. There is a lot of literature, usually available at Internet, so, reader is referred to the independent research. The use of available computer programs is also recommended. There are reasons to think that geometric algebra is mathematics for future. Paradoxically, it has been established since the mid-19th century, but was ignored as a result of a series of (unfortunate) circumstances. It's hard to believe that those who have made careers will easily accept something new, hence belief that this text is mainly for young people. The background in physics and mathematics at the undergraduate level is necessary for some parts of the text, but it is somewhat possible to follow the exposure using Internet to find explanation for the less familiar terms. A useful source is the book [35], which can certainly help to those who are just starting with algebra and geometry. The book [20] is hard one and it is recommended to those who think seriously. But, read Hestenes' articles first.

It is important for the reader to adopt the idea that the vector multiplication here exposed is natural and justified. The rest are the consequences of such a multiplication. The reader can independently come up with arguments to justify the introduction of the geometric product. The goal is to understand that the geometric product is not just a "neat trick", but that naturally arises from the concept of vector. That changes a lot of mathematics. A simple setting that parallel vectors commute while orthogonal anti-commute produces an incredible amount of mathematics and unites many different mathematical disciplines into the language of geometric algebra.

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## Geometric product (of vectors)

Vectors will generally be denoted in small letters in *italic* format, wherever there's no possibility of confusion. We will use **bold** format also, if necessary. We will use the Greek alphabet for real numbers. Multivectors are denoted in uppercase *italic* format. If we define the orthonormal base in the vector space then the number of unit vectors which square to 1 is denoted with  $p$ , the number of those with square -1 with  $q$  and the number of those with square 0 with  $r$ . Then the common designation for such a vector space is  $\mathfrak{R}(p, q, r)$  or  $\mathfrak{R}^{p,q,r}$ , while triplet  $(p, q, r)$  is referred as the *signature* (in literature it is also the sum  $p + q$ ). For geometric algebra of 3D Euclidean vector space  $\mathfrak{R}^3$  we use the abbreviation *Cl3*, which is motivated by the surname Clifford.

Here, when we say “vector”, we do not refer to elements of an abstract vector space, we rather take that concept as “oriented straight line”. To add vectors we use the parallelogram rule. Vectors  $a$  and  $b$  that satisfy the relation  $b = \alpha a$ ,  $\alpha \in \mathfrak{R}$ ,  $\alpha \neq 0$ , are *parallel*. For parallel vectors we say that they have the same *direction (attitude)*, but could have the same or opposite *orientation*. We can resolve any vector  $b$  into the component in the direction of the vector  $a$  (*projection*) and the component without any part parallel to the vector  $a$  (*rejection*)

$$b = b_{\parallel} + b_{\perp}, \quad b_{\parallel} = \alpha a, \quad \alpha \in \mathfrak{R}, \quad \alpha \neq 0.$$

Here we can immediately anticipate objections, like: „Yes, but if we talk about orthogonal vectors we need a scalar product ...“. Although we use the character „ $\perp$ “, here we are **not** talking about the orthogonality of vectors, yet. Simple, by the fact that vectors can be added, we conclude that any vector can be written as a vector sum of two vectors, in an infinite number of ways. One of these possibilities is just given by the previous relation, so it can be seen as a question of **existence**, and not how to practically implement it. Namely, for  $b_{\perp} = b - b_{\parallel} = b - \alpha a$ , if we assume that the vector  $b_{\perp}$  contains a component parallel to  $a$  we can write  $b'_{\perp} + \beta a = b - \alpha a$ , but then vector  $b'_{\perp}$  is our rejection. If there is no  $b'_{\perp}$  then vector  $b$  is parallel to the vector  $a$ . After, eventually, we succeed to define the product of vectors, we can return to the question how to find  $b_{\perp}$  practically, and that is what the new product of vectors should certainly enable to us.

Let's ask the question: **how to multiply vectors**? We will need to “forget” everything we have learned about the multiplication of vectors (i.e. scalar and cross products). Well, before we “forget” them, let's look at some of their properties. Can we uniquely solve the equation  $a \cdot x = \alpha$  (here  $a \cdot x$  is a scalar product)? The answer is, clearly, we cannot, because if we imagine a plane perpendicular to the vector  $a$  and passing through the vector endpoint, each vector whose start point coincides with the start point of the vector  $a$  and ending on the plane will be the solution. What about the equation  $a \times x = b$  (cross product)? It cannot be uniquely solved, because if  $x$  is a solution then each vector of form  $x + \beta a$  is a solution, too. But, interesting, if we take into account both equations then we can find a unique solution. Notice that scalar product is commutative, while cross product is anti-commutative. For two unit vectors  $m$  and  $n$  in 3D we have

$$m \cdot n = \cos \alpha \quad \text{and} \quad |m \times n| = \sin \alpha,$$

which suggests that these two products are somehow related, because of

$$\sin^2 \alpha + \cos^2 \alpha = 1.$$

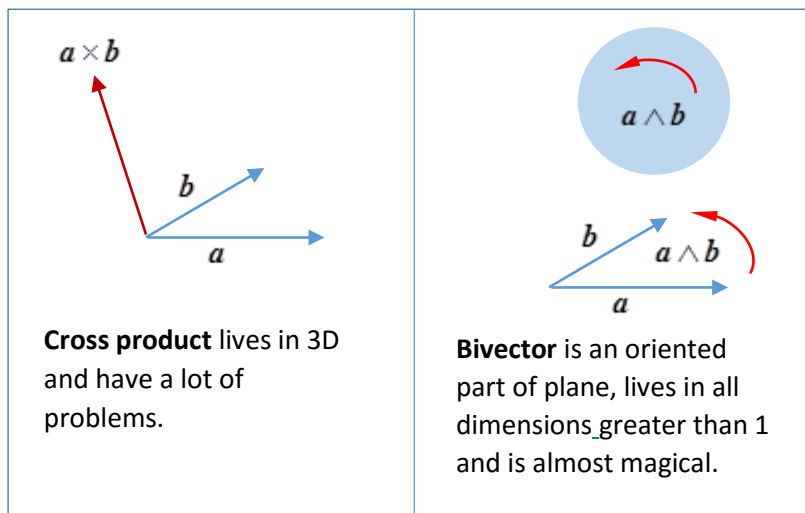
An interconnection could be anticipated if we look at multiplication tables in 3D ( $e_i$  are orthonormal basis vectors):

•	$e_1$	$e_2$	$e_3$	×	$e_1$	$e_2$	$e_3$	
	$e_1$	1	0	0	$e_1$	0	$e_3$	$-e_2$
	$e_2$	0	1	0	$e_2$	$-e_3$	0	$e_1$
	$e_3$	0	0	1	$e_3$	$e_2$	$-e_1$	0

We see that scalar product has a values different from zero only on the diagonal, while cross product has zeros on diagonal (due to anti-commutativity). Multiplication tables simply lure us to unite them. The form of both products suggests similarity with complex numbers that can be elegantly written in the trigonometric form, but for this we need a quantity which gives -1 squared, like imaginary unit. But, it is not clear how to naturally relate the cross product to the imaginary unit like quantity. On the other hand, cross product is anti-commutative, which suggests that it "should" have the feature to give -1 when squared. Namely, if we imagine any quantities that give positive real values when squared and whose products are anti-commutative and associative we would have

$$(AB)^2 = ABAB = -ABBA = -A^2B^2 < 0.$$

Let's look at an orthonormal basis in 3D, we can say that the vector  $e_1$  is polar vector, while  $e_2 \times e_3 = e_1$  is axial vector. So, what is  $e_1$  like? Of course, we could play with more general definitions invoking tensors, but it is strange that in such an elementary example we immediately have a problem. Mathematicians would argue that the cross product can generally be defined in the dimensions different from 3, but if you think about it a little and require a natural and simple definition, some questions arise immediately.



Let's look at a 2D world where flatbed physicists want to define the torque. If they do not wish to look for the new dimensions outside "their world", they will not even try to define a cross product, there is no vector orthogonal to their world. But, we can see that the torque makes sense in 2D world: it is proportional to the amount of both force and force arm, the two possible orientations of rotation are clearly defined, therefore, how to multiply a force arm vector and a force vector to provide the desired torque? The answer to that question is found already in 19th century by great mathematician Grassmann, underestimated and neglected in his time. He defined the anti-commutative exterior product of vectors and so got a **bivector**, an object contained in a plane, with orientation and module, so, it is ideal for our 2D problem. In addition, it can be easily generalized to higher dimensions.

Grassmann himself and Clifford a little later managed to unite scalar and outer (exterior) product into one: *geometric product*, exactly what we are talking here about. The scalar product of vectors is not changed, but the cross product is replaced by outer product and artificial difference between "axial" and "polar" vectors disappeared. All "axial" vectors are bivectors (magnetic field vector, for example, see in text).

Alright, now "forget" scalar and cross product and let's find how to define a **new** one. It is reasonable to require associativity and distributivity of the new multiplication (like for real numbers), i.e.

$$a(bc) = (ab)c \text{ and } (\beta b + \gamma c)a = \beta ba + \gamma ca, \quad a(\beta b + \gamma c) = \beta ab + \gamma ac, \quad \beta, \gamma \in \mathfrak{R}.$$

Of course, we do not expect commutativity of vector multiplication, except for scalars (real numbers). After all, definition of the cross product is motivated by the need for such a non-commutative constructs (like torque, or Lorentz force, ...).

- 1) Let's consider the term  $a^2$  first ( $a$  is a vector). We will assume that  $a^2 \in \mathfrak{R}$ . Clarify immediately that we do **not** imply that  $ab \equiv a \cdot b$ , as usual, where we have the scalar product denoted by dot. This is important to note, as it would lead to confusion otherwise. We expect that the square of the vector does not depend on the vector direction, but depends on its length (we exclude the possibility of nonzero vectors with the length zero, for now).
- 2) We expect that the multiplication of the vector by real scalar is commutative, which immediately results in that the multiplication of parallel vectors ( $a \parallel b$ ) is commutative:

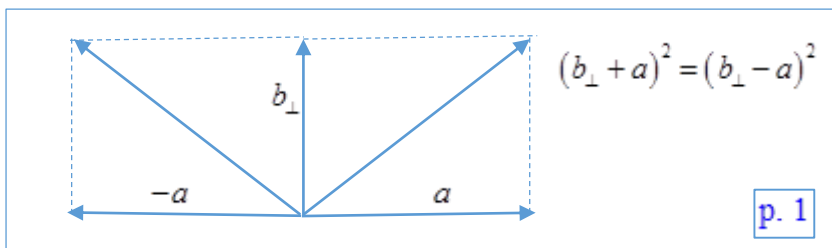
$$\lambda a = a\lambda \Rightarrow ab = a\lambda a = \lambda aa = ba, \quad \lambda \in \mathfrak{R}.$$

Actually, we could call the principles of symmetry to help us, we immediately see that multiplication of parallel vectors must be commutative, because we have no criterion to distinguish which vector is the "first" and which is the "second". It is obvious if vectors have the same orientation, but if vectors have the opposite orientations we can refer to the fact that all the orientations in space are equal (isotropy). Our new product should also include multiplication of reals by reals.

- 3) Due to the independence of the square of the vector on direction we have (recall,  $b_{\perp}$  has no component in the direction of  $a$ )

$$(b_{\perp} + a)^2 - (b_{\perp} - a)^2 = 0 = 2(b_{\perp}a + ab_{\perp}),$$

meaning that vectors  $b_{\perp}$  and  $a$  anti-commute. You can design other "arguments", but recall, we do not assume scalar or cross product, we are looking for properties of the **new** product of vectors "from scratch". This example is **not** the proof, just an idea how we could think about it. In figure p. 1 we can easily see what we demanded: that the square of vectors does not depend on the direction.



We can, of course, after we assumed non-commutative multiplication, just use

$$(a + b_{\perp})^2 = a^2 + b_{\perp}^2 + ab_{\perp} + b_{\perp}a$$

and immediately conclude that it must be  $ab_{\perp} + b_{\perp}a = 0$  because we expect the Pythagorean theorem is true. But figure p. 1 show us that we have symmetry here, namely, vectors  $a$  and  $-a$  define a "straight line", here "right" and "left" is not important concept and we see that direction of vector  $b_{\perp}$  suggests the symmetry in accordance with our intuitive concept of orthogonality. Without this symmetry we enter the "skew land", but let pure mathematicians to go there.

- 4) Let us show now that, according to 3),  $a^2$  commutes with  $b$  (without any assumption what  $a^2$  is):

$$a^2b = a^2(b_{\parallel} + b_{\perp}) = b_{\parallel}a^2 - ab_{\perp}a = b_{\parallel}a^2 + b_{\perp}a^2 = ba^2,$$

which justifies our (previous) assumption that  $a^2 \in \mathfrak{R}$ . Again, it is important to understand that we are **not** giving proofs, we are to **justify** the new product of vectors. It follows immediately that  $ab_{\parallel}$  commutes with  $b$ , because of  $b_{\parallel} = \alpha a$ ,  $\alpha \in \mathfrak{R}$ . Now we have

$$ab + ba = ab_{\perp} + b_{\perp}a + 2ab_{\parallel} = 2ab_{\parallel},$$

so  $ab + ba$  commutes with  $b$ . It is clear that commutes with  $a$  also, which means that commutes with any vector.

We can always decompose any non-commutative product into symmetric and anti-symmetric part:

$$ab = \frac{ab + ba}{2} + \frac{ab - ba}{2}.$$

Symmetric part, we have seen, commutes with all vectors. It is also seen from

$$ab + ba = a^2 + b^2 - (a + b)^2,$$

because the square of a vector is commutative. Note that we have not defined yet precisely what  $a^2$  is, but it is obvious that regardless of explicit value of  $a^2$  we have for vectors  $a$  and  $b_{\perp}$

$$(a + b_{\perp})^2 = a^2 + b_{\perp}^2 + ab_{\perp} + b_{\perp}a = a^2 + b_{\perp}^2 + ab_{\perp} - ab_{\perp} = a^2 + b_{\perp}^2,$$

i.e. we have the Pythagorean theorem, here expressed through the new multiplication of vectors. If we define the term "orthogonal" as the relation between vectors in which the projection of one on the other is zero ( $b_{\perp} = a - b_{\parallel}$ ), we get the Pythagorean theorem, which now applies to orthogonal vectors regardless of the specific value of  $a^2$ , if we accept the arguments from the part 3). Let us recall that the Pythagorean theorem is, as a rule, expressed over the scalar product of vectors and that in this way we have a problem with negative signature (meaning that there are vectors whose square is negative), as is customary in the special theory of relativity. For any two vectors, the relation

$$(a + b)^2 = a^2 + b^2 + ab + ba$$



can be taken as the *scalar theorem*, because the symmetric part of the new product commutes with all vectors, and thus is the "scalar", that is easily justified here, because the symmetric part of the product of vectors depends only on squares of vectors.

We assumed that  $a^2$  is real number equal to  $\pm|a|^2$ , where  $|a|$  is the absolute length of the vector  $a$  (we say that we are introducing *metrics*). Now we can write for the symmetric part

$$a \cdot b \equiv (ab + ba) / 2,$$

that we call the *inner product*. We see that it coincides with the usual scalar product of vectors, but here we need a little bit of caution: in geometric algebra we generally distinguish several types of "scalar" products, one of them is *scalar product* (generally different than that of Gibbs), and there are more: *dot product*, *left contraction*, etc. For vectors, all types of "scalar" products coincide, but generally they are a little different (see literature). Here we are to work with the *inner product* and the *left contraction* (see in text).

For the unit vectors of the orthonormal basis we have  $e_i^2 = \pm 1$  (null-vectors are not included here), which means

$$e_i e_j + e_j e_i = \pm 2\delta_{ij}.$$

**Caution:** do not confuse  $e_i e_j$  with  $e_i \cdot e_j$ ! If you are wondering what is  $e_i e_j$  the answer is: a **completely new type of object**, we will see it in the text.

Let's look at a 2D examples:

$$\mathfrak{R}^2: e_1^2 = e_2^2 = 1 \Rightarrow (e_1 + e_2)^2 = 1 + 1 + e_1 e_2 + e_2 e_1 = 2 = e_1^2 + e_2^2,$$

$$\mathfrak{R}^{1,1}: e_1^2 = -e_2^2 = 1 \Rightarrow (e_1 + e_2)^2 = 1 - 1 + e_1 e_2 + e_2 e_1 = 0 = e_1^2 + e_2^2,$$

we see that in both cases the Pythagorean theorem is valid, but with the new multiplication of vectors.

For  $\mathfrak{R}^3$  we have:

$$e_1^2 = e_2^2 = e_3^2 = 1, \quad e_i e_j + e_j e_i = 2\delta_{ij},$$

but, here's a magic, there are known mathematical objects that meet precisely these relations: *Pauli matrices*, discovered in the glorious years of the development of quantum mechanics. We can say that the Pauli matrices are 2D matrix representation of the unit vectors in  $\mathfrak{R}^3$ , we only need vectors to be multiplied in a new manner, just described. That is to say, the Pauli matrices have the same multiplication table as orthonormal basis vectors. Let's make sure of that. Pauli matrices are defined as

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

so, for example

$$\hat{\sigma}_2 \hat{\sigma}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \hat{\sigma}_1 \hat{\sigma}_2 + \hat{\sigma}_2 \hat{\sigma}_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Designation  $\hat{\sigma}_i$  is often used for Pauli matrices, so here we use  $\sigma_i$  for unit vectors in  $\mathfrak{R}^3$ . Pauli matrices are important to describe the spin in quantum mechanics, so we see that vectors could serve to this purpose as well, but with our new product of vectors. Indeed, quantum mechanics can be nicely formulated by such mathematics, **without matrices and imaginary unit** (see below).

Note that by transposition followed by complex conjugation of Pauli matrices we get again the same matrix (*Hermitian adjoint*), for example

$$\hat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \xrightarrow{T} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

or simply  $\hat{\sigma}_2^\dagger = \hat{\sigma}_2$ . Also we have, for example,  $(\hat{\sigma}_2 \hat{\sigma}_3)^\dagger = \hat{\sigma}_3 \hat{\sigma}_2$  (*antiautomorphism, show that*). This exactly matches the operation *reverse* (see below) on vectors, for example  $e_1 e_2 e_3 \rightarrow e_3 e_2 e_1$ . Therefore, the character  $\dagger$  is often used to denote the *reverse* operation (we will do so here).

Here we can immediately spot the important feature of the new multiplication of vectors. The vector is geometrically clear and intuitive concept, and the new product of vectors also has a clear geometric interpretation (see below). For example, we can clearly geometrically present the product  $e_1 e_2$  as the oriented area, it has the ability to rotate, unambiguously defines the plane spanned by vectors  $e_1$  and  $e_2$ , etc. All this we can immediately conclude at a glance. For comparison, consider now the matrix representation of vectors  $\sigma_1$  and  $\sigma_2$  with their product:

$$\hat{\sigma}_1 \hat{\sigma}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Can we derive similar conclusions about geometric interpretation just by looking into the resultant matrix? Just looking certainly not, it would take a lot of effort, but we will often fail to get the clear geometrical interpretation. Which plane the resultant matrix defines (if any is to be defined)? **Pauli matrices cannot do all that vectors can.** In this text we will, hopefully, illuminate such a things in order to get an idea of the importance of the new multiplication of vectors.

It is time for the new multiplication of vectors to get the name "officially" (*Clifford, Hestenes, ...*): *geometric product*. Symmetric and anti-symmetric parts of the geometric product of vectors have special insignia:  $a \cdot b$  and  $a \wedge b$  ( $a \cdot b$  is *inner* and  $a \wedge b$  is *outer product*), so we can write for the vectors

$$ab = a \cdot b + a \wedge b.$$

An important concept, that we will often use, is the *grade*. Real numbers have grade zero, vectors have grade 1, all elements that are linear combinations of products  $e_i \wedge e_j$ ,  $i \neq j$ , have grade 2, and so on. Notice that geometric product of two vectors is a combination of grades 0 and 2, it is *even*, because its grades are even. [What grades generally has the geometric product of three vectors?](#)

A vector space over the real field with geometric product (GP in text) becomes an algebra (*geometric algebra*, GA in text). Elements of geometric algebra obviously are not the vectors only. Note that inner product is zero for orthogonal vectors, for example, for orthonormal basis vectors we have

$$e_1 \cdot e_1 = \frac{e_1 e_1 + e_1 e_1}{2} = 1, \quad e_1 \cdot e_2 = \frac{e_1 e_2 + e_2 e_1}{2} = 0 \Rightarrow e_1 e_2 = e_1 \cdot e_2 + e_1 \wedge e_2 = e_1 \wedge e_2,$$

so for orthogonal vectors geometric product is the same as outer product. How about the anti-symmetric part? We have

$$e_1 \wedge e_2 = \frac{e_1 e_2 - e_2 e_1}{2} = \frac{e_1 e_2 + e_1 e_2}{2} = e_1 e_2, \quad e_1 \wedge e_1 = \frac{e_1 e_1 - e_1 e_1}{2} = 0,$$

Obviously,  $e_1 e_2$  is not a scalar, it doesn't commute with all other vectors, for example

$$(e_1 \wedge e_2)e_1 = (e_1 e_2)e_1 = -e_1 e_1 e_2 = -e_1 (e_1 \wedge e_2),$$

but is neither a vector in  $\mathfrak{R}^3$ , it squares to -1:

$$(e_1 e_2)^2 = e_1 e_2 e_1 e_2 = -e_1 e_1 e_2 e_2 = -1,$$

so, we have a **new** type of mathematical object, it is like **imaginary unit**, except that is non-commutative. The name for this object is *bivector*. Generally, we will define a bivector as element of algebra of form  $a \wedge b$ . Let's look at some more properties of the bivector  $e_1 e_2$ . We have

$$(e_1 e_2)e_1 = -e_1 e_1 e_2 = -e_2, \quad (e_1 e_2)e_2 = e_1,$$

so, acting from the left on vectors it rotates them by  $-\pi/2$ . [How it rotates vectors if acting from the right?](#)

Recall the *reverse* operation on geometric product of vectors:  $x = abc\dots d \rightarrow d\dots cba = x^\dagger$ , so we have

$$(e_1 e_2)(e_1 e_2)^\dagger = (e_1 e_2)(e_2 e_1) = -(e_1 e_2)^2 = 1,$$

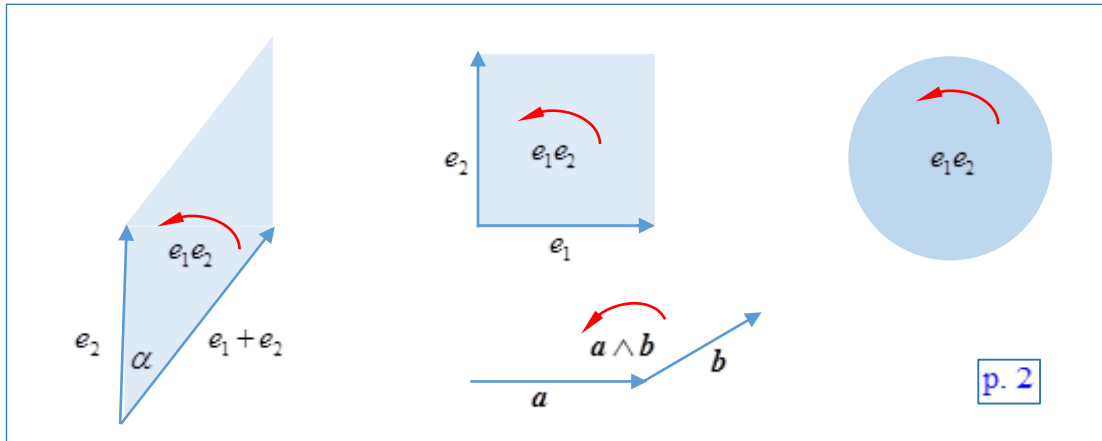
therefore we call it a *unit bivector*. Generally, it is possible to find a module of bivectors, so, bivectors have the module and orientation. Furthermore, unit bivectors, like  $e_1 e_2$ , except for the module, orientation ( $e_1 e_2 \neq e_2 e_1 = -e_1 e_2$ ) and the ability to rotate vectors, have another important feature, which imaginary unit does not have, namely, it **defines** the plane spanned by vectors (here  $e_1$  and  $e_2$ ). Later we will see how this is implemented in practice by outer product.

Now let's see how we can graphically present (unit) bivector. The obvious option is to try with oriented parallelogram (square for  $e_1 e_2$ ). But, the shape of area which represents

bivector is not important, we should keep the amount of area and orientation, therefore it is often a practical choice an oriented circle of radius  $|e_1 e_2| / \sqrt{\pi}$ . To justify our claims, look at

$$e_1 e_2 = e_1 \wedge e_2 \Rightarrow (e_1 + e_2) \wedge e_2 = e_1 e_2,$$

it can illustrate the fact that shape is not important.



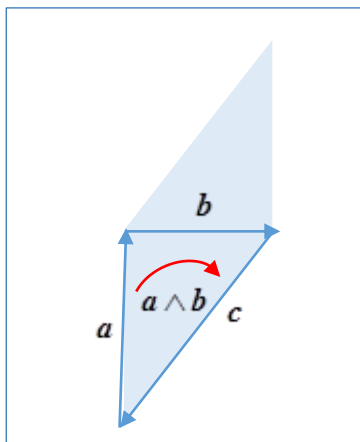
Notice immediately that two vectors, except that define a plane, generally define a parallelogram, too. The outer product of such vectors (bivector) has module just equal to the parallelogram area (see below), while direction we define as in figure p. 2. [Find the area of the leftmost parallelogram in p. 2. Notice that bivector is just  \$e\_1 e\_2\$ , but show that formula](#)

$$1 = |e_1 e_2| = |e_1 + e_2| |e_2| \sin \alpha$$

[gives the area of the parallelogram.](#)

Let's look at three vectors in  $\mathfrak{R}^3$  which sum is zero (triangle), from  $\mathbf{a} + \mathbf{b} + \mathbf{c} = 0$  it follows that

$$\mathbf{a} \wedge \mathbf{b} = \mathbf{b} \wedge \mathbf{c} = \mathbf{c} \wedge \mathbf{a},$$



to see this it is enough to look at expressions  $(\mathbf{a} + \mathbf{b} + \mathbf{c}) \wedge \mathbf{a}$  and  $(\mathbf{a} + \mathbf{b} + \mathbf{c}) \wedge \mathbf{b}$  (check), but we can see it easy without calculation, it is enough to look at the figure on the left: each pair of vectors defines the parallelogram of area equal to the double area of the triangle, and all pairs give the same orientation of bivector. This is important, often we can draw conclusions simply from the geometric observations, without calculation. In the formula for the area of a parallelogram appears the sine function, so we see that the previous equalities are just the *sine theorem*. If we recall that bivector is not shape depended, we see that all three our bivectors have the same factor  $I$  (the unit bivector). Now we have

$$I a b \sin \gamma = I b c \sin \alpha = I a c \sin \beta .$$

Bivectors define a plane. Consider the outer product in  $Cl_3$

$$(e_1 \wedge e_2) \wedge (a_1 e_1 + a_2 e_2 + a_3 e_3) = (e_1 \wedge e_2) \wedge (a_1 e_1 + a_2 e_2 + a_3 e_3) = a_3 e_1 e_2 e_3,$$

so, we can see that outer product of bivector with a vector gives the possibility to eliminate the components of vector that do not belong to the plane defined by the bivector. Therefore, the plane of the bivector  $B$  is defined by the relation

$$B \wedge x = 0.$$

In our example, this would be all vectors of form  $x = a_1 e_1 + a_2 e_2$ .

Imagine a unit bivector in  $Cl_3$ . It defines a plane and have the properties of (non-commutative) imaginary unit (in that plane). It is powerful: we can use the formalism of complex numbers in any plane, in any dimension. How? Let's take back our bivector  $e_1 e_2$  and the vector  $x e_1 + y e_2$ . If we multiply our vector by  $e_1$  from the left we get

$$e_1 (x e_1 + y e_2) = x + y e_1 e_2 = x + y I, \quad I = e_1 e_2,$$

so, we have a complex number. [What we get if we multiply from the right?](#) For more details see below.

[The reader may show that any linear combination of unit bivectors in  \$Cl\_3\$  can be expressed as an outer product of two vectors.](#) This is not necessarily true in 4D, take for example  $e_1 e_2 + e_3 e_4$ . [Prove that there are no two vectors in 4D with the property  \$a \wedge b = e\_1 e\_2 + e\_3 e\_4\$ .](#) In 3D, for each plane we have exactly one orthogonal unit vector (up to the sign), while that is not true in the higher dimensions. For example, in 4D, plane defined by bivector  $e_1 e_2$  has orthogonal unit vectors  $e_3$  and  $e_4$  (their linear combinations too). Take the bivector  $e_1 e_2$  in  $\mathfrak{R}^3$  and multiply it by  $-e_1 e_2 e_3 \equiv -j$ :  $-e_1 e_2 j = e_3$ , one can see that we get exactly the cross product of vectors  $e_1$  and  $e_2$ , or, for arbitrary vectors

$$\mathbf{a} \times \mathbf{b} = -j \mathbf{a} \wedge \mathbf{b}.$$

This is valid in 3D, but expression  $-I \mathbf{a} \wedge \mathbf{b}$  is valid in any dimension, where  $I$  is a general pseudoscalar. In 2D  $-I \mathbf{a} \wedge \mathbf{b}$  is just a real scalar, while in 4D or higher we can take advantage of the concept of *duality*. The cross product of vectors (*Gibbs*) requires the right hand rule and use of perpendiculars to the surfaces. With bivectors it will not be necessary, so, for example, we can completely omit objects such

as "rotation axis", etc. [Find the geometric product of two vectors  \$\mathbf{a} = \sum\_{i=1}^3 a\_i e\_i\$  and  \$\mathbf{b} = \sum\_{i=1}^3 b\_i e\_i\$  in  \$\mathfrak{R}^3\$  and show that it can be expressed as](#)

$$\mathbf{a} \mathbf{b} = \mathbf{a} \cdot \mathbf{b} + (\mathbf{a} \times \mathbf{b}) e_1 e_2 e_3 = \mathbf{a} \cdot \mathbf{b} + (\mathbf{a} \times \mathbf{b}) j.$$

## Algebra

Let's look again at 2D example. All possible outer products of vectors expressed in the orthonormal basis can provide a linear combination of "numbers"  $1, e_1, e_2$  and  $e_1 \wedge e_2 = e_1 e_2$  (any linear combination of these "numbers" we will refer as *multivector*). Outer product is anti-commutative, so, all terms that have some unit vector repeated disappear. "Numbers"  $1, e_1, e_2$  and  $e_1 e_2$  form the basis of  $2^2$  – dimensional linear space. In fact, we have the basis of the algebra (*Clifford algebra*). When geometric meaning is in the forefront we refer it as *geometric algebra* (due to Clifford himself). Element 1 is a real scalar. We have two vectors and one bivector (in the terminology of geometric algebra it is referred as *pseudoscalar* in algebra, namely, member of the algebra with the maximum grade). In  $\mathfrak{R}^3$  we have the basis of the algebra (*Cl3*):

$$1, e_1, e_2, e_3, e_1 e_2, e_1 e_3, e_2 e_3, e_1 e_2 e_3,$$

here  $j \equiv e_1 \wedge e_2 \wedge e_3 = e_1 e_2 e_3$  is the unit *pseudoscalar*. [Show that  \$j\$  commutes with all elements of Clifford basis in \*Cl3\* and that  \$j^2 = -1\$ .](#) Pseudoscalars in any dimension are all proportional to some unit pseudoscalar. [Prove it, at least for  \$j\$ .](#) So, pseudoscalar  $j$  is a perfect (commutative) imaginary unit in *Cl3*. Such a pseudoscalar will appear also in *Cl7, Cl11, ...* This has far-reaching consequences. But here one should be careful, commutativity property of pseudoscalar means geometric product, while in terms with other products one should be cautious. Real scalars do not have this "problem", they can "walk" through all products. For pseudoscalar we have, for example

$$j e_1 e_3 = e_1 j e_3 = e_1 e_3 j = (e_1 j) e_3 = e_1 (j e_3),$$

i.e. geometric product allows "walking", but this is not generally valid with the, say, inner product

$$(e_1 \cdot e_3) j = 0 \neq e_1 \cdot (j e_3) = e_1 \cdot (e_1 e_2),$$

here we have a *mixed product* (see below).

In 3D, for arbitrary four vectors we have  $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} \wedge \mathbf{d} = 0$ . Outer product has distributivity and associativity properties also ([see literature or prove itself](#)). If any two vectors here are parallel, relation is true due to anti-commutativity of outer product. Otherwise we have, for example,  $\mathbf{d} = \alpha \mathbf{a} + \beta \mathbf{b} + \gamma \mathbf{c}$ ,  $\alpha, \beta, \gamma \in \mathfrak{R}$ , so, our statement is true due to distributivity and anti-commutativity.

Maximum grade of multivector cannot be larger than the dimension of the vector space ([show that](#)). [Show that number of elements in Clifford basis with the grade  \$k\$  equals to binomial coefficient](#)

$$\binom{n}{k},$$

where  $n$  is the dimension of the vector space. For real scalars we have  $k = 0$ , so, there it is just one real scalar in the basis (i.e. 1). The same is for  $k = n$ , there is just one element with the grade  $n$  in the basis, which gave rise to the term "*pseudoscalar*". [Show that the number of elements in the Clifford basis for  \$n\$ -dimensional vector space equals to  \$2^n\$ .](#)

An important concept is the *parity* of multivector and refers to the parity of grades. All elements with even grades define the subalgebra ([geometric product of any two of these elements is even, too, show that!](#)), while this is not true for the odd part of the algebra.

Grades of multivector  $M$  are usually written as  $\langle M \rangle_r$ , where  $r$  is the grade. For the grade 0 we use just  $\langle M \rangle$ , for example  $a \cdot b = \langle ab \rangle$ . Grade 0 is a real number and it does not depend on the order of multiplication, so we have  $\langle AB \rangle = \langle BA \rangle$ , which leads to the possibility of cyclical changes, like  $\langle ABC \rangle = \langle CAB \rangle$ . This is beneficial relation, for example, consider the inner product  $a \cdot b$  and ask ourselves what would happen if we apply the transformations  $a \rightarrow nan$  and  $b \rightarrow nbn$  ( $n$  is a unit vector). Note that the result of such transformation is a vector ([resolve the vector  \$a\$  on components parallel and orthogonal to  \$n\$](#) ). The inner product of two vectors is just the zero grade of their geometric product, so we have, using cyclical changes

$$(nan) \cdot (nbn) = \langle nannbn \rangle = \langle nabn \rangle = \langle abnn \rangle = \langle ab \rangle = a \cdot b.$$

Such a transformation doesn't change the inner product, so we have an example of an orthogonal transformation (this one is a reflection). Transformation  $X \rightarrow nXn$  ( $n$  is a unit vector) generally doesn't change the grade. For example, if we have  $X = ab$  then

$$nabn = nannbn = (nan)(nbn),$$

i.e. we have a geometric product of two vectors again. This is a very important conclusion. To see that it is generally valid, recall that each multivector is a linear combination of elements of Clifford basis. So we have, for example  $e_1(e_1e_3)e_1 = e_3e_1 = -e_1e_3$ , so, grade is still 2. If grade of element is changed by a transformation then we obtain the new type of element, but we don't want that generally. Rather, frequently we want transform vectors to vectors, bivectors to bivectors, etc.

Let us now discuss some important formulas in which mixed products appear. For example, let's look at the product

$$a(b \wedge c) = a(bc - cb) / 2 = (abc - acb) / 2.$$

We can take advantage of the obvious (and useful) relation  $ab = 2a \cdot b - ba$  and show that ([left to the reader](#))

$$a(b \wedge c) - (b \wedge c)a = 2(a \cdot b)c - 2(a \cdot c)b,$$

Here we have a situation in which grade of bivector is downgraded, so it is customary to write such a relationship as the inner product, i.e. a kind of **contraction**

$$a \cdot B = (aB - Ba) / 2,$$

( $B$  is a bivector) or,

$$a \cdot (b \wedge c) = (a \cdot b)c - (a \cdot c)b = a \cdot bc - a \cdot cb,$$

where it is understood that the inner product is executed first. This is a useful and important formula. It is not difficult to show that

$$\begin{aligned} a \wedge B &= (aB + Ba) / 2, \\ aB &= a \cdot B + a \wedge B. \end{aligned}$$

Find  $e_1 \cdot (e_1 e_2)$  and  $e_1 \wedge (e_1 e_2)$ .

Here is one more useful relation (without proof)

$$e_1 \cdot (\mathbf{a}_1 \wedge \dots \wedge \mathbf{a}_n) = \sum_{k=1}^n (-1)^{k+1} e_1 \cdot \mathbf{a}_k (\mathbf{a}_1 \wedge \dots \wedge \check{\mathbf{a}}_k \wedge \dots \wedge \mathbf{a}_n),$$

where  $\check{\mathbf{a}}_k$  means that factor  $\mathbf{a}_k$  is missing in the outer product. Find  $e_1 \cdot (\mathbf{a} \wedge \mathbf{b})$ .

It is straightforward to find *projection* and *rejection* (we announced this possibility earlier), for example of vector  $a$  using the orientation of the unit vector  $n$

$$a = n^2 a = n(n \cdot a + n \wedge a) = nn \cdot a + nn \wedge a = a_{\parallel} + a_{\perp},$$

where geometric product is to be executed last. For general formulas (for any elements of algebra) see literature.

## Important concepts

Before we dive into *C3* let's look at some more general terms.

- versor* → geometric product of any number of vectors
- blade* → outer product of any number of vectors
- involution* → any function with the property  $f(f(x)) = x$
- inverse* → for element  $x$  it is element  $y$  such that  $xy = 1$ ,  $y = x^{-1}$
- nilpotent* →  $x^2 = 0$
- idempotent* →  $x^2 = x$
- zero divisors*

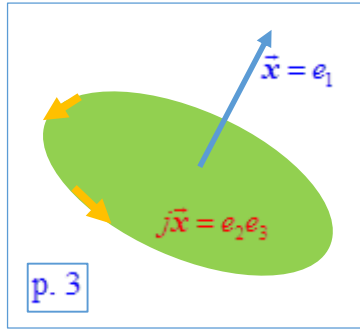
Let's explain those terms in more details.

- Example of versor is  $abc$ , if factors are vectors. For geometric product of two vectors we have generally grades 0 and 2. For verification techniques that some multivector is a versor see *Bouma* and [19]. Show that geometric product of versor and its reverse is a real number.
- Example of blade is  $a \wedge b \wedge c$ , if factors are vectors. For verification techniques that some multivector is a blade see *Bouma* and [19]. Blade is *simple* if it can be reduced to the outer product of basis vectors (up to the real factor).

While versor  $ab$  generally have grades 0 and 2, blade  $a \wedge b$  has grade 2 and defines the 2D subspace. Show that any homogeneous versor (has single grade only) is a blade. Show that any blade can be transformed to a versor with orthogonal vectors as factors. Any blade in *C3* which is outer product of three linearly independent vectors is proportional to the unit pseudoscalar (show that, if you have not done it already).

Consider the arbitrary set of indices of unit vectors of orthonormal basis, some of which can be repeated. Find an algorithm for sorting indices, so as to take into account skew-symmetry for different indices. The goal is to find the overall sign. After sorting, the unit vectors of the same index are multiplied and thus reduce to one unit vector or a real number. Example:  $e_2 e_3 e_1 e_2 = e_1 e_2 e_3 e_2 = -e_1 e_2 e_2 e_3 = -e_1 e_3$ .





Elements of Clifford basis are *simple* blades. We have seen that in  $Cl_3$  any linear combination of the unit bivectors defines a plane (i.e. can be represented as an outer product of two vectors). Multiply every element of the Clifford basis by pseudoscalar  $j$ . What you get? Figure p. 3 can help in thinking. You can use GAViewer and see how your products look like.

- c) In geometric algebra the most commonly used are three *involutions*, and all of them come down to change the sign of the components in the Clifford basis.

*Grade involution* is obtained by changing the sign of each basis vector of the vector space. In this way all even elements remain unchanged, while odd ones change the sign. Consider general multivector  $M$  in  $Cl_3$ :

$$M = t + x_1e_1 + x_2e_2 + x_3e_3 + B_1e_{12} + B_2e_{13} + B_3e_{23} + bj,$$

where  $e_{12} \equiv e_1e_2$ . Grade involution gives

$$\hat{M} = t - x_1e_1 - x_2e_2 - x_3e_3 + B_1e_{12} + B_2e_{13} + B_3e_{23} - bj.$$

Grade involution is an *automorphism* (show that), which means

$$(MN)^\wedge = \hat{M}\hat{N}.$$

Elements  $(M + \hat{M})/2 \equiv \langle M \rangle_+$  and  $(M - \hat{M})/2 \equiv \langle M \rangle_-$  give even and odd part of the multivector  $M$  (find them for  $M$ ).

*Reverse involution* is an *anti-automorphism* ( $(MN)^\dagger = N^\dagger M^\dagger$ , show that):

$$M^\dagger = t + x_1e_1 + x_2e_2 + x_3e_3 - X_1e_{12} - X_2e_{13} - X_3e_{23} - bj.$$

Elements  $(M + \hat{M})/2 \equiv \langle M \rangle_R$  and  $(M - \hat{M})/2 \equiv \langle M \rangle_I$  give real and imaginary part of multivector  $M$  (see below, find them for  $M$ ).

*Clifford conjugation (involution)* is an *anti-automorphism* ( $\overline{MN} = \overline{N}\overline{M}$ , show that):

$$\overline{M} = t - x_1e_1 - x_2e_2 - x_3e_3 - X_1e_{12} - X_2e_{13} - X_3e_{23} + bj.$$

Elements  $(M + \hat{M})/2 \equiv \langle M \rangle_S$  and  $(M - \hat{M})/2 \equiv \langle M \rangle_V$  give (complex) scalar and (complex) vector part of multivector  $M$  (see below, find them for  $M$ ).

What we get applying all three involutions on multivector, and what we get applying any two of them? Each involution changes sign of some grades. If overall sign of the grade is given in the form  $(-1)^{f(r)}$ ,  $r$  is grade, find function  $f$  for each involution. Often we need to check the properties of some product, sum, etc. What is multivector if  $M = \text{inv}(M)$ , where  $\text{inv}$  stands for any of three defined involutions? Show that for versors  $V$  relation  $V = v_1 v_2 \dots v_k \Rightarrow \hat{V} = (-v_1)(-v_2) \dots (-v_k)$  is valid. Show that multivector  $\hat{V}_x V^\dagger$  is a vector if  $x$  is a vector.

- d) An important consequence of the geometric multiplication of vectors is the existence of the inverse of vector (and many other elements of algebra), i.e. we can divide by vector. For vectors (null-vectors do not have an inverse) we have

$$a^{-1} = a / a^2,$$

which means that the unit vector is inverse to himself. The existence of the inverse has far-reaching consequences and significantly distinguishes geometric product from ordinary scalar and cross product. Now we can solve the equation:

$$ab = c \Rightarrow a = bc^{-1},$$

etc. We can define the inverses of other multivectors, for example, it is easy to see what the inverse of versor is:

$$(e_1 e_2)^{-1} = -e_1 e_2 / (e_1 e_2 e_2 e_1) = -e_1 e_2 = e_2 e_1.$$

Here we are using the fact that geometric product of versor and his reverse is just a real number. There exist multivectors without the inverse, we will see it a little later. Existence and definition of an inverse isn't always simple and obvious, but in  $Cl_3$  that task is relatively easy. It is important to note that existence of an inverse depends on possibility to define module (norm) of multivector, and that is not always unique. For general approach see references cited.

- e) Geometric product allows existence of multivectors different from zero, but whose square is zero. They are *nilpotents* in algebra and have an important role here, for example, when formulated in  $Cl_3$ , an electromagnetic wave in vacuum is just a nilpotent in the algebra. For example, we have

$$(e_1 + e_1 e_2)^2 = e_1 (1 + e_2) e_1 (1 + e_2) = e_1 e_1 (1 - e_2) (1 + e_2) = 0.$$

Nilpotents don't have an inverse. If  $N \neq 0$  is a nilpotent and  $M$  is its inverse, than from  $NM = 1$  we have  $N^2 M = N$ , i.e.  $0 = N$ .

- f) *Idempotents* have the simple property  $p^2 = p$ . Show that multivector  $(1 + e_1) / 2$  is the idempotent. In fact, every multivector of the form  $(1 + f) / 2$ ,  $f^2 = 1$ , is an idempotent. Later in text we will find the general form of idempotent in  $Cl_3$ . Trivial idempotent is 1. Show that trivial idempotent is the only one with the inverse.
- g) Multiply  $(1 + e_1)(1 - e_1)$ . There are multivectors different from zero that multiplied give zero (*zero divisors*). Although it differs from the properties of real numbers, it turns out to be very useful in many applications.

We should mention that the addition of quantities like  $x$  and  $jn$  (or other expressions of different grades) is not a problem, as some people complain, we add objects of different grades, so, as with complex numbers, such a sum preserves separation of grades. Here sum is to be understood as a **relation** between different subspaces. Let us clarify this a little bit for  $C\mathbb{3}$ . Real numbers have grade zero and define subspace of "points". Vectors define oriented lines, bivectors define oriented planes and pseudoscalars define oriented volumes. For example, bivector  $B$  defines oriented plane by relation  $B \wedge x = 0$ . In that plane we can find a unit bivector  $\hat{B}$  which has a number of interesting properties: squares to -1, it is oriented, rotates the vectors in the plane, etc. As example,  $B = e_1 e_2 + e_2 e_3 = e_2 \wedge (e_3 - e_1)$ , so vectors  $e_2$  and  $e_3 - e_1$  span the plane. Relation  $B \wedge x = 0$  gives vectors  $x$  as linear combinations of vectors  $e_2$  and  $e_3 - e_1$ . Find  $BB^\dagger$ . We see, the (unit) bivector  $\hat{B} = B / \sqrt{2}$  has a clear geometric interpretation, but it is also the operator which rotates vectors in the plane it defines. It can also serve as an imaginary unit for complex numbers defined in the plane it defines. Multivector of the form  $\alpha + B$  is the sum of different grades, but there is no way to "blend" real scalars and bivectors in sums: they are always separated. But together, as a sum, they are powerful, as *rotors* or *spinors*, for example (see below).

Finally, any multivector can be expressed as a list of coefficients in Clifford basis. As an example we can use the multivector  $3 - e_2 + e_1 e_2$  in 2D, list of coefficients is  $(3, 0, -1, 1)$ . It is clear that we can add and subtract such lists, find a rule to multiply them, etc.

## Examples of solving equations

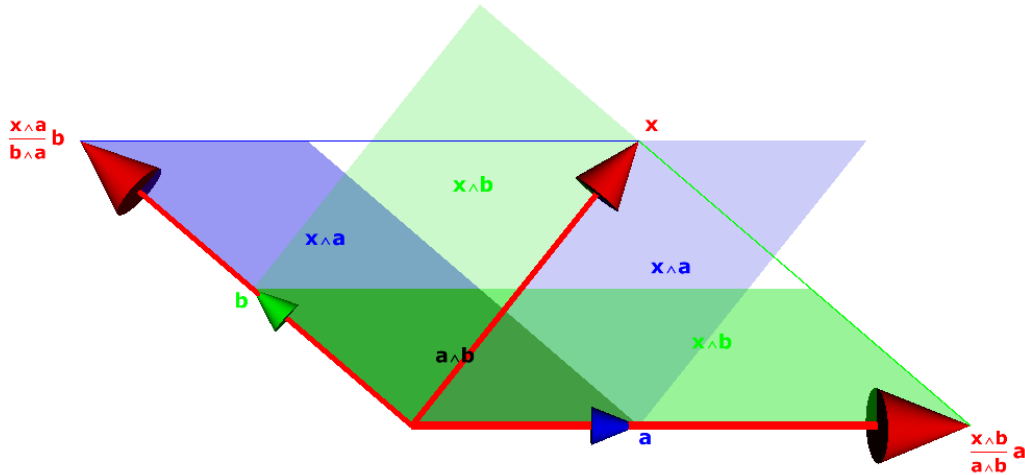
Let's find real numbers  $\alpha$  and  $\beta$  such that  $x = \alpha a + \beta b$  in  $\mathfrak{R}^3$ . We have

$$\begin{aligned} x \wedge a &= \alpha a \wedge a + \beta b \wedge a = \beta b \wedge a, \\ x \wedge b &= \alpha a \wedge b + \beta b \wedge b = \alpha a \wedge b. \end{aligned}$$

Note that bivectors  $x \wedge a$  and  $b \wedge a$  define the same plane and both are proportional to the unit bivector in that plane, i.e. their ratio is real number (unit bivector divided by itself gives 1). Therefore we have

$$x = \frac{x \wedge b}{a \wedge b} a + \frac{x \wedge a}{b \wedge a} b.$$

Let's use the GAViewer to show it graphically:



Now let's look the quadratic equation:

$$x^2 + x + 1 = 0.$$

Show that  $x = -e^{\pm i\pi/3}$ ,  $i = e_1 e_2$ , is the solution. Can you find a solution for an arbitrary quadratic equation? Pay attention to the fact that the expression  $x^2 + x + 1$ , with the above solution, we can interpret as the operator which acting on some vector  $v$  gives zero. That means that we have sum of vector ( $v$ ), rotated vector ( $xv$ ) and twice rotated vector ( $x^2v$ ), three vectors that we can arrange in the triangle. About rotations and exponential form see below, here you can feel free to treat expressions like complex numbers with the imaginary unit  $i = e_1 e_2$  (i.e. you can use trigonometric form of the complex number). In the next chapter you will find an explanation for this approach.

## Geometric product of vectors in the trigonometric form

Let's look at the product in  $\mathfrak{R}^n$  (for other signatures see literature, main ideas are the same),

$$\begin{aligned} (a \wedge b)(a \wedge b) &= (ab - a \cdot b)(a \cdot b - ba) = \\ &= -ab^2a - (a \cdot b)^2 + a \cdot b(ab + ba) = \\ &= (a \cdot b)^2 - a^2b^2 = -a^2b^2 \sin^2 \theta, \end{aligned}$$

where we used  $(a \cdot b)^2 = a^2b^2 \cos^2 \theta$ . We see that in  $\mathfrak{R}^n$  the square of bivector is negative real number. Now we can define the module of a bivector as

$$|a \wedge b| = |a||b| \sin \theta.$$

We got a general expression for the square of bivector, so we see that the geometric product of two vectors can be written as

$$ab = |a||b|\hat{a}\hat{b} = |a||b|(\hat{a} \cdot \hat{b} + \hat{a} \wedge \hat{b}) = |a||b|(\cos \theta + \hat{B} \sin \theta), \quad \hat{B} = \frac{\hat{a} \wedge \hat{b}}{|\hat{a} \wedge \hat{b}|}, \quad \hat{B}^2 = -1,$$

or

$$ab = |a||b|e^{\hat{B}\theta}.$$

Notice that we have a similar formula for complex numbers, but the situation is quite different here: unit bivector  $\hat{B}$  is not just an „imaginary unit“, it defines the plane spanned by vectors  $a$  and  $b$ . This is a great advantage compared to ordinary complex numbers, it brings the clear geometric meaning to expressions. For example, formulation of quantum mechanics in geometric algebra uses real numbers, there is no need for  $\sqrt{-1}$ , and in every expression we can see the geometric meaning directly. This makes the new formulation more powerful, it provides new insights, which would otherwise be hidden or difficult to reach.

Here we have the opportunity to answer the question about multiplication tables. We have seen how the multiplication tables for scalar and cross product are almost complement. We know, geometric product of two vectors can be decomposed into symmetric and anti-symmetric parts, then we can find their modules, they have functions sine and cosine as factors and that gives us “united” multiplication table. Here it is (note that, for example,  $e_3 = -je_1e_2$ )

$$\begin{array}{c}
 \bullet \\
 e_1 \\
 e_2 \\
 e_3
 \end{array}
 \begin{array}{ccc}
 e_1 & e_2 & e_3 \\
 1 & 0 & 0 \\
 0 & 1 & 0 \\
 0 & 0 & 0
 \end{array}
 \oplus
 \begin{array}{c}
 \times \\
 e_1 \\
 e_2 \\
 e_3
 \end{array}
 \begin{array}{ccc}
 e_1 & e_2 & e_3 \\
 0 & e_3 & -e_2 \\
 -e_3 & 0 & e_1 \\
 e_2 & -e_1 & 0
 \end{array}
 \rightarrow
 \begin{array}{c}
 \text{GP} \\
 e_1 \\
 e_2 \\
 e_3
 \end{array}
 \begin{array}{ccc}
 e_1 & e_2 & e_3 \\
 1 & e_1e_2 & e_1e_3 \\
 -e_1e_2 & 1 & e_2e_3 \\
 -e_1e_3 & -e_2e_3 & 1
 \end{array}$$

and we can see that the new multiplication table has bivectors as non-diagonal elements ( $\oplus$  is just for fun). In fact, looking at those tables one can get nice insights about our 3D space and geometric algebra in general.

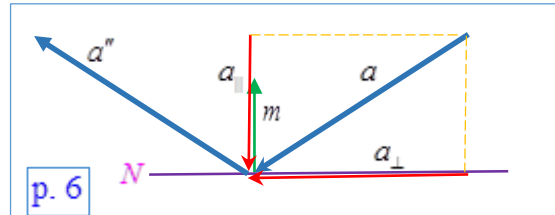
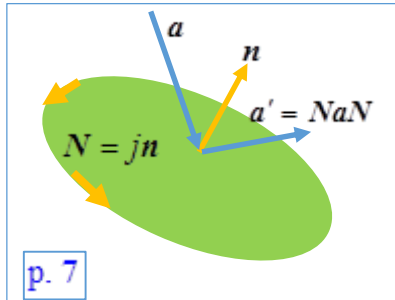
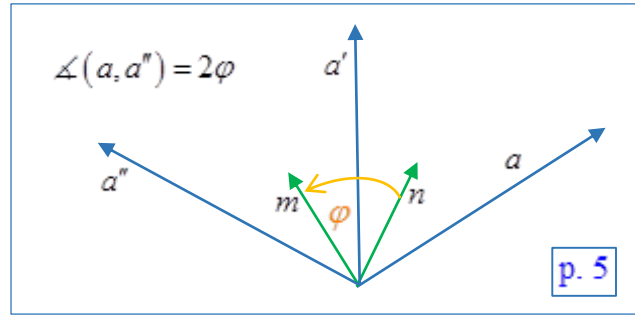
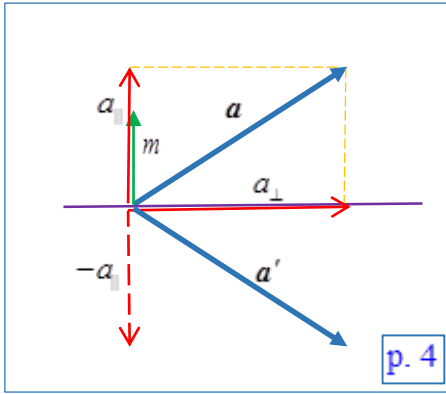
## Reflections, rotations, spinors, quaternions ...

The reader is now, perhaps, convinced that the geometric product is really natural and, actually, inevitable way to multiply vectors. One way or another, magic is still to come.

Consider now powerful formalism of geometric algebra applied to reflections and rotations (we are still in  $\mathfrak{R}^n$ , details of the other signatures can be found in the literature). For the vector  $a$  and the unit vector  $n$  in  $\mathfrak{R}^3$  (just to imagine things easier, generalization is straightforward) we can find projection (parallel to  $n$ ) and rejection (orthogonal to  $n$ ) of vector  $a$ , so,  $a = a_{\parallel} + a_{\perp}$ . Now we have

$$a' = -nan = -n(a_{\parallel} + a_{\perp})n = -(a_{\parallel} - a_{\perp})n = a_{\perp} - a_{\parallel},$$

which means that vector  $a$  is reflected on the plane orthogonal to  $n$  (generally a hyper plane, figure p. 4). We can omit the minus sign, then reflection is on the vector  $n$ . Recall, reflection does not change the grade of reflected object.



We should mention that in physics we are often interested in reflections on surfaces in 3D, so we can slightly adjust the pictures (p. 6, p. 7). We use the fact that  $j^2 = -1$ , so

$$a' = -nan = j^2 nan = jnajn = NaN,$$

where the unit bivector  $N$  defines the reflection plane.

What if we apply two consecutive reflections, using two unit vectors  $m$  and  $n$ ? There is a well-known theorem, which states that two consecutive reflections provide rotation. In figure p. 5 we see that after reflection on  $n$  we have  $a \rightarrow a'$ , then by reflection on  $m$  we have  $a' \rightarrow a''$ . If the angle between unit vectors  $m$  and  $n$  is  $\varphi$  then the rotation angle of vector  $a$  is  $2\varphi$ . Respectively, if we want to rotate the vector by angle  $\varphi$  we need to use the unit vectors the angle between which is equal to  $\varphi/2$ . We see how the half angle appears, so characteristic in the description of spin in quantum mechanics. Here we see that there is nothing "quantum" in half angle, it is simply a part of the geometry of our 3D space. This will be discussed later.

Now we can write an expression for the rotation as

$$a'' = m(nan)m = mnanm.$$

Another way to rotate the vector is to construct an operator which rotates and operates from the left. Thanks to the existence of an *inverse* of the vector this is easy to achieve:

$$a'' = (a''a^{-1})a \equiv Oa, \quad O = a''a^{-1}.$$

But the method that uses reflections is very general and elegant (rotates any element of the algebra), has a "sandwich" form, which is actually common and preferable in geometric algebra, especially for generalization to higher dimensions. Let's look more closely the term  $mnanm$ . Geometric products of two unit vectors consist generally of grades 0 and 2, so, it belongs to the even part of the algebra and makes subalgebra, which means that the product of any two of these elements will result in an element of the even part of algebra. We denote it as  $R = mn$  (rotor in text). Now we have

$$a'' = RaR^\dagger, \quad RR^\dagger = mnmm = 1 = R^\dagger R,$$

where  $R^\dagger = R^{-1}$  means *reverse* ( $mn \rightarrow nm$ ). For the rotation angle  $\varphi$  we need unit vectors with the angle  $\varphi/2$  between them. We have

$$mn = m \cdot n + m \wedge n,$$

where  $|m \wedge n| = \sin(\varphi/2)$ . Using the unit bivector  $\hat{B} \equiv n \wedge m / |n \wedge m|$  (note the order of vectors), we have

$$mn = m \cdot n + m \wedge n = \cos(\varphi/2) - \hat{B} \sin(\varphi/2) = \exp(-\hat{B} \varphi/2),$$

minus sign here is due to the convention (positive rotation is counter clockwise). In *Cl3* we can write unit bivector  $\hat{B}$  as  $j\mathbf{w}$ , where  $\mathbf{w}$  is the unit vector defining the axis of rotation. Rotor inverse is

$$R^\dagger = nm = \exp(\hat{B} \varphi/2),$$

so rotation is finally

$$a'' = RaR^\dagger = e^{-\frac{\varphi}{2}\hat{B}} a e^{\frac{\varphi}{2}\hat{B}}.$$

This is the general formula. If  $a$  commutes with  $\hat{B}$  rotation transformation has no effect on  $a$ . If  $a$  anti-commutes with  $\hat{B}$  we have operator form

$$a'' = e^{-\varphi \hat{B}} a,$$

For example, for  $\hat{B} = e_1 e_2$  vector  $e_3$  commutes with  $\hat{B}$ , while vector  $e_1$  anti-commutes.

Bivector  $\hat{B}$  defines the rotation plane and it is clear that vectors orthogonal to that plane are not changed by rotor. Notice, we do not need rotation matrices, Euler angles, or any other known mechanism. Once you define a unit bivector it will do all the necessary job. You can imagine it like a small spinning top that does exactly what we need. Notice that two different consecutive rotations make the rotation again ([show that](#)). This produces a group structure, but here we will not talk about it.

**Example.** Rotate vector  $e_1 + e_2 + e_3$  in the plane  $e_1 e_2$  by angle  $\varphi$ . We have

$$e^{-\frac{\varphi}{2}e_1 e_2} (e_1 + e_2 + e_3) e^{\frac{\varphi}{2}e_1 e_2},$$

so take advantage of the fact that the vector  $e_3$  commutes with bivector  $e_1 e_2$ , while  $e_1$  and  $e_2$  anti-commute:

$$\begin{aligned} e^{-\frac{\varphi}{2}e_1 e_2} (e_1 + e_2 + e_3) e^{\frac{\varphi}{2}e_1 e_2} &= e_3 e^{-\frac{\varphi}{2}e_1 e_2} e^{\frac{\varphi}{2}e_1 e_2} + e^{-\frac{\varphi}{2}e_1 e_2} e^{-\frac{\varphi}{2}e_1 e_2} (e_1 + e_2) = \\ &= e_3 + (\cos \varphi - e_1 e_2 \sin \varphi)(e_1 + e_2) = e_3 + (e_1 \cos \varphi + e_2 \sin \varphi) + (-e_1 \sin \varphi + e_2 \cos \varphi), \end{aligned}$$

and for the vectors in the plane  $e_1 e_2$  we recognize the rotation matrix

$$\begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix},$$

where the columns represent the images of the unit vectors. Rotation by angle  $-\varphi$  we get using the bivector  $e_2 e_1 = -e_1 e_2$ .

Consider rotation

$$e^{\frac{0.7\pi}{2}e_1e_2} a e^{\frac{0.7\pi}{2}e_1e_2}$$

and the corresponding rotation matrix

$$\begin{pmatrix} -0.588 & -0.809 \\ 0.809 & -0.588 \end{pmatrix}.$$

What can be said about the geometrical interpretation, that is, what you can conclude looking at the matrix? Try now to make a rotation matrix for an arbitrary plane. Try to repeat all in 4D. The easiness with which we perform rotations in geometric algebra is unseen before. There are no special cases, no vague matrices, just follow the simple application of the rotors to any multivector. Many prefer quaternions, but they do not have the geometric clarity. And they are limited to 3D! If only elegance and power of rotations was the result of using geometric algebra it would be worth of effort. But it gives us much, much more.

Notice how any rotor can be factored in small rotations

$$R = e^{I\phi/2} = \underbrace{e^{I\phi/2n} \dots e^{I\phi/2n}}_n,$$

which can be used in practice, for example, when interpolating.

Let's look at the rotation of vector  $e_2$  for small angle in the plane  $e_1e_2$  (p. 8, p.9). Recall the definition

$$e^x = \lim_{n \rightarrow \infty} \left( 1 + \frac{x}{n} \right)^n,$$

and let's construct the operator  $1 + \varepsilon e_1e_2$ ,  $\varepsilon$  is a small real number. Acting from the left we have

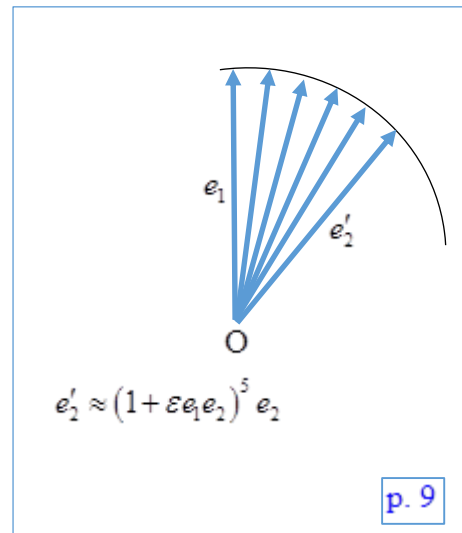
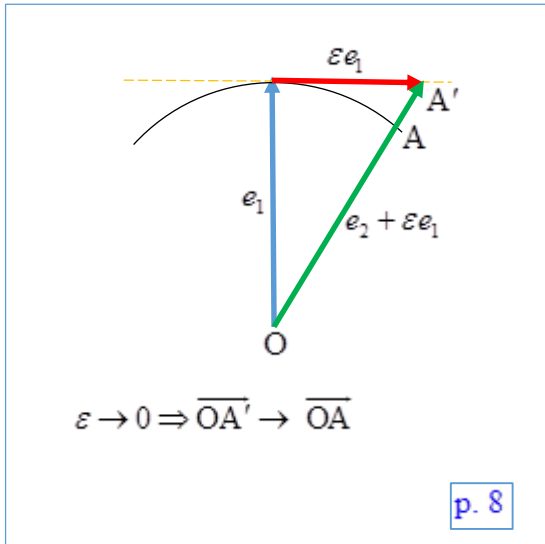
$$(1 + \varepsilon e_1e_2)e_2 = e_2 + \varepsilon e_1,$$

so we get an approximate small rotation of vector  $e_2$ . Note the sign of the number  $\varepsilon$ , for  $\varepsilon < 0$  we would have a counterclockwise rotation. Operator  $1 + \varepsilon e_1e_2$  rotates all vectors in the plane for the same angle, so, by successive application on  $e_2$  we get rotated  $e_2$  first, then rotated newly established vector, etc. This justifies the definition of exponential form of the rotor: each rotation is the composition of a large number of small successive rotations. Of course, all this is well defined for infinitely small rotations, and for bivector  $B$  we have

$$e^B = \lim_{n \rightarrow \infty} \left( 1 + \frac{B}{n} \right)^n.$$

Notice (or show it) that rotor will not change the bivector  $\hat{B}$ , for example, so it is an invariant of rotation. The fact that the blade can be invariant directly leads to the notion of the *proper blade* with real eigenvalues, which is a generalization of the common concept of eigenvectors and eigenvalues (see in *linear transformations*). Rotate the bivector  $e_1e_2$  in the plane spanned by vectors  $e_1$  and  $e_2$ . What do you notice?



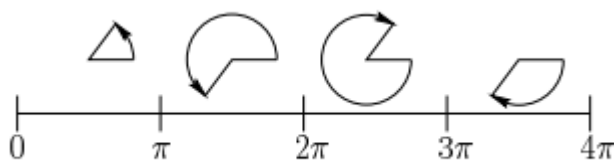


Rotations are linear (orthogonal) transformations that are usually described by matrices in linear algebra. To find the invariants of these transformations we study the results of action of matrices on vectors **only**. For matrix  $A$  (that represents a linear transformation) we seek for vectors  $x$  such that  $Ax = \lambda x$ , which provides solutions for the eigenvalues  $\lambda \in \mathbb{C}$ . Here we see that in geometric algebra we can find invariants with respect to a bivector (or any blade). Instead of the concept of eigenvector we can introduce the concept of the eigenblade (which includes eigenvectors). This allows reducing of the set of eigenvalues of transformation to the set of real numbers and giving a geometric meaning to the concept of eigenvalues. Linear transformations will be discussed later in the text.

Rotor  $-R$  has the same effect as the rotor  $R$ , but, the direction of rotation is not the same, for example, vector  $e_1$  can be rotated to  $-e_2$  clockwise by  $\pi/2$  or counter clockwise by  $3\pi/2$ , so we see that rotor clearly shows the direction of rotation ([try it with matrices!](#)). For example

$$-e^{I\varphi/2} = e^{-I\pi} e^{I\varphi/2} = e^{-I(2\pi-\varphi)/2},$$

minus disappears due to the "sandwich" form. For each rotation we have two possible rotors ([find what double cover of a group is](#)).



Note that, due to the half-angle, rotor

$$e^{\frac{\varphi}{2}\hat{B}} = \cos(\varphi/2) - \hat{B} \sin(\varphi/2)$$

has periodicity of  $4\pi$  instead of  $2\pi$ . Often for such objects we are using the name *unit spinor*. Geometric algebra is an ideal framework to study all unusual properties of rotations, but it would take a lot of space.

**Example:** Let's rotate (see [18]) some object in 3D around  $e_1$  by  $\pi/2$ , then around  $e_2$  by  $\pi/2$ , what we get? [Do that using matrices also](#).

$$e^{je_1\pi/4} e^{je_2\pi/4} = \frac{1}{\sqrt{2}}(1+je_1) \frac{1}{\sqrt{2}}(1+je_2) = \dots = \frac{1}{2} + \frac{1}{2}\sqrt{3}j \frac{e_1+e_2-e_3}{\sqrt{3}} = e^{jv\pi/3}, \quad v = \frac{e_1+e_2-e_3}{\sqrt{3}},$$

so we have rotation by  $2\pi/3$  around the vector  $v$ .

**Question:** What is the meaning of  $e^{i\pi} = -1$ ? In 2D for  $i = e_1e_2$  we have ( $v$  is a vector in the  $e_1e_2$  plane, you can choose  $v = e_1$  if you like)

$$e^{i\pi/2} v e^{-i\pi/2} = -v,$$

and using anti-commutativity

$$e^{i\pi/2} v e^{-i\pi/2} = e^{i\pi} v = -v,$$

then multiplying by  $v^{-1}$  on the right we get a clear meaning. Rotor  $e^{i\pi/2}$  transforms the vector  $v$  to the vector  $-v$ , i.e. rotates it by  $-\pi$  (sign is not important here). Of course, we also recognize the rotational properties of the imaginary unit in the complex plane (selected in advance), but bivector **defines** the rotation plane and we could write identical relations, without change, in any dimension, in any plane. In fact, bivector in the exponent of the rotor could depend on time, formulas are still valid, rotation plane changes with the bivector. Try to do that with the "square root of minus one".

Let's say you want to find the rotor in 3D that will transform the orthonormal coordinate basis  $e_i$  to orthonormal coordinate basis  $f_i$ . We need a rotor with the property  $f_i = R e_i R^\dagger$ . Let's define  $R = \alpha - \beta \hat{B}$ , where  $\hat{B}$  is a unit bivector, then  $R^\dagger = \alpha + \beta \hat{B}$ . Notice two simple and useful relations in 3D

$$\sum_i e_i^2 = 3 \quad \text{and} \quad \sum_i e_i \hat{B} e_i = -\hat{B},$$

(prove it). It follows

$$\sum_i e_i R^\dagger e_i = 3\alpha - \beta \hat{B} = 4\alpha - R^\dagger,$$

and

$$\sum_i f_i e_i = \sum_i R e_i R^\dagger e_i = R(4\alpha - R^\dagger) = 4\alpha R - 1,$$

so

$$R = \frac{1 + \sum_i f_i e_i}{\left| 1 + \sum_i f_i e_i \right|} = \frac{A}{\sqrt{AA}}, \quad A = 1 + \sum_i f_i e_i.$$

Rotation by  $\pi$  can be treated as a special case. Show that the rotor can be expressed using Euler angles as

$$e^{-e_{12}\phi/2} e^{-e_{23}\theta/2} e^{-e_{12}\psi/2}.$$

Let's comment the historical role of Hamilton, who in the 19th century found a similar mechanism for rotations: *quaternions*. There is a connection between quaternions and formalism described here, namely, quaternions can be easily related to the unit bivectors in  $C/3$ . However, quaternions are like extended complex numbers, they do not have a clear geometrical interpretation. Moreover, they exist only in 3D. (Hamilton wanted to give a geometric meaning to unit quaternions, and was trying to treat them as vectors, which did not give the expected results, but unit vectors

$i, j, k$  inherited their names due to these attempts.) The formalism of geometric algebra is valid for any dimension. Every calculation in which we use quaternions can be easily translated into the language of geometric algebra, while the reverse is not true. However, quaternions are still successfully used in the applications for calculating rotation, for example, in computers of military and space vehicles, as in robotics. If you implement the geometric algebra on computer quaternions are not needed.

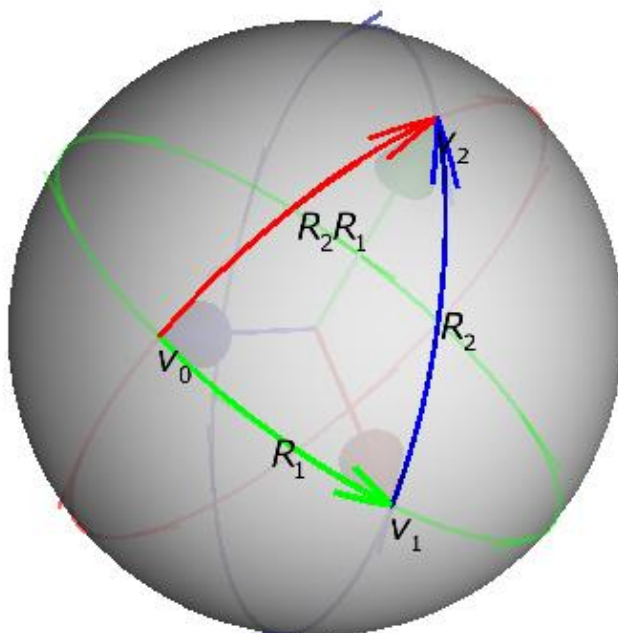
Unit quaternions have the property  $ijk = -1$  and the square of each of them is  $-1$ . It was enough to come up with object that squares to  $-1$  and anti-commute to describe the rotations in 3D successfully. [The reader can check that replacements  \$i \rightarrow -e\_{23}, j \rightarrow e\_{13}, k \rightarrow -e\_{12}\$  generate quaternion multiplication table.](#)

Certainly it is good to understand that bivector  $-e_{12} = e_2e_1$  has a very clear geometrical interpretation, while unit quaternion  $k$  (like imaginary unit or matrix) has not. Unfortunately, the concept of geometric objects like bivector is often strange to traditionally oriented people.

Once we know how to rotate vectors we can rotate any element of geometric algebra. Note especially nice feature of geometric algebra: objects that perform transformations (“operators”) are also elements of the algebra. Let's look at the rotation of versor

$$RabcR^\dagger = RaR^\dagger RbR^\dagger RcR^\dagger = (RaR^\dagger)(RbR^\dagger)(RcR^\dagger),$$

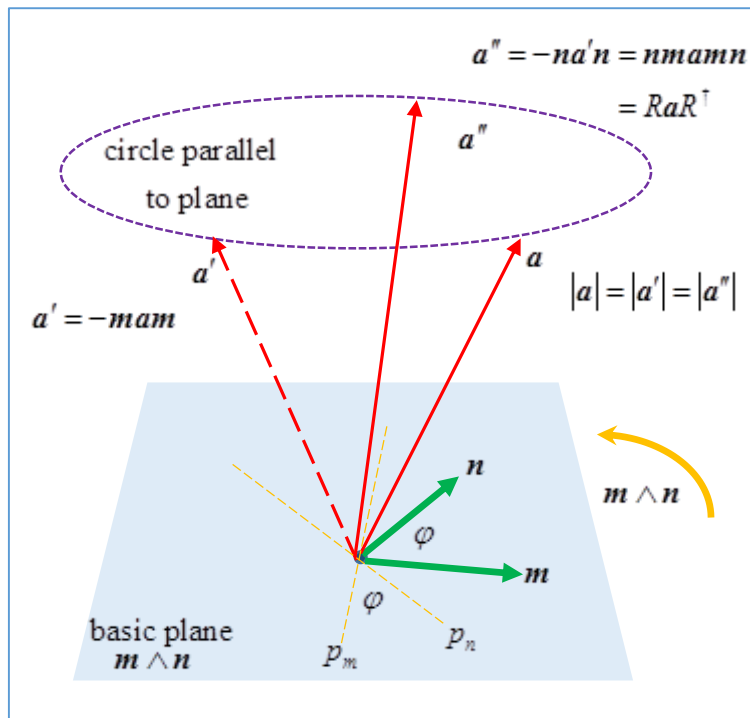
which clearly shows how the rotation of versor can be reduced to the rotation of the individual vectors and vice versa. Every multivector is a linear combination of elements of Clifford basis which elements are simple blades, so, they are versors. We see that our last statement is always true, due to linearity. [The reader is advised to do rotations of different objects in Cl3. Find on Internet the term „gimbal lock“.](#)



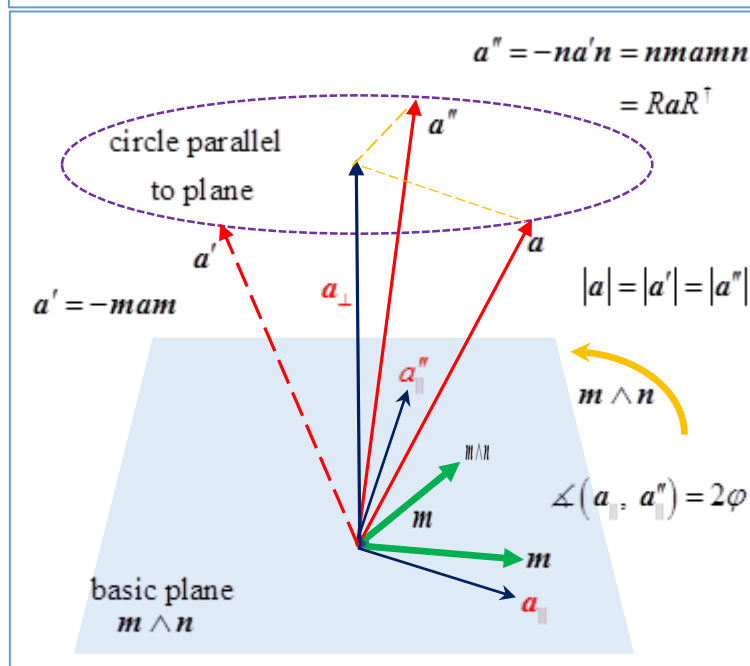
It is interesting to look at the unit sphere in 3D and unit vectors starting at the origin of the sphere. Each rotation of the unit vector defines the arc on some main circle. Such arches, if we take into account their orientation, can become a kind of vectors on the sphere, and composition of two rotations can be reduced to a (non-commutative) addition of such vectors. See [4].

If we take an arbitrary element of even part of algebra (for example in 3D), not only the rotors, with the rotation we get the additional effect: dilatation, which is exactly the property of spinors. Spinors are closely associated with the even part of algebra. Geometric algebra hides within itself an unusual amount of mathematics which is branched out in different disciplines. It's amazing how the redefinition of the multiplication of vectors integrates into

a single formalism many different branches of mathematics. Spinors, tensors, Lie groups and algebras, various theorems of integral and differential calculus are united, ..., theory of relativity (special and general), quantum mechanics, theory of quantum information, ... one almost cannot believe. Many complex results of physical theories here become simple and get a new meaning. Maxwell's equations are reduced to three letters, with the possibility of inverting the derivation operator over the Green functions, hard problems in electromagnetism become solvable (see [2]), the Kepler problem is elegantly reduced to the problem of the harmonic oscillator, Dirac theory in  $C/3$  or the minimal standard model in  $C/7$  are nicely formulated ([34]), not to list further. Geometric algebra has a good chance to become mathematics of future. Unfortunately, it is difficult to break through the traditional university (and especially high school) programs.



One can study following pictures to better understand rotations.



- $m \wedge n$  defines the plane, direction of rotation and the rotation angle
- $a_{\perp}$  is invariant to rotation, only  $a_{\parallel}$  is rotated by  $2\phi$
- The same picture is valid in any dimension (in dimensions higher than 3 there is a subspace invariant to rotation).
- It is easy to obtain any composition of rotations in the same manner.
- Geometric product of vectors gives us the possibility to maintain rotations easy.

## Contractions

We defined the inner product that for vectors coincides with the usual scalar multiplication of vectors. In general, in geometric algebra we can define various products that lower grades of elements (outer product raises it). It appears that the best choice is *left contraction*. For the vectors it is just as the inner product, but generally allows avoiding various special cases, such as, for example, the inner product of vector with the real number. Here we will mention just a few properties of left contraction, see [19] for more details. The idea is that for any two blades (including real numbers) we define a „scalar“ multiplication that will generally reduce the grade of the blade that is on the right in the product:

$$\text{grade}(A \rfloor B) = \text{grade}(B) - \text{grade}(A),$$

whence immediately follows that the left contraction is zero if  $\text{grade}(B) < \text{grade}(A)$ . For vectors we have

$$a \rfloor b \equiv a \cdot b,$$

and generally for blades we have

$$(A \wedge B) \rfloor C = A \rfloor (B \rfloor C).$$

Useful relation for vectors is

$$x \rfloor (a \wedge b) = (x \cdot a)b - (x \cdot b)a,$$

while in general we can write for any multivector

$$A \rfloor B = \sum_{k,l} \langle \langle A \rangle_k \langle B \rangle_l \rangle_{l-k},$$

where we have geometric product between homogeneous (of the same grade) parts of the multivectors. The left contraction for blades  $A$  and  $B$  ( $A \rfloor B$ ) is the subspace in  $B$  orthogonal to  $A$ . If vector  $x$  is orthogonal to all vectors from the subspace defined by the blade  $A$  then  $x \rfloor A = 0$ . The left contraction can help us to define the angle between subspaces. Because of the generality, clear geometric interpretation and benefits for use on computers (there are no exceptions, so *if* loops are not needed) left contraction should be used instead of "ordinary" inner product. You can also define the right contraction, however, due to the properties of duality, it is not really necessary.

## Commutators and orthogonal transformations

Let's define the *commutator* as a new kind of product of multivectors (here we use the character  $\otimes$  to avoid possible confusion with the cross product)

$$A \otimes B \equiv (AB - BA) / 2.$$

This product is not associative, i.e.  $(A \otimes B) \otimes C = A \otimes (B \otimes C)$  is not valid, but we have Jacobi identity

$$(A \otimes B) \otimes C + (C \otimes A) \otimes B + (B \otimes C) \otimes A = 0.$$

We have (prove it) general formulas ( $A$  is bivector, not necessarily a blade,  $X$  is multivector,  $\alpha$  is real scalar,  $x$  is a vector)

$$\alpha X = \alpha \wedge X$$

$$xX = x \lrcorner X + x \wedge X$$

$$AX = A \lrcorner X + A \wedge X + A \otimes X.$$

Here we are particularly interested in commutators with bivector as one of the factors. Namely, commutators with the bivector keep the grade of multivector (if they do not commute with it):

$$\text{grade}(B) = 2 \Rightarrow \text{grade}(X \otimes B) = \text{grade}(X), \quad X \otimes B \neq 0.$$

Instead of proving it let us look at the examples. If bivector is  $B = e_1 e_2$ , then vector  $e_3$  commutes with  $B$ , but for the vector  $e_1$  (grade 1) we have

$$B \otimes e_1 = (e_1 e_2 e_1 - e_1 e_1 e_2) / 2 = -e_2,$$

grade 1 again. Let us take the series expansion

$$e^{-B/2} X e^{B/2} = X + X \otimes B + (X \otimes B) \otimes B / 2 + ((X \otimes B) \otimes B) \otimes B / 3! + \dots,$$

so if we take a small bivector of the form  $\varepsilon \hat{B}$ ,  $\hat{B}^2 = -1$ , we see that we can keep only two terms

$$e^{-\varepsilon \hat{B}/2} X e^{\varepsilon \hat{B}/2} \approx X + \varepsilon X \otimes \hat{B}.$$

Preservation of grades is important here, because we want to, after the transformation, have a geometric object of the same type. The last transformation we see as an orthogonal transformation which will slightly change the initial multivector. Here we must mention that we look for the orthogonal transformation connected to the identity transformation, which means that they can be implemented in the small steps. Reflections do not meet this requirement, we cannot perform "a little of reflection". Such small transformations are called perturbations. Therefore, we can conclude that the small perturbations of elements of geometric algebra are to be performed by rotors.

Note that orthogonal transformations do not permit to just add a small vector  $\delta x$  to the vector  $x$ , orthogonal transformations must keep the vector length. So we must have  $x \cdot \delta x = 0$ . Generally, such an element ( $\delta x$ ) of geometric algebra has the form  $\delta x = x \lrcorner \delta B$ , where  $\delta B$  is a small bivector. We can show it

$$x \cdot (x \lrcorner \delta B) = x \lrcorner (x \lrcorner \delta B) = (x \wedge x) \lrcorner \delta B = 0.$$

It follows now that

$$\delta x = x \lrcorner \delta B = (x \delta B - \delta B x) / 2 = x \otimes \delta B,$$

and we have the desired shape in the form of a commutator. It may seem that the restriction on the rotations is too strict, it looks as if we cannot do a simple translation of a vector. However, here it just means that we need to find a way to describe translations by rotations. It is possible in geometric algebra, but we will not show it here (see [19]).

Here we will stop, but noting that a small portion of formalism just shown leads to Lie groups and algebras. It can be shown that every finite Lie group or algebra can be directly described in the context of geometric algebra. The infinite case is not yet absolutely clear, but it would be unusual for a result to be different. Anyway, another nice part of mathematics fits perfectly into the geometric algebra. Anyone who seriously studies the geometric algebra was initially probably astonished by the fact that different branches of mathematics show a new light in the language of geometric vector

multiplication, but with time one gets used to it and does not expect exceptions. One cannot help wonder what our science would look like that the power of this magical language of mathematics was understood and accepted a century ago. And it was all at our fingertips.

## Complex numbers

Let's specify the vector in 2D  $\mathbf{r} = xe_1 + ye_2$ . Let's use the existence of the inverse and write

$$\mathbf{r} = e_1(x + ye_1e_2) = e_1(x + yi), \quad i = e_1e_2,$$

and we can see that we get a complex number  $x + yi$ , but with non-commutative "imaginary unit". The first thing to complain about is: „Yes, but your imaginary unit is not commutative, and quantum mechanics cannot be formulated without imaginary unit ...“. Immediately you see that the "critic" commented something he knows almost nothing about, because, first, quantum mechanics works nicely (and even better) with real numbers, without the imaginary unit, but one should learn geometric algebra, then learn the formulation of quantum mechanics in the language of geometric algebra ... Not only that we can without using imaginary unit, but many relations obtain a clear geometric meaning and thus provide a new insights into the theory in the language of geometric algebra. And second, non-commutativity of our bivector  $i = e_1e_2$  actually becomes an advantage, it enriches the theory of complex numbers and, as we are repeating until you get bored, gives it a clear geometric meaning. For our complex number  $z = e_1\mathbf{r}$  we have (due to anti-commutativity)  $z^* = \mathbf{r}e_1$ , so

$$\begin{aligned} zz^* &= e_1\mathbf{r}\mathbf{r}e_1 = r^2e_1e_1 = r^2 = x^2 + y^2, \quad \text{or} \\ z + z^* &= e_1\mathbf{r} + \mathbf{r}e_1 = 2e_1 \cdot \mathbf{r} = 2x, \\ z - z^* &= e_1\mathbf{r} - \mathbf{r}e_1 = 2e_1 \wedge \mathbf{r} = 2yi, \end{aligned}$$

etc. We see that the operations on complex numbers are, without any problem, confined to the operations in geometric algebra. Define derivative operator in 2D

$$\nabla \equiv e_1 \frac{\partial}{\partial x} + e_2 \frac{\partial}{\partial y},$$

and introduce a complex field  $\psi = u + iv$ ,  $i = e_1e_2$ . Simple calculation shows ([do it](#)) that derivation of the field is

$$\nabla \psi = e_1 \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) + e_2 \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right).$$

So, if we want the derivative to be identically zero (*analyticity*), *Cauchy-Riemann* equations immediately follow. Note how anti-commutativity of unit vectors gives correct signs. So, analyticity condition in geometric algebra has a simple form  $\nabla \psi = 0$ , and we can immediately generalize it to higher dimensions. And yes, this is just a right moment to stop and think. Let the advocates of the traditional approach do all that using just commutative imaginary unit. Actually, it's amazing how this old, good imaginary unit has made a lot of work, given the modest possibilities! But, it is time to rest a little, let bivectors, pseudoscalars ... do the job. It should be noted, to make no confusion, the choice of the plane  $e_1e_2$  is unimportant here. We can take the bivector like  $(e_1 + e_2)(e_1 - e_2)$ , normalize it

and we get the new „imaginary unit“, but in the new plane. We can do that in 4D also, and take, for example  $i = e_3 e_4$ , all formulas will be valid. The plane  $e_1 e_2$  is just one of infinity of them, but geometrical relationships in each of them are the same. We can solve the problem in the plane  $e_1 e_2$ , and then rotate all to the plane we want, we have powerful rotors in the geometric algebra. And when it is said „powerful“ then it literally means that we do not have to be experts in the matrix calculations, here something like that an advanced high school student can make. We can rotate any object, not only the vectors. Linear algebra is the mathematics of vectors and operators, geometric algebra is mathematics of subspaces and operations on them. Anyone who uses mathematics should understand how important it is.

We will show here that one can get solutions of the equation  $\nabla \psi = 0$  by using series in  $z$ . Notice first an easy relation for vectors

$$abc + bac = (ab + ba)c = 2a \cdot bc,$$

wherein the inner product has priority. Operator  $\nabla$  is acting as vector (expressions like  $\mathbf{r}\nabla$  are possible, but then we usually write  $\dot{\mathbf{r}}\nabla$ , which does not mean the time derivative, but indicates the element the derivation operator acts on, and gives desired order in products of unit vectors), so take advantage of the previous relation (a very useful calculation)

$$\nabla z = \nabla(e_1 \mathbf{r}) = 2e_1 \cdot \nabla \mathbf{r} - e_1 \nabla \mathbf{r} = 2e_1 - 2e_1 = 0.$$

Now we have

$$\nabla(z - z_0)^n = n \nabla(e_1 \mathbf{r} - z_0)(z - z_0)^{n-1} = 0,$$

so, Taylor expansion about  $z_0$  automatically gives the analytical function. Again, in any plane, in any dimension. It is not only that geometric algebra contains all the theory of functions of complex variables (including integral theorems, as a special case of the fundamental theorem of integral calculus in geometric algebra), but also extends and generalizes it to any dimension. Is not this a miracle? And we were just wondering how to multiply vectors. If you still have a desire to pronounce the sentence „Yes, but ...“, please, go back to the beginning of the text and see how all this began. Time of geometric algebra is yet to come, hopefully. The children of Adam and Eve will again speak one language, we will have one language of mathematics. Dark Ages of matrices and coordinates will disappear and will be replaced by the time of synergy of algebra and intuitively clear geometry. Students will learn much faster and be superior to today's "experts". And when we learn computers to "think" in this magical language (imagine a computer that knows how to perform operations on subspaces) children will be able to play with geometric shapes as now play a car racing or other computer games. The properties of triangles, circles, spheres and other shapes we will learn through play, on computers, interactive. Language of geometric algebra is so powerful that it can "automate" even the process of proving the theorems (there's still a lot of work to do, but the possibilities are there). We have reasons to think that geometric algebra is not just "another formalism", but it offers the possibility of deep questioning the very concept of number.

## Spinors

Let's look at the elements of algebra which in the "sandwich" forms do not change the grade of vector (i.e. vector transform to vector). Among them are the transformations which rotate and



dilate vectors, we usually call them *spinors*. Let's look at multivectors  $\psi$  with the property ( $\mathbf{v}$  is a vector)

$$\psi \mathbf{v} \psi^\dagger = \rho R \mathbf{v} R^\dagger, \quad \rho \in \mathfrak{R}, \quad R R^\dagger = 1,$$

which is precisely the rotation of the vector with dilatation. If we define  $U \equiv R^\dagger \psi$ , previous relation becomes

$$U \mathbf{v} U^\dagger = \rho \mathbf{v},$$

and we will find the element  $U$ . [Show that pseudoscalars of odd dimensions commute and of even dimensions anti-commute with vectors. Other grades do not possess such a general property \(real scalars commute\).](#) We see that element  $U$  induces a pure dilation of the vector  $\mathbf{v}$  and that is possible if it commutes or anti-commutes with  $\mathbf{v}$ , so it follows that element  $U$  is, generally, real scalar, or pseudoscalar, or combination of both:  $U = \lambda_1 + \lambda_2 I$ . Now, using definition of  $U$ , we get

$$U = \lambda_1^2 \mathbf{v} + \lambda_1 \lambda_2 (I \mathbf{v} + \mathbf{v} I^\dagger) + \lambda_2^2 I \mathbf{v} I^\dagger = \rho \mathbf{v}.$$

In  $Cl_3$  ( $p=3, q=0$ ) pseudoscalar  $I = j$  commutes with all elements of the algebra and reverse is  $I^\dagger = -j$ , middle term disappears, so we have

$$\lambda_1^2 + \lambda_2^2 = \rho \Rightarrow \psi = R(\lambda_1 + j\lambda_2),$$

and it is easy to check

$$\psi \mathbf{v} \psi^\dagger = R(\lambda_1 + j\lambda_2) \mathbf{v} (\lambda_1 - j\lambda_2) R^\dagger = (\lambda_1 + j\lambda_2)(\lambda_1 - j\lambda_2) R \mathbf{v} R^\dagger = (\lambda_1^2 + \lambda_2^2) R \mathbf{v} R^\dagger = \rho R \mathbf{v} R^\dagger.$$

In general, note that

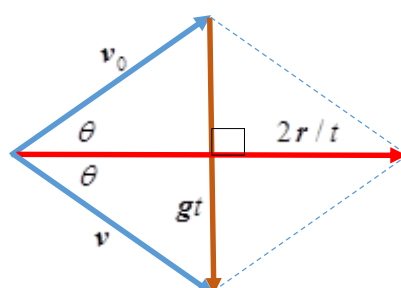
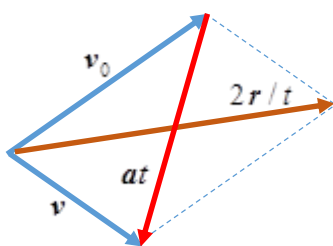
$$\mathbf{v} I^\dagger = (-1)^{n-1} I^\dagger \mathbf{v}, \quad \mathbf{v} I^\dagger = (-1)^{(n-1)(n-2)/2} I \mathbf{v}, \quad I I^\dagger = (-1)^q,$$

(prove it, at least for the signatures (3, 0) and (1, 3)) and we can find solutions (find them) dependent on the parity of number  $(n-1)(n-2)/2$ .

Spinors in geometric algebra, as elsewhere, can be defined by (left) ideals of the algebra, but here we will not deal with it ([7]).

## A little of "ordinary" physics

Let's see how we can solve the kinematic problem in its generality using simple calculations and intuitively clear. Consider the problem of accelerated motion with a constant acceleration.



The problem is easily reduced to relations

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{a}t, \quad \mathbf{v} + \mathbf{v}_0 = 2\mathbf{r} / t,$$

wherein the second relation defines the average speed vector  $\bar{\mathbf{v}} = \mathbf{r} / t$ , so we have

$$(\mathbf{v} + \mathbf{v}_0)(\mathbf{v} - \mathbf{v}_0) = 2\mathbf{r}\mathbf{a} \Rightarrow$$

$$v^2 - v_0^2 + \mathbf{v}_0\mathbf{v} - \mathbf{v}\mathbf{v}_0 = v^2 - v_0^2 + 2\mathbf{v}_0 \wedge \mathbf{v} = 2(\mathbf{r} \cdot \mathbf{a} + \mathbf{r} \wedge \mathbf{a}),$$

where by comparison of the scalar and bivector parts we get

$$v^2 - v_0^2 = 2\mathbf{r} \cdot \mathbf{a},$$

$$\mathbf{v}_0 \wedge \mathbf{v} = \mathbf{r} \wedge \mathbf{a},$$

i.e. the law of conservation of energy and the surface of the parallelogram theorem. For the projectile motion problem, ( $\mathbf{a} = \mathbf{g}$ ) we have (figure on the right)

$$\mathbf{r} \cdot \mathbf{g} = 0 \Rightarrow v^2 = v_0^2 \Rightarrow |\mathbf{v}_0 \wedge \mathbf{v}| = v_0^2 \sin(2\theta) = |\mathbf{r} \wedge \mathbf{g}| = rg \Rightarrow$$

$$r = \frac{v_0^2}{g} \sin(2\theta),$$

and this is the known relation for the range. Notice how properties of geometric product lead to simple manipulations. Another example is the Kepler problem. Immediately after setting the problem, after a few lines, we obtain non-trivial conclusions that textbooks usually put as hard part at the end. Examples here are to show how to obtain solutions without coordinate systems and coordinates. Unfortunately, research shows ([21]) that many physics students see vectors mainly as a series of numbers (coordinates) and it is a sad reflection of the current education systems, regardless of place on the planet. Such "attitude" does not provide a good start for a serious professionals. The connection of linear algebra and geometry is usually quite neglected. With the geometric product algebra and geometry go hand in hand. Instead of treating the vectors as the key elements of algebra we have a whole range of objects that are not vectors and have a very clear geometric meaning. We are calculating with the subspaces! And in any dimension. Something like that is impossible to achieve just manipulating by coordinates. Emphasize this, impossible! Russian physicist Landau, famous for math skills, ended up in Stalin's prison. After his release from prison, he said that his prison was welcome, because he had learned to run tensor calculus "in the head". Physicists of the future will be more skilled than Landau, they will use linear transformations in geometric algebra instead of tensor calculus. They will calculate faster, regardless of the dimension of space, without using coordinates and with a clear geometric interpretation at every step. Landau was also famous by the method of accepting students. He would said to the young candidate: "Here, solve the integral." Many have failed. In geometric algebra, there is a theorem (fundamental theorem) about integration that combines all known integral theorems used in physics, including complex area. Just imagine, Landau would be really surprised! He was a typical representative of the mathematics of the 20th century, although in his time already existed the new mathematics. It existed, but almost completely neglected and forgotten. Part of the price paid (and we still pay it) is a rediscovery of what is neglected and forgotten. Pauli discovered its matrices – we have continued to use matrices. They say that the geometric algebra is non-commutative and that this discourages people. What about matrices? Not only that they are non-commutative, they are unintuitive. Then Dirac discovered his matrices, ideal for geometric algebra. Again, we continued with matrices. And many authors, on various occasions, rediscovered spinors, even giving them different names. Then we decided to make fast spacecrafts equipped with computers and found that we have problems with matrices. Then we started to use quaternions and improved things in some

extent. We can find a number of other indications, and, after all, it is obvious that many of the problems simply disappear when geometric product is introduced instead of products of Gibbs. In spite of everything, one of the great authors in the field of geometric algebra, Garret Sobczyk, wrote in an e-mail:

*„I am surprised that after 45 years working in this area, it is still not generally recognized in the scientific community. But I think that you are right that it deserves general recognition ... Too bad Clifford died so young, or maybe things would be different now.“*

## Words and sentences

Let's look, just for illustration, how „words“ in geometric algebra can have a geometric content. For example, „word“ **abba**.

$$\begin{aligned} abba &= a^2b^2 = ab(ab)^\dagger = \\ (a \cdot b + a \wedge b)(a \cdot b - a \wedge b) &= (a \cdot b)^2 - (a \wedge b)^2 = \\ (a \cdot b)^2 + |a \wedge b|^2 &= a^2b^2 (\cos^2 \theta + \sin^2 \theta), \end{aligned}$$

and we have well known trigonometric identity. This is, of course, just a game, but in geometric algebra it is important to develop intuition about the geometric content written in expressions. Due to the properties of geometric product structure of expressions is quickly manifested, as for relations between the subspaces, to be an element of subspace, orthogonality, to be parallel, etc.

Let's compare exposed to the matrix approach. We have seen that in 3D we can represent vectors by Pauli matrices. Try to imagine that we are not aware of it, but we know about the Pauli matrices (from quantum mechanics). We could write the word **abba** in the language of matrices, we could resolve matrices in symmetric and anti-symmetric parts (it is custom), but try to derive the sine and cosine of the angle and the basic trigonometric identity. If you succeed (it is possible), how would you interpret that angle? And more important, how to even come up with the idea to look for an angle, just looking at matrices? It is hard, for sure, but with vectors it is natural and straightforward. That is the main idea: language of matrices hides important geometric content. True, physicists know that Pauli matrices have to do something with the orientation of spin, but generally, problem of geometric interpretation still remains. Here is one more example. We have unit vectors  $m = (e_1 + e_2)/\sqrt{2}$  and  $n = (e_2 + e_3)/\sqrt{2}$  in 3D. It is not difficult to imagine or draw them, there is the plane spanned and bivector  $m \wedge n$  in it (bivector defines the plane). Image again that we are using Pauli matrices, but, as before, without awareness that they represent vectors in 3D (we cannot even know it if we do not accept the geometric product of vectors). Someone could really investigate a linear combinations of the Pauli matrices, even come to the idea to look at anti-symmetric part of products of matrices, something like  $(\hat{\sigma}_m \hat{\sigma}_n - \hat{\sigma}_n \hat{\sigma}_m)/2$ , where  $\hat{\sigma}_m = (\hat{\sigma}_1 + \hat{\sigma}_2)/\sqrt{2}$ , and  $\hat{\sigma}_n = (\hat{\sigma}_2 + \hat{\sigma}_3)/\sqrt{2}$ . We should now calculate it, so, we can compare needed calculation with matrices and simple calculation of the outer product (in fact, there is no need to calculate the outer product, we have the geometric picture without effort). Whatever, bivector is

$$m \wedge n = (e_1 + e_2) \wedge (e_2 + e_3) / 2 = (e_1 e_2 + e_1 e_3 + e_2 e_3) / 2.$$

Fortunately, computer can help here with matrices (you see the problem?), so, anti-symmetric part of the matrix product is

$$\begin{pmatrix} i & -1+i \\ 1+i & -i \end{pmatrix} / 2.$$

Now, how, without connecting with vectors in 3D, to interpret this matrix as the plane? Or find the angle between - what? It is easy to express formulas from  $Cl_3$  via Pauli matrices, but matrix form to vectors – it could be tricky, especially for the blades of higher grades, or general multivectors. Language of matrices blurs the geometric content! In quantum mechanics with Pauli matrices we need the imaginary unit, and people say that imaginary unit is necessary to formulate the theories of subatomic world. This often leads to a philosophical debates and questions about the „real nature“ of the world we live in. In the language of geometric algebra imaginary unit is absolutely not necessary, quantum mechanics can be beautifully and elegantly formulated using real numbers, with the clear geometric interpretation. Besides the real numbers, complex numbers and quaternions could be of interest in quantum mechanics, but it is clear now, they all are natural part of  $Cl_3$ , as we discussed earlier. In the article [1], author comments: „... instead of being distinct alternatives, real, complex and quaternionic quantum mechanics are three aspects of a single unified structure.“ There are useful remarks on the Frobenius–Schur indicator in this article. True, there is no geometric algebra in the cited article, although there is term, “division algebra” in the title. Rather than comment, here is the sentence from [28], one that should be known to all mathematician and physicists. Unfortunately, it is not.

“Geometric algebra is, in fact, the largest possible associative division algebra that integrates all algebraic systems (algebra of complex numbers, vector algebra, matrix algebra, quaternion algebra, etc.) into a coherent mathematical language that augments the powerful geometric intuition of the human mind with the precision of an algebraic system.” To be honest, division algebra or not – it is unimportant. It unifies and it works!

## Linear transformations

Often we are interested in the transformations of the elements of algebra (eg, vectors, bivectors, ...) to other elements in the same space. Among them are certainly the most interesting linear transformations. Let's look linear transformation  $F$  which translates vectors into vectors, with property

$$F(\alpha a + \beta b) = \alpha F(a) + \beta F(b), \quad \alpha, \beta \in \mathfrak{R}.$$

We can imagine that the result of such a transformation is, for example, the rotation of the vector with the dilatation. For such a simple picture we do not need vector components. Another example may be a rotation:

$$F(a) = R(a) \equiv RaR^\dagger.$$

We have seen that the effect of rotation of the blade is the same as action of the rotation on each vector in the blade, so we require that all of our linear transformations have that property, which means

$$F(a \wedge b) = F(a) \wedge F(b).$$

Considering the linear transformation that gives back a vector we see that the form of outer product is preserved. Such transformation have a special name: *outermorphism*. The action of two successive transformations can be written as  $F(G(a)) \equiv FG(a)$ , which is handy for manipulating expressions.

If for linear transformation  $F:V \rightarrow W$  there is an adequate linear transformation  $\bar{F}:W \rightarrow V$ , we'll call it *transposed transformation (adjoint)*. Here we will restrict to transformations  $F:V \rightarrow V$ . We say that they are transposed because we can always find a matrix representation in some basis and see that matrix of  $\bar{F}$  is just transposed matrix of  $F$  (see [22]). Here is an implicit definition of adjoint

$$a \cdot \bar{F}(b) = F(a) \cdot b,$$

for any two vectors  $a$  and  $b$ . Define now reciprocal base vectors  $e^i$  with the property

$$e^i \cdot e_j = \delta_{ij}.$$

Here we are using orthonormal bases of positive signature, so

$$e^i = e_j \Rightarrow e^i \cdot e_j = e^i e_j = \delta_{ij},$$

and definition is motivated by two facts: first, we want to use the Einstein summation convention

$$e^i e_i \equiv \sum_{i=1}^n e^i e_i,$$

and, second, we want the ability to generalize easily. Explicit form of the transposed transformation can be found using the orthonormal basis

$$e_i \cdot \bar{F}(a) = F(e_i) \cdot a,$$

so we have

$$\bar{F}(a) = e^i a \cdot F(e_i),$$

where summation is understood and the inner product has priority. Designation  $\bar{F}$  is not common,  $F^T$  or  $F^\dagger$  is, but sometimes we use  $\underline{F}$  for linear transformations, so nice symmetry in expressions could occur if we use  $\bar{F}$ . Furthermore,  $\bar{F}(a)$  is not a matrix or tensor, so designation highlights the difference. There cannot be confusion with Clifford conjugation in the text, we are consistently using format *italic* for multivectors. For transposed transformation of the "product" of transformations we have

$$\overline{FG}(a) = \bar{G}\bar{F}(a),$$

(see literature). Transformations with the property  $\bar{F} = F$  are *symmetric*. Important symmetric transformations are  $\bar{F}F$  and  $F\bar{F}$  (show that).

Let  $I$  to be the unit pseudoscalar. Determinant of linear transformation is defined as

$$F(I) \equiv I \det F, \quad \det F \in \mathfrak{R}.$$

This definition is in full compliance with the usual definition. Notice that this relation looks like eigenvalue relation. In fact, that is true, pseudoscalar is invariant (eigenblade) and determinant is an eigenvalue (real!). An example is 3D rotation

$$R(j) = RjR^\dagger = jRR^\dagger \Rightarrow \det R = RR^\dagger = 1, \quad j = e_{123},$$

what we expect for rotors (for rotation matrices, too). Again, notice the power of formalism: without components, without matrices, by simple manipulation, we get an important result. Pseudoscalar

represents an oriented volume, so linear transformation of pseudoscalar is simply reduced to his multiplication by real number. Determinant of transposed transformation is

$$F(I) = I \det F \Rightarrow \\ \det F = F(I) I^{-1} = \langle F(I) I^{-1} \rangle = \langle \bar{I} F(I^{-1}) \rangle = \det \bar{F},$$

where we take advantage of the fact that determinant is a real number, therefore has the grade zero. For the composition of transformations we have

$$(FG)(I) = FG(I) = F(I \det G) = (\det G) F(I) = I \det F \det G,$$

and it is well known rule for determinants, but recall how much effort we need to prove that in the matrix theory. Here, proof is almost trivial. Beginner needs pretty much time to become skilled with matrices. Finally she(he) gets a tool that cannot effectively cope even with rotations. That time he could use to learn the basics of geometric algebra and get a powerful tool for many branches of mathematics. And geometric algebra today, thanks to Grassmann, Clifford, Artin, Hestenes, Sobczyk, Baylis and many other smart and hardworking people (see detailed list at the end of the text) has become a well-developed theory, with applications in many areas of mathematics, physics, engineering, including biology, studies of brain functions, computer graphic, robotics, etc.

We will state without proof ([the reader can prove it](#)) some useful relations. For bivectors we have

$$B_1 \cdot \bar{F}(B_2) = F(B_1) \cdot B_2.$$

This can be extended to arbitrary multi vectors as

$$\langle A \bar{F}(B) \rangle = \langle F(A) B \rangle.$$

Now we will define the inverse of linear transformation. For multivector  $M$  we have

$$IM \det F = F(I) M = F(\bar{I} F(M)),$$

where we used the fact that inner product with pseudoscalar can be replaced by geometric product, namely, there is no additional grades in geometric product ([show it](#)). Let's take the multivector  $A = IM$  so we get

$$A \det F = F(\bar{I} F(I^{-1} A)),$$

and similar relation can be written for  $\bar{F}$ . It follows

$$F^{-1}(A) = \bar{I} F(I^{-1} A) (\det F)^{-1}, \\ \bar{F}^{-1}(A) = I F(I^{-1} A) (\det F)^{-1}.$$

For rotors in  $Cl_3$  we have  $R(a) = RaR^\dagger$ , applied to any multivector gives  $R(M) = RMR^\dagger$  and  $\bar{R}(M) = R^\dagger MR$ , and using  $\det R = 1$

$$R^{-1}(M) = jR^\dagger j^{-1}MR = R^\dagger MR = \bar{R}(M),$$

ie, the inverse of rotation is equal to the transposed rotation. This is actually the definition of each orthogonal transformation (transformation with determinant  $\pm 1$ ). For nice examples see [18].

## Eigenvectors and eigenblades

Concept of eigenvalues and eigenvectors should be known to the reader. Briefly, for an operator (matrix)  $m$  we can define eigenvalues  $\lambda_i$  and eigenvectors  $v_i$  as follows

$$mv_i = \lambda_i v_i, \quad \lambda_i \in \mathbb{C}.$$

In geometric algebra, we say that linear transformation (function) has *eigenvector*  $e$  and *eigenvalue*  $\lambda \in \mathfrak{R}$  if

$$F(e) = \lambda e,$$

which entails

$$\det(F - \lambda I) = 0,$$

so, we have a polynomial equation (secular equation). Generally, secular equation has roots over the complex field, but, we have algebra over the field of real numbers and it is not desirable to spread to a complex area. For example, how to interpret the product  $\sqrt{-1}e_1$ , which is not an element of the algebra. Fortunately, this is not necessary in geometric algebra, because we can give a whole new meaning to complex solutions. For this purpose, we introduce the concept of *eigenblade*. Namely, vectors are just elements of the algebra with grade 1, but we have grades 2, 3, ... in geometric algebra, which are not defined in the ordinary theory of vector spaces. It is therefore natural to extend the definition of eigenvalues to the other elements of algebra. For a blade  $B_r$  with grade  $r$  we define

$$F(B_r) = \lambda B_r, \quad \lambda \in \mathfrak{R}.$$

In fact, we already have such a relationship, namely, for  $B_r = I$  we have an eigenvalue  $\det F$ , because of  $F(I) = I \det F$ . Accordingly, pseudoscalars are eigenblades of linear transformations. To explain the concept of the eigenblade let's look at the following example (see [18]). Let's specify a linear function with the property

$$F(e_1) = e_2, \quad F(e_2) = -e_1,$$

(recognize rotation?) so, it is not difficult to find a solution using matrices. Matrix of transformation is

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

with the eigenvalues  $\pm i$ ,  $i = \sqrt{-1}$ , and eigenvectors  $e_1 \pm ie_2$  (use secular equation and prove). In geometric algebra, for the blade  $e_1 \wedge e_2$  we have (notice elegance)

$$F(e_1 \wedge e_2) = F(e_1) \wedge F(e_2) = e_2 \wedge (-e_1) = e_1 \wedge e_2,$$

so, blade  $e_1 \wedge e_2$  is eigenblade with the (real) eigenvalue 1. Our blade is invariant, but we know that from rotors formalism! There is no need for imaginary unit, we have our blade. Notice that vectors in the plane defined by  $e_1 \wedge e_2$  are changed by transformation, but bivectors are not. You see simple mathematics and important result. In standard methods, using matrices, there is no "blades" at all. Why? Simple, there is no geometric product. So, try to find such a result using matrices. All those who like to comment on geometric algebra by sentences as „Yes, but imaginary unit in quantum mechanics ...“ should think twice about this simple example, and when they come to the conclusion that „it does not make sense ...“, well, what to say? Just think again. This is the question of how do we understand

the very concept of number. Probably, Grassmann and Clifford directed us well and their time is yet to come.

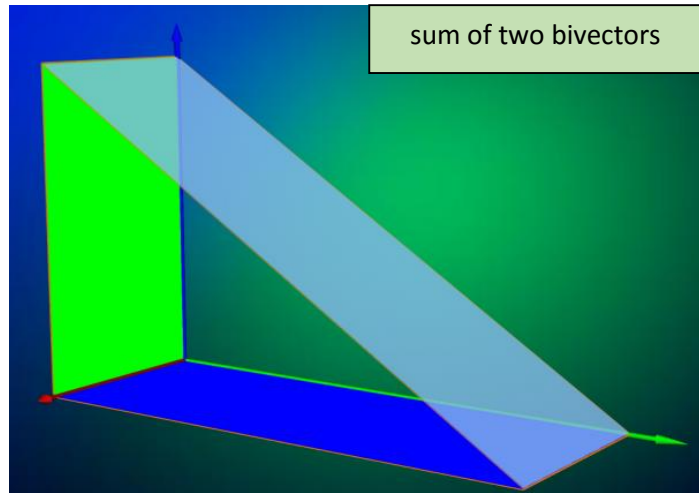
If orthonormal basis vectors  $e_i$  and  $e_j$  are eigenvectors of linear transformation  $F$ , then

$$e_i \cdot F(e_j) = e_i \cdot (\lambda_j e_j) = \lambda_j e_i \cdot e_j.$$

Apply the previous relation to the symmetric linear transformations and show that their eigenvectors with different eigenvalues must be orthogonal.



## Euclidean 3D geometric algebra (C/3)



Generally, a multivector in  $C/3$  can be rewritten as

$$M = t + \mathbf{x} + \mathbf{jn} + bj, \quad t, b \in \mathfrak{R}, \quad j = e_1 e_2 e_3,$$

where for three-dimensional vectors we are using **bold** format here. We have seen already that the unit pseudoscalar  $j$  commutes with all elements of the algebra and squares to -1, making it an ideal replacement for the imaginary unit (there are many “imaginary units” in GA). Pseudoscalar with such properties will appear again in  $C/7$ ,  $C/11$ , ... Here we use one more, very useful, form of multivector:

$$M = Z + \mathbf{F}, \quad Z = t + bj, \quad \mathbf{F} = \mathbf{x} + \mathbf{jn}.$$

Element  $Z$  obviously commutes with all elements of the algebra (belongs to the *center of the algebra*). This feature makes it a *complex scalar*. Complex scalar is really acting as a complex number, as we shall see below. This is the reason that we write  $Z \in \mathbb{C}$ , although, obviously, we have to change the meaning of the symbol  $\mathbb{C}$ , i.e. we replace the ordinary imaginary unit by pseudoscalar. Element  $\mathbf{F}$  is a *complex vector*, with the *real vectors* as components. The choice of designation ( $\mathbf{F}$ ), as well as for complex scalars, is not without significance, namely, due to a complex mixture of electric and magnetic field in electromagnetism. Here, when we say “real”, we mean real scalar, or 3D vector, or their linear combination. When a real element is multiplied by pseudoscalar  $j$  we get an imaginary element, so, sum of real and imaginary elements gives a complex one. For example,  $\mathbf{x}$  (vector) is real,  $t + \mathbf{x}$  (paravector) is real,  $t + \mathbf{jn}$  (spinor) is complex,  $\mathbf{jn}$  (bivector) is imaginary,  $\mathbf{F} = \mathbf{x} + \mathbf{jn}$  (complex vector) is complex, etc. Note that the multivector could be written as

$$M = t + \mathbf{x} + \mathbf{jn} + bj = t + \mathbf{x} + j(b + \mathbf{n}),$$

so, it is just a complex number, with real components (paravectors). Use involution (which?) to extract the real (imaginary) part of multivector. How about  $Z$  and  $\mathbf{F}$ ? Or  $t + \mathbf{jn}$ ?

The reader is suggested to write all three described involutions in this new form. You can use a complex conjugation. As an example we look at Clifford involution (i.e. *Clifford conjugation*, main involution)  $\bar{M} = Z - \mathbf{F}$

$$(M + \bar{M})/2 = Z \equiv \langle M \rangle_s, \quad (\text{scalar part})$$

$$(M - \bar{M})/2 = \mathbf{F} \equiv \langle M \rangle_v, \quad (\text{vector part}).$$

Due to commutativity of the complex scalar  $Z$  we have

$$M\bar{M} = (Z + F)(Z - F) = Z^2 - F^2 = (Z - F)(Z + F) = \bar{M}M,$$

where

$$Z^2 = t^2 - b^2 + 2tbj, \quad F^2 = x^2 - n^2 + j(\mathbf{xn} + \mathbf{nx}) = x^2 - n^2 + 2j\mathbf{x} \cdot \mathbf{n}.$$

Here is result to remember: square of the complex vector is a complex scalar. It means that element  $M\bar{M}$  is a complex scalar. It can be shown that  $M\bar{M}$  is the **only** element of form  $M\bar{M}$  (here  $\bar{M}$  stands for **any** involution of  $M$ ,  $M\bar{M}$  is referred as the square of the *amplitude*) that satisfies  $M\bar{M} = \bar{M}M \in \mathbb{C}$ . We have

$$(Z + F)(\check{Z} + \check{F}) = Z\check{Z} + Z\check{F} + \check{Z}F + F\check{F},$$

so we have two possibilities

$$\check{Z} = Z, \quad \check{F} = -F \quad \text{or} \quad \check{Z} = -Z, \quad \check{F} = F,$$

which differ only in the overall sign. Any involution that changes the complex vector the other way, changes (up to overall sign) bivector or vector part, so

$$F\check{F} = (\mathbf{x} + j\mathbf{n})(\mathbf{x} - j\mathbf{n}) = x^2 + n^2 + j(\mathbf{nx} - \mathbf{xn}) = x^2 + n^2 - 2j\mathbf{x} \wedge \mathbf{n},$$

and we get outer product of real vectors which cannot be canceled, it is absent in  $Z\check{Z} + Z\check{F} + \check{Z}F$ . So, must be  $\check{M} = \bar{M}$ . We already found that  $M\bar{M} = \bar{M}M$ , but we can show that from demand that the amplitude (any) belongs to the center of the algebra follows commutativity

$$M\check{M} \in \mathbb{C} \Rightarrow M(M\check{M}) = (M\check{M})M = M(\check{M}M) \Rightarrow M(M\check{M} - \check{M}M) = 0,$$

due to associativity and distributivity. In a special case the expression in parentheses need not to be zero because there are zero divisors in the algebra, but we need general commutativity, so it must be zero. Scalar  $M\bar{M}$  is referred as the *amplitude of multivector* (MA in text, in fact this is the square of amplitude, but that will not make confusion).

Using MA we can define inverse of multivector, if  $M\bar{M} \neq 0$ :

$$M^{-1} \equiv \bar{M} / M\bar{M}.$$

To find  $1 / M\bar{M}$  we use complex numbers technique

$$\frac{1}{M\bar{M}} = \frac{(M\bar{M})^*}{M\bar{M}(M\bar{M})^*},$$

where \* stands for complex conjugation, which means  $j \rightarrow -j$ . Technique is the same, but interpretation is not, namely, pseudoscalar  $j$  is oriented unit volume, it has intuitive geometric interpretation.

**Example:**  $1/(1+j)$ ? We have  $1/(1+i) = (1-i)/2 \Rightarrow 1/(1+j) = (1-j)/2$ .

Of course, this „trick“ is justified

$$\frac{1}{1+j} = \frac{1-j}{(1+j)(1-j)} = \frac{1-j}{2},$$

We'll see that this procedure sometimes is not enough to find all possible solutions in geometric algebra, e.g. solutions for the roots of a complex numbers can be extended to complex vectors. A simple example is  $\sqrt{1} = e_1$ .

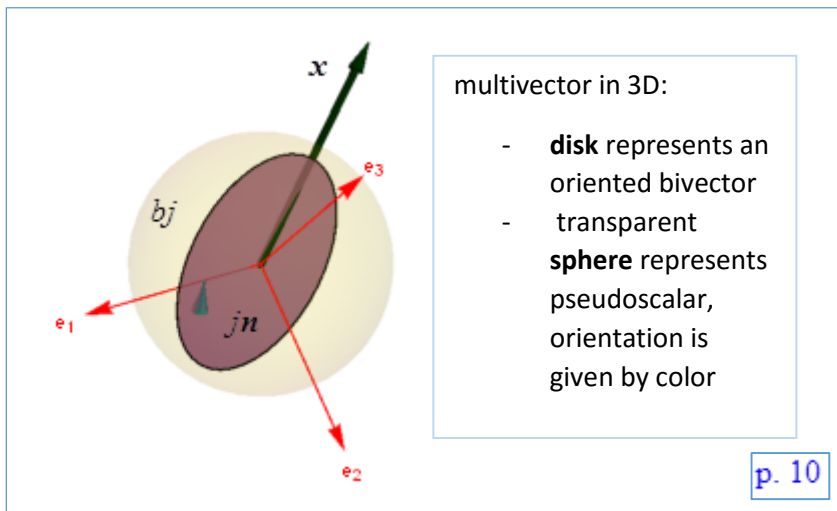
Important concept is *dual* of multivector  $M$  defined as

$$M^* \equiv -jM$$

(do not confuse with complex conjugation \*). Note that with the dual operation real scalar becomes pseudoscalar, vector becomes bivector (and vice versa). As mentioned, element  $j\mathbf{n}$  is a bivector. We suggest reader to express  $j\mathbf{n}$  in orthonormal basis and interpret it. Also, take any two vectors in orthonormal basis and make their outer product. Then find the dual of obtained bivector and check that this dual is just cross product of your vectors. It follows that cross product is

$$\mathbf{x} \times \mathbf{y} = -j\mathbf{x} \wedge \mathbf{y},$$

but, we can use it in 3D only, although the term on the right can be defined in any dimension.



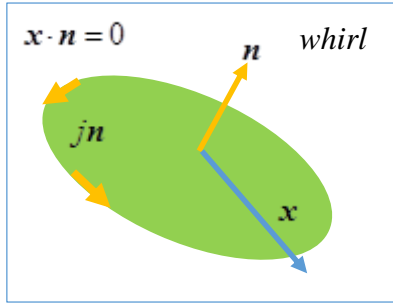
From the general form of multivector in  $C\mathbb{3}$

$$M = t + \mathbf{x} + j\mathbf{n} + bj = Z + F$$

we see that it is essentially determined by two real numbers ( $t, b$ ) and two vectors ( $\mathbf{x}, \mathbf{n}$ ). Bivectors are usually represented by oriented disks, while pseudoscalar can be represented by sphere with two possible colors to give the orientation, so we can imagine a simple image that represents multivector (p. 10). It helps a lot. Figure p. 10 is created in the program *Mathematica*. For the reader, except imagination, we certainly suggest *GAViewer*.

Let's look at properties of complex scalar  $F^2 = x^2 - n^2 + 2j\mathbf{x} \cdot \mathbf{n}$ . In particular, for orthogonal vectors ( $\mathbf{x} \cdot \mathbf{n} = 0$ ) we have  $F^2 \in \mathfrak{R}$ , and values -1, 0 and 1 are of particular interest.

Recall that  $j\mathbf{n}$  is a bivector which defines the plane orthogonal to vector  $\mathbf{n}$ , so, for  $\mathbf{x} \cdot \mathbf{n} = 0$  vector  $\mathbf{x}$  belongs to that plane. This is often used situation (e.g. complex vector of the electromagnetic field in empty space), so it is important to imagine a clear picture. Note that in this case the real value of  $F^2 = x^2 - n^2$  is determined by the lengths of the vectors  $\mathbf{x}$  and  $\mathbf{n}$ . On the next picture you can see the situation described. There is no a special name for this kind of complex vector in the literature (probably?), so we suggest the term *whirl* (short of whirligig).



## Nilpotents and dual numbers

$$1) \quad \mathbf{F}^2 = 0$$

This means that such a complex vector is a *nilpotent*. Let's find the general form of nilpotents in  $Cl_3$  (recall,  $\mathbf{F}^2 \in \mathbb{C}$ ):

$$\begin{aligned} (\mathbf{Z} + \mathbf{F})^2 &= \mathbf{Z}^2 + 2\mathbf{Z}\mathbf{F} + \mathbf{F}^2 = 0 \Rightarrow \\ \mathbf{Z} = 0 &\Rightarrow \mathbf{F}^2 = 0 \Rightarrow x = n, \quad \mathbf{x} \cdot \mathbf{n} = 0, \end{aligned}$$

(we excluded the trivial case  $\mathbf{F} = 0$ ). Notice how often we use the form  $\mathbf{Z} + \mathbf{F}$  here to draw conclusions, it is not a coincidence. It is good practice to avoid habits of some authors to frequently express multivectors by components, so formulas look opaque. Here the focus is on the structure of multivector, and that structure reflects geometrical properties.

One simple example of nilpotent is  $e_1 + je_2$  (check it). Functions with a nilpotent as an argument is easy to find using series expansion, almost all terms just disappear. For example, from  $\mathbf{N}^2 = 0$  follows  $e^{\mathbf{N}} = 1 + \mathbf{N}$  (see below).

Nilpotents are welcome in physics, for example, electromagnetic wave in vacuum is a nilpotent in  $Cl_3$  formulation, field is a complex vector  $\mathbf{F} = \mathbf{E} + j\mathbf{B}$ ,  $E = B$ ,  $c = 1$ , here  $\mathbf{E}$  and  $\mathbf{B}$  are vectors of electric and magnetic field. We can define the *direction of the nilpotent*  $\mathbf{N} = \mathbf{x} + j\mathbf{n}$  as  $\hat{\mathbf{k}} = -j\hat{\mathbf{x}} \wedge \hat{\mathbf{n}} = \hat{\mathbf{x}} \times \hat{\mathbf{n}}$ ,  $\hat{\mathbf{x}}^2 = \hat{\mathbf{n}}^2 = 1$ , so we have

$$\hat{\mathbf{k}}\mathbf{N} = -\mathbf{N}\hat{\mathbf{k}} = \mathbf{N}, \quad (1 + \hat{\mathbf{k}})\mathbf{N} = 2\mathbf{N}.$$

All this is not difficult to prove whether we recall that  $\mathbf{x} \perp \mathbf{n} \Rightarrow \mathbf{x} \wedge \mathbf{n} = \mathbf{x}\mathbf{n}$ ,  $\hat{\mathbf{x}} = \mathbf{x} / x$ ,  $\mathbf{x} = n \cdot \hat{\mathbf{x}}$ . There are many other interesting relations (see literature). These relations have a direct application in electromagnetism, for example.

Let us now comment the possibility of defining the dual numbers. For nilpotent  $\mathbf{N} = \mathbf{x} + j\mathbf{n}$  we have  $x = n$ ,  $\mathbf{x} \cdot \mathbf{n} = 0$ , so let's define a „unit nilpotent“ (nilpotents have zero MA)

$$\boldsymbol{\varepsilon} \equiv \mathbf{N} / x = \hat{\mathbf{x}} + j\hat{\mathbf{n}}, \quad \boldsymbol{\varepsilon}^2 = 0.$$

Now we can define the dual numbers as  $\alpha + \beta\boldsymbol{\varepsilon}$ ,  $\alpha, \beta \in \mathbb{R}$ . Addition of these numbers is similar to the complex numbers, while for multiplication we have

$$(\alpha_1 + \beta_1\boldsymbol{\varepsilon})(\alpha_2 + \beta_2\boldsymbol{\varepsilon}) = \alpha_1\alpha_2 + (\alpha_1\beta_2 + \alpha_2\beta_1)\boldsymbol{\varepsilon},$$

so, for  $\alpha_1\beta_2 + \alpha_2\beta_1 = 0$  it is a real number. If  $\alpha_1 = 0$ ,  $\alpha_2 = 0$  product is zero, which distinguishes dual and complex numbers. For a dual number  $z$  specified as  $z = \alpha + \beta\boldsymbol{\varepsilon}$  we define the conjugation  $\bar{z} = \alpha - \beta\boldsymbol{\varepsilon}$  (notice, it is again just the Clifford involution), it follows

$$z\bar{z} = (\alpha + \beta\boldsymbol{\varepsilon})(\alpha - \beta\boldsymbol{\varepsilon}) = \alpha^2,$$

and the module of dual number is  $|z| = \alpha$  (could be negative). Notice that there is no dependence on  $\beta$ . For  $\alpha \neq 0$  we have the polar form

$$z = \alpha + \beta\boldsymbol{\varepsilon} = \alpha(1 + \varphi\boldsymbol{\varepsilon}), \quad \varphi = \beta / \alpha,$$

here  $\varphi$  is an *argument* of dual number. [Check that](#)

$$(1 + \varphi\epsilon)(1 - \varphi\epsilon) = 1 \quad \text{and} \quad (1 + \varphi\epsilon)^n = (1 + n\varphi\epsilon), \quad n \in \mathbb{N}.$$

For polynomials we have ([check it](#))

$$\begin{aligned} P(\alpha + \beta\epsilon) &= p_0 + p_1(\alpha + \beta\epsilon) + \dots + p_n(\alpha + \beta\epsilon)^n = \\ &P(\alpha) + \beta P'(\alpha)\epsilon, \end{aligned}$$

where  $P'$  is the first derivation of the polynomial. This may be extended to analytic functions (see below), or to maintain the *automatic derivation*. Division by dual numbers is also defined as

$$\frac{\alpha + \beta\epsilon}{\gamma + \delta\epsilon} = \frac{(\alpha + \beta\epsilon)(\gamma - \delta\epsilon)}{(\gamma + \delta\epsilon)(\gamma - \delta\epsilon)} = \frac{(\alpha + \beta\epsilon)(\gamma - \delta\epsilon)}{\gamma^2}, \quad \gamma \neq 0.$$

Especially,

$$\begin{aligned} \frac{1}{1 + \varphi\epsilon} &= \frac{1 - \varphi\epsilon}{(1 + \varphi\epsilon)(1 - \varphi\epsilon)} = 1 - \varphi\epsilon \Rightarrow \\ \left( \frac{1}{1 + \varphi\epsilon} \right)^n &= (1 - \varphi\epsilon)^n = 1 - n\varphi\epsilon, \end{aligned}$$

and we see that Moivre's formula is valid

$$z^n = \alpha^n (1 + \varphi\epsilon)^n = \alpha^n (1 + n\varphi\epsilon), \quad n \in \mathbb{Z}.$$

Dual numbers are of some interest in physics, for example, let's define the special dual number („event“)  $t + x\epsilon$ , where the coordinates of time and position are introduced, and the *proper velocity* („boost“) as  $u \equiv 1 + v\epsilon$ ,  $u\bar{u} = 1$ . Speed  $v$  is the argument of the dual number,  $\varphi = x/t$ . It follows

$$(t + x\epsilon)u = (t + x\epsilon)(1 + v\epsilon) = t + (x + vt)\epsilon = t' + x'\epsilon,$$

which means  $t' = t$ ,  $x' = x + vt$ , so we have the Galilean transformations. Velocity addition rule follows immediately

$$\begin{aligned} u &= 1 + v\epsilon = u_1 u_2 = (1 + v_1\epsilon)(1 + v_2\epsilon) = 1 + (v_1 + v_2)\epsilon \Rightarrow \\ v &= v_1 + v_2. \end{aligned}$$

Here we have a problem, namely, velocity vector is not defined properly (there is no orientation), but if we recall nilpotent direction, we can use it to specify velocity vector  $\mathbf{v} = -jv\epsilon$ . Proper velocity now becomes  $u \equiv 1 - jv\epsilon$ ,  $u\bar{u} = 1$ . For an „event“ we then use  $t - jx\epsilon$ . It follows

$$(t - jx\epsilon)(1 - jv\epsilon) = t - j(x + vt)\epsilon = t' - jx'\epsilon,$$

and we have the Galilean transformations again.

So, as for Lorentz transformations (hyperbolic numbers, vectors and complex vectors, see below) and rotors (complex numbers, bivectors), Galilean transformations (dual numbers, nilpotents) also have a common place in the geometry of  $\mathfrak{R}^3$ . Because all this is a part of larger structure ( $CB$ ), one can get an idea that Galilean transformations are not just approximation of Lorentz transformations for small velocities, but some deeper physical content, independent of speed. But, such an idea is just due to our special choice of components of dual number  $(x, t)$ . Dual numbers like  $t + x\epsilon$  could be useful in non-relativistic physics, but certainly they are not in accordance with the special theory of relativity. In the chapter on special relativity it is shown that Galilean transformations

(with rotations and Lorentz transformations) follow from simple symmetry assumptions about our world (homogeneity and isotropy). If there is a deeper physics behind this formalism then it certainly does not include an explicit space-time events. But what if we choose differently? For example, typical nilpotent is the electromagnetic wave in vacuum. If we define  $\phi = E = B$  and  $\varepsilon = (\mathbf{E} + j\mathbf{B})/\phi$  one could investigate dual numbers like  $\psi + \phi\varepsilon$ ,  $\psi \in \mathfrak{R}$ , but then there is a question: how to interpret  $\psi$ ? According to the structure of expression it could be some sort of scalar field, but then we have another question: what is an argument of such a dual number, the ratio of vector field (complex vector) and scalar field would be a "velocity"? Ok, let's stop.

## Idempotents and hyperbolic structure

2)  $\mathbf{F}^2 = 1$

For  $\mathbf{F}^2 = x^2 - n^2 = 1$  we can find a general form using the relation  $\cosh^2 \varphi - \sinh^2 \varphi = 1$ , so, generally, we have  $\mathbf{F} \equiv \mathbf{f} = \mathbf{n} \cosh \varphi + j\mathbf{m} \sinh \varphi$ ,  $\mathbf{n}^2 = \mathbf{m}^2 = 1$ ,  $\mathbf{n} \perp \mathbf{m}$ , where  $\mathbf{f}$  is a unit complex vector. Example:  $\mathbf{f} = e_1 \cosh \varphi + je_2 \sinh \varphi$ . Such a complex vector can be obtained using  $\sqrt{\mathbf{F}^2}$ , check that multivector  $\mathbf{f} \equiv \mathbf{F} / \sqrt{\mathbf{F}^2}$  has requested properties. Check that  $p = (1 + \mathbf{f})/2 \Rightarrow p^2 = p$ , so, we have an idempotent.

**Theorem 1.** All idempotents in  $Cl_3$  have the form  $p = (1 + \mathbf{f})/2$ .

*Proof:*

$$(Z + \mathbf{F})^2 = Z^2 + 2Z\mathbf{F} + \mathbf{F}^2 = Z + \mathbf{F} \Rightarrow Z = 1/2 \Rightarrow \mathbf{F}^2 = 1/4 \Rightarrow \mathbf{F} = \mathbf{f}/2.$$

Notice again the "Z, F" form. The general form of idempotents is now

$$p = (1 + \mathbf{n} \cosh \varphi + j\mathbf{m} \sinh \varphi)/2, \quad \mathbf{n}^2 = \mathbf{m}^2 = 1, \quad \mathbf{n} \perp \mathbf{m}.$$

Idempotents like  $(1 + \mathbf{n})/2$ ,  $\mathbf{n}^2 = 1$  ( $\mathbf{n}$  is a unit vector) are referred as *simple*.

**Theorem 2:** Each idempotent in  $Cl_3$  can be expressed as the sum of simple idempotent and a nilpotent.

*Proof:*

For the simple idempotent  $p = (1 + \mathbf{n})/2$  and a nilpotent  $N$  we have

$$(p + N)^2 = p + pN + Np = p + N + (\mathbf{n}N + N\mathbf{n})/2,$$

so we can see that the statement is correct if  $\mathbf{n}N + N\mathbf{n} = 0$ , which means that the vector  $\mathbf{n}$  must to anti-commutate with vectors which are defining  $N$ , i.e. must be orthogonal to them, or, parallel to the vector of the nilpotent direction:  $\hat{\mathbf{n}} = \pm \hat{\mathbf{k}}$ . Theorem is proved and we found conditions for the nilpotent.

**Example:**  $p = (1 + e_1)/2$ ,  $N = (e_2 + je_3)/2$ ,  $\hat{\mathbf{k}} = e_2 \times e_3 = e_1$ .

## Spectral decomposition and functions of multivectors

Let's define  $u_{\pm} = (1 \pm f)/2$ , with the properties

$$u_+ + u_- = 1, \quad u_+ - u_- = f, \quad u_+ u_- = u_- u_+ = 0, \quad u_{\pm}^2 = u_{\pm}, \quad \bar{u}_+ = u_-.$$

Note that idempotents  $u_{\pm}$  do not make a basis in  $Cl_3$  (for details about *spectral basis* see [33]), and that we should write  $f = f(M)$  and  $u_{\pm} = u_{\pm}(M)$ , but we omit that. We can express a general multivector with  $F^2 \neq 0$  as

$$M = Z + F = Z + \sqrt{F^2} f \equiv Z + Z_f f, \quad f^2 = 1,$$

so if we define a complex scalars  $M_{\pm} = Z \pm Z_f$  we get a form

$$M = M_+ u_+ + M_- u_-.$$

We say that we have a *spectral decomposition* of multivector. Spectral decomposition gives us a magic opportunity

$$M^2 = (M_+ u_+ + M_- u_-)^2 = M_+^2 u_+ + M_-^2 u_-,$$

and we can immediately generalize this to any positive integer in exponent, but to negative integers also if the inverse of multivector exists. **Prove that in spectral basis form  $MM^{-1} = M_+ M_-^{-1}$  is valid.**

For analytic functions we can utilize series expansion to find

$$f(M) = f(M_+) u_+ + f(M_-) u_-.$$

Recall, to find  $f(M_{\pm})$  we use the complex numbers theory, switch  $j \rightarrow i = \sqrt{-1}$ , find our function and switch again  $i \rightarrow j$ . For multivectors  $M = F = \sqrt{F^2} f$  we have

$$M_{\pm} = \pm \sqrt{F^2} \Rightarrow f(M) = f(\sqrt{F^2}) u_+ + f(-\sqrt{F^2}) u_-.$$

Now for even functions follows

$$f(F) = f(\sqrt{F^2})(u_+ + u_-) = f(\sqrt{F^2}) \in \mathbb{C},$$

and for odd functions

$$f(F) = f(\sqrt{F^2})(u_+ - u_-) = f(\sqrt{F^2}) f.$$

Multivectors of form  $M = z + N$ ,  $F^2 = N^2 = 0$  haven't the spectral decomposition, but using

$$M^n = (z + N)^n = z^n + n z^{n-1} N,$$

we have

$$f(z + N) = f(z) + \frac{df(z)}{dz} N.$$

We can look at some special cases

$$f(u_{\pm}) = f(\pm 1) u_{\pm},$$

$$f(\mathbf{f}) = f(u_+ - u_-) = f(1)u_+ + f(-1)u_-,$$

$$f(-j\mathbf{f}) = f(-ju_+ + ju_-) = f(-j)u_+ + f(j)u_-.$$

For the inverse functions we have

$$f^{-1}(y) = x \Rightarrow f(x) = y \Rightarrow f(x_{\pm}) = y_{\pm} \Rightarrow x_{\pm} = f^{-1}(y_{\pm}).$$

If  $M\bar{M} = 0$  (light-like multivector) we have

$$M = z + \sqrt{\mathbf{F}}f = z + z_F f, \quad z^2 - \mathbf{F}^2 = 0 = (z - z_F)(z + z_F),$$

so we have two options:

- 1)  $z = z_F \Rightarrow M_+ = 2z_F, M_- = 0 \Rightarrow f(M) = f(2z_F)u_+,$
- 2)  $z = -z_F \Rightarrow M_+ = 0, M_- = -2z_F \Rightarrow f(M) = f(-2z_F)u_-.$

Let us now see some examples of elementary functions.

Inverse of multivector ( $M\bar{M} \neq 0$ ) is found easily

$$M^{-1} = \frac{1}{M_+u_+ + M_-u_-} = \frac{M_+u_- + M_-u_+}{(M_+u_+ + M_-u_-)(M_+u_- + M_-u_+)} = \frac{M_+u_- + M_-u_+}{M_+M_-} = \frac{u_+}{M_+} + \frac{u_-}{M_-},$$

with the power

$$M^{-n} = \frac{1}{(M_+u_+ + M_-u_-)^n} = \frac{u_+}{(M_+)^n} + \frac{u_-}{(M_-)^n}, \quad n \in \mathbb{N}.$$

Square root is simple, too (see [13] for different form)

$$\sqrt{M} = S = S_+u_+ + S_-u_- \Rightarrow M = M_+u_+ + M_-u_- = (S_+)^2 u_+ + (S_-)^2 u_- \Rightarrow S_{\pm} = \pm\sqrt{M_{\pm}},$$

or

$$M^{\pm 1/n} = S \Rightarrow S_{\pm} = (M_{\pm})^{\pm 1/n}, \quad n \in \mathbb{N}.$$

**Example:**

$$\sqrt{e_i} = \pm(j + e_i) / \sqrt{2j}.$$

Exponential function is

$$e^M = e^{M_+}u_+ + e^{M_-}u_-,$$

so logarithmic function is obtained as

$$\log M = X \Rightarrow e^X = M = M_+u_+ + M_-u_- = \exp(X_+)u_+ + \exp(X_-)u_- \Rightarrow X_{\pm} = \log M_{\pm}.$$

With definition  $\mathbf{I} \equiv \mathbf{F} / |\mathbf{F}| = -j\mathbf{f}$ ,  $\mathbf{I}^2 = -1$ , logarithmic function has a form (Chappell)



$$\log M = \log |M| + \varphi \mathbf{I}, \quad \varphi = \arctan(|\mathbf{F}|/Z),$$

but we can show that these two formulas are equivalent:

$$\begin{aligned} \log(M_+)u_+ + \log(M_-)u_- &= \frac{\log(M_+) + \log(M_-)}{2} + \frac{\log(M_+) - \log(M_-)}{2} \mathbf{f} = \\ \log |M| - j\mathbf{I} \log \left( \sqrt{\frac{1-j(|\mathbf{F}|/z)}{1+j(|\mathbf{F}|/z)}} \right) &= \log |M| + \mathbf{I} \arctan(|\mathbf{F}|/z) = \log |M| + \varphi \mathbf{I}. \end{aligned}$$

**Examples:**

$$e_1^{e_1} = X \Rightarrow e_1 \log e_1 = \log X, \quad (u_{\pm} = (1 \pm e_1)/2), \quad e_1 u_{\pm} = \pm u_{\pm},$$

$$e_1 = u_+ - u_- \Rightarrow \log e_1 = u_+ \log 1 + u_- \log(-1) = j\pi u_- \Rightarrow$$

$$e_1 \log e_1 = -j\pi u_- \Rightarrow X = \exp(-j\pi u_-) = \exp(-j\pi)u_- = -u_-.$$

We leave to the reader to explore the possibilities, and to find expressions for trigonometric functions.

We can now take the example of the polynomial equation

$$M^2 + 1 = 0,$$

where solutions are all multivectors whose square is -1. We could try

$$(Z + \mathbf{F})^2 + 1 = 0 \Rightarrow Z^2 + 2Z\mathbf{F} + \mathbf{F}^2 + 1 = 0 \Rightarrow Z = 0 \Rightarrow \mathbf{F}^2 = -1,$$

and we know (see next chapter) the general solution. Using the spectral decomposition we have

$$M^2 + 1 = (M_+u_+ + M_-u_-)^2 + u_+ + u_- = (M_+^2 + 1)u_+ + (M_-^2 + 1)u_- = 0 \Rightarrow$$

$$M_+^2 + 1 = 0, \quad M_-^2 + 1 = 0,$$

so we get two equations with the complex numbers. This was just a little demonstration of possibilities, but reader should do complete calculation.

We have already pointed out that  $Cl_3$  has the complex and hyperbolic structures, the complex one due to  $j$  and other elements that square to -1, and hyperbolic due to elements that square to 1, unit vectors are hyperbolic, for example. There are also dual numbers here (using nilpotents). It is possible to efficiently formulate the special relativity theory using hyperbolic (*double, split-complex*) numbers, so, it should not be a surprise if it turns out that the theory is easy to formulate in  $Cl_3$  (see below). Unit complex vector  $\mathbf{f}$  is the most general element of the algebra with features of hyperbolic unit. For two multivectors that have the same unit complex vector  $\mathbf{f}$  (the same „direction“)

$$M_1 = z_1 + z_{1\mathbf{F}}\mathbf{f} \quad \text{and} \quad M_2 = z_2 + z_{2\mathbf{F}}\mathbf{f},$$

we can define the square of *distance of multivectors* as

$$\bar{M}_1 M_2 \equiv (z_1 - z_{1\mathbf{F}}\mathbf{f})(z_2 + z_{2\mathbf{F}}\mathbf{f}) = z_1 z_2 - z_{1\mathbf{F}} z_{2\mathbf{F}} + (z_1 z_{2\mathbf{F}} - z_2 z_{1\mathbf{F}})\mathbf{f} \equiv h_1 + h_0 \mathbf{f},$$

where  $h_i$  and  $h_o$  are *hyperbolic inner* and *hyperbolic outer* products. If  $M_1 = M_2 = M$  we have the *multivector amplitude*. For  $h_o = 0$  we say that multivectors are  $h$ -parallel, while for  $h_i = 0$  are  $h$ -orthogonal.

**Lemma:** Let  $\bar{M}_1 M_2 = 0$  for  $M_1 \neq 0$  and  $M_2 \neq 0$ . Then  $M_1 M_2 \neq 0$  and vice versa.

$$M_1 M_2 = (M_{1+} u_+ + M_{1-} u_-)(M_{2+} u_+ + M_{2-} u_-) = M_{1+} M_{2+} u_+ + M_{1-} M_{2-} u_- ,$$

$$\bar{M}_1 M_2 = (M_{1+} u_- + M_{1-} u_+)(M_{2+} u_+ + M_{2-} u_-) = M_{1-} M_{2+} u_+ + M_{1+} M_{2-} u_- ,$$

so  $M_{1-} M_{2+} = 0$  or  $M_{1+} M_{2-} = 0$ , which means  $M_{1-} = 0$  &&  $M_{2-} = 0$  or  $M_{1+} = 0$  &&  $M_{2+} = 0$ , but both cases imply  $M_1 M_2 \neq 0$ . The reverse statement is similar to prove.

What is  $\sqrt{-1}$ ?

3)  $F^2 = -1$

Generally, this kind of complex vector can be obtained by  $\sqrt{-F^2} = \sqrt{FF} \equiv |F|$ , we have  $I \equiv F / |F| = -jf$ ,  $I^2 = -1$ . General form is

$$I = n \sinh \varphi + jm \cosh \varphi, \quad n^2 = m^2 = 1, \quad n \perp m .$$

Note that we have a non-trivial solution for  $\sqrt{-1}$ . In order to further substantiate we can look for all possible solutions for  $\sqrt{z} = \sqrt{c + jd}$ , so we need to solve the equation  $M^2 = z$ . One solution is just the ordinary square root of complex number (for  $F = 0$ ), but more generally

$$(Z + F)^2 = z \Rightarrow Z^2 + 2ZF + F^2 = z \Rightarrow Z = 0 \Rightarrow F = \sqrt{z} = v + jw ,$$

so

$$\sqrt{c + jd} = v + jw \Rightarrow c + jd = v^2 - w^2 + 2jv \cdot w ,$$

and  $c = v^2 - w^2$ ,  $d = 2v \cdot w$ . Amazing, the square root of a complex number is a complex vector (and this is expected because the square of a complex vector is a complex scalar)! [The reader is proposed to explore the different possibilities.](#)

## Trigonometric forms of multivectors

Recall that for  $F^2 = 0$  we defined the dual numbers  $z = \alpha + \beta \varepsilon$ ,  $\varepsilon^2 = 0$ ,  $\alpha, \beta \in \mathfrak{R}$  and that for  $\alpha \neq 0$  we found the polar form

$$z = \alpha + \beta \varepsilon = \alpha(1 + \varphi \varepsilon), \quad \varphi = \beta / \alpha ,$$

where  $\varphi$  is an *argument* of the dual number.

Elements  $f$  and  $I$  can be utilized to define trigonometric forms of general multivectors. To take the advantages of the theory of complex numbers we use  $I$ . So, we define the *argument of multivector* as

$$\varphi = \arg M \equiv \arctan\left(\frac{|\mathbf{F}|}{Z}\right).$$

Now we have (with the conditions of existence),  $|M| \equiv \sqrt{M\bar{M}}$ ,

$$\cos \varphi = \frac{Z}{|M|}, \quad \sin \varphi = \frac{|\mathbf{F}|}{|M|},$$

which gives

$$M = Z + \mathbf{F} = |M|(\cos \varphi + \mathbf{I} \sin \varphi).$$

Recalling that  $\mathbf{I}^2 = -1$ , generalized Moivre's formula is valid

$$M^n = |M|^n [\cos(n\varphi) + \mathbf{I} \sin(n\varphi)].$$

Notice that we have a form as for complex numbers, but there is a substantial difference: element  $\mathbf{I}$  has a clear geometric meaning, it contains the properties that are determined by the vectors which define the vector part of the multivector. Using  $\mathbf{F} = \mathbf{I}|\mathbf{F}|$  and series expansion we have

$$e^M = e^{Z+\mathbf{F}} = e^Z e^{\mathbf{F}} = e^Z (\cos|\mathbf{F}| + \mathbf{I} \sin|\mathbf{F}|),$$

which is possible due to commutativity of complex scalar  $Z$ . The case  $\mathbf{F}^2 = 0$  we discussed earlier. There is an interesting article where the multivector functions are defined starting right from the properties of the complex vector  $\mathbf{I}$  ([13]).

To take the advantages of the theory of hyperbolic numbers we use  $\mathbf{f}$  :

$$M = Z + \mathbf{F} = Z + Z_f \mathbf{f} = \rho \left( \frac{Z}{\rho} + \frac{Z_f}{\rho} \mathbf{f} \right) = \rho (\cosh \varphi + \mathbf{f} \sinh \varphi), \quad \rho = \sqrt{M\bar{M}} = \sqrt{Z^2 - Z_f^2}.$$

If  $M\bar{M} = 0$  there is no polar form (*light-like* multivectors), but then we have  $M = Z(1 \pm \mathbf{f})$ . Let's define „velocity“  $\mathcal{G} \equiv \tanh \varphi$ , then follows

$$M = \rho (\cosh \varphi + \mathbf{f} \sinh \varphi) = \rho \gamma (1 + \mathcal{G} \mathbf{f}), \quad \gamma^{-1} = \sqrt{1 - \mathcal{G}^2}.$$

If we define the *proper velocity*  $u = \gamma(1 + \mathcal{G} \mathbf{f})$ ,  $u\bar{u} = 1$ , it follows „velocity addition rule“ as

$$\begin{aligned} \gamma_1 \gamma_2 (1 + \mathcal{G}_1 \mathbf{f})(1 + \mathcal{G}_2 \mathbf{f}) &= \gamma_1 \gamma_2 (1 + \mathcal{G}_1 \mathcal{G}_2 + (\mathcal{G}_1 + \mathcal{G}_2) \mathbf{f}) = \\ \gamma_1 \gamma_2 (1 + \mathcal{G}_1 \mathcal{G}_2) (1 + \mathbf{f} (\mathcal{G}_1 + \mathcal{G}_2) / (1 + \mathcal{G}_1 \mathcal{G}_2)) &\Rightarrow \\ \gamma &= \gamma_1 \gamma_2 (1 + \mathcal{G}_1 \mathcal{G}_2), \quad \mathcal{G} = (\mathcal{G}_1 + \mathcal{G}_2) / (1 + \mathcal{G}_1 \mathcal{G}_2), \end{aligned}$$

which are formulas of special theory of relativity. Proper velocity in „rest reference system“  $\mathcal{G} = 0$  is  $u_0 = 1$ , so we can transform to a new reference frame by  $u_0 u = u$ , or, as in the previous example  $u_0 u_1 u_2 = u_1 u_2$ . These formulas represent geometric relations and are more general than those of the special theory of relativity, namely, for SR we usually need just the real part of multivector (*paravectors*, see next chapter), here we have bivectors too.

Using the spectral decomposition we have

$$M = \rho\gamma(1 + \mathcal{G}f) = k_+u_+ + k_-u_- \Rightarrow k_{\pm} = \rho\gamma(1 \pm \mathcal{G}) = \rho K^{\pm 1},$$

where (here we use  $\ln x \equiv \log_e x$ )

$$K = \sqrt{(1 + \mathcal{G}) / (1 - \mathcal{G})}, \quad \varphi = \ln K,$$

is generalized *Bondi factor*. It follows

$$(K_1u_+ + u_- / K_1)(K_2u_+ + u_- / K_2) = K_1K_2u_+ + u_- / (K_1K_2) \Rightarrow K = K_1K_2,$$

which is the exact formula from the special theory of relativity and it is analogous to velocities addition rule.

It goes without saying that the geometric product gave us the possibility of writing a "relativistic" formulas **without the use of Minkowski space**. If Einstein knew that ...

## Special theory of relativity

The reader could take advantage of the previous chapter and apply it to multivectors of form  $\hat{t} + \mathbf{x}$  (*paravectors*), and so immediately get the necessary formulas. Anyway, we have a lot to comment.

The Special Theory of Relativity (SR), in its classic form, is the theory of coordinates, and especially important is the concept of the velocity. If we find physical phenomena in which the notion of velocity becomes questionable then application of SR would be questionable, too. Geometric algebra does not substantially depend on the specific coordinates, which gives the opportunity to consider the general geometric relationships, not only relations between the coordinates, which is certainly desirable because physical processes do not depend on the coordinate systems in which they are formulated. Unfortunately, many authors who use geometric algebra cannot resist to use coordinates, and that makes formulas non-transparent and blurs geometric content. It's hard to get rid of old habits. There are many texts and comments about SR, there is a lot of opponents too, which often only show a lack of understanding of the theory. So, for example, they say that Einstein "wrote nonsense" because in formulas uses the "speed of photons" as  $c$  and  $c \pm v$ , not realizing important and simple fact that speed of the photon is  $c$  in any inertial reference system, but if we want to find the time photon needs to reach the wall of the wagon that runs away from photon (viewed from the rails system, *collision time*) we must use  $c + v$ . Why? Because it is the relative velocity of the photon and the wall of the wagon in the rails system. Speed of the photon and speed of the wall are both measured in the same reference system, so added simple, without relativistic addition rule. It is quite another matter when we have the man in the wagon which walks in the direction of movement of the train with the speed  $u$ , relative to the train. Velocity of the man as measured in the rails system is  $(v + u) / (1 + uv / c^2)$ , but here the speed  $u$  is measured in the train system, while the speed  $v$  (speed of the train) is measured in the rails system. So, relativistic velocity addition formulas we use for velocities measured in different frames of reference. Quantities from one system of reference we are not to transform, so there is no formulas that arise from transformations (here Lorentz transformations).

Before we proceed it may be useful to clarify some terms. We say that laws of physics need to be *covariant*, meaning that in different reference frames have the same form, so, a formula  $A = B$  leads to  $A' = B'$ . A physical quantity is a *constant* if it does not depend on coordinates, for example, number 3 or the charge of electron. The speed of light is **not** a constant in that sense, it is an *invariant*.

It means that it **depends on coordinates** ( $dr/dt$ ), but has the same value in any inertial reference frame. The speed of light is a constant of nature in the sense that it is limiting speed, but related to Lorentz transformations it is an invariant (scalar).

Another common misconception is about the postulates of the special theory of relativity. Let the covariance postulate be first and invariance of the speed of light postulate the second one. From the first we have, for example,  $v = dx/dt$  and  $v' = dx'/dt'$ . The second postulate is mainly motivated by the Maxwell electromagnetic theory, which predicts invariance of the light velocity in inertial reference frames. Now, it is important to note that we need **the first postulate only** to derive the Lorentz transformations (LT) ([it is not hard to find the references, so we highly recommend to do it, see \[26\]](#)). Once we have LT immediately follows the existence of the maximum speed ( $v_g$ ), invariant one. It means that we don't need the second postulate to have that in the theory. Accordingly, in relativistic formulas we can use  $v_g$  instead of  $c$ . Einstein simply assumed that  $v_g = c$ , relying mainly on the Maxwell's theory. However, the existence of the speed limit does not necessarily mean that there must be an object that is moving at such a speed. We think that light is such an object. But we can imagine that the limit speed is  $1 \text{ mm/s}$  larger than  $c$ . What experiment could show the difference? But, if that were so, photon would have to have a mass, no matter how small it was. We could then imagine a reference system that moves along with the photon, so that the photon is at rest in it. But light is a wave too, so, we would see a wave that is not moving. Wave phase would be constant to us (maximum amplitude, for example), so we couldn't see any vibrations. Now, without the change of the electric field in time, there is no magnetic field, so we see an electrostatic field. However, there is no a charge distribution in space that could create such a field (*Einstein*). So, instead of  $v_g$  we use  $c$ , but that does not mean that the assumption of the invariance of the speed of light is necessary for validity of SR. Our first postulate is certainly deeply natural and typical for Einstein, who was among the first which stressed the importance of symmetries in physics, and this is certainly the question of symmetry. True, it is easier to make sense of the thought experiments and derive formulas using the postulate of the speed of light. It is done so in almost all textbooks, so students get the impression that there is no the theory without the second postulate. Let us also mention that there are numerous tests that confirm SR, and none (as far as is known to the author) that refutes it, although many are trying to show things differently, even make up stories about "relativists conspiracy". Let us mention two important facts. First, quantum electromagnetic theory (QED) is deeply based on the special theory of relativity, and it is known that the predictions of QED are in unusually good agreement with experiments. Second, we have the opportunity almost every day to monitor what is happening at speeds comparable to the speed of light, namely, we have particle accelerators. They are built using formulas of special theory of relativity, and it is really hard to imagine that would operate if SR is not valid.

There is one more thing to be discussed. Usually in textbooks is inertial coordinate system defined as "un-accelerated system", but that implies homogeneity, in agreement with the first Newton law only, not all Newton laws, as authors state. To include the third Newton law we have to introduce the concept of isotropy (of inertia). Why? Consider two protons at rest and let them to move freely. Then we expect that protons move in the opposite orientations due to repulsion, but we also expect that both protons have exactly the same kinematical properties. All orientations in space are equal. Without that we have not the third Newton law. Isotropy is directly connected to possibility to synchronize the clocks. It is also natural to expect that light speed is equal in all possible orientations. Then we have inertial coordinate systems (ICS) with the homogeneity and isotropy (of inertia) included. Class of inertial coordinate systems (rotated, translated) that are not moving relative to some inertial coordinate system we call inertial reference frame (IRF). Now, with homogeneity and isotropy included we do not need light speed postulate, symmetries are enough to obtain Lorentz transformations.

Let's see how to do that. Due to linearity we expect transformations like ( $v$  is a relative velocity between systems)

$$x' = Ax + Bt \quad t' = Cx + Dt.$$

For  $x' = \text{const}$  we have  $dx' = 0$ , so,  $B = -vA$ . Inverse transformations are

$$x = \frac{Dx' - Bt'}{AD - BC} \quad t = \frac{-Cx' + At'}{AD - BC},$$

then from  $x = \text{const}$  we have  $B = -vD$ , so,  $D = A$ . If we denote

$$\delta = \sqrt{AD - BC} \quad \text{and} \quad \lambda = A/\kappa$$

we have transformations

$$x' = \delta\lambda(x - vt) \quad t' = \delta\lambda\left(t - \frac{\lambda^2 - 1}{v^2\lambda^2}vx\right),$$

$$x = \frac{\lambda}{\delta}(x' + vt') \quad t = \frac{\lambda}{\delta}\left(t' + \frac{\lambda^2 - 1}{v^2\lambda^2}vx'\right).$$

If we replace  $v$  with  $-v$  these two transformations should be exchanged (due to isotropy) and we have  $\delta = 1$  (note that it means that transformation is orthogonal). If we denote

$$\kappa = \frac{\lambda^2 - 1}{v^2\lambda^2}$$

we have

$$x' = \lambda(x - vt) \quad t' = \lambda(t - \kappa vx),$$

$$x = \lambda(x' + vt') \quad t = \lambda(t' + \kappa vx').$$

Now we have

$$\lambda = 1/\sqrt{1 - \kappa v^2},$$

which gives general transformations in form

$$x' = \frac{x - vt}{\sqrt{1 - \kappa v^2}} \quad t' = \frac{t - \kappa vx}{\sqrt{1 - \kappa v^2}}.$$

Reader is encouraged to show (using three inertial coordinate systems) that  $\kappa(v) = \text{const}$ . Using appropriate physical units we get only three interesting possibilities for  $\kappa$ : -1, 0, 1. Looks familiar?

For  $\kappa = -1$  we have pure Euclidean rotation in the  $(x, t)$  plane, by angle  $\tan^{-1}(v)$ . For  $\kappa = 0$  we have Galilean transformations. For  $\kappa = 1$  we have the Lorentz transformations. Experiments in physics teaching us that we have to use  $\kappa = 1$ , but notice that Galilean relativity is the valid relativity theory, all of this is consequence of our definition of the ICS. Direct consequence of the Lorentz transformations is existence of maximum speed, but we discussed this already.

Recall that we have already seen numbers -1, 0, 1 here in text, we discussed rotations, dual numbers and hyperbolic numbers obtained from general multivectors in  $C\mathbb{3}$ .

A paravectors in  $C\mathbb{3}$ , like  $t + \mathbf{x}$  (multivector with grades 0 and 1), give paravector again when squared (check it), therefore module of paravector is to be defined differently. For complex and hyperbolic numbers (or quaternions) we have a similar obstacle, so we use conjugations. For

paravectors we don't need any *ad hoc conjugation* as we already have the Clifford involution, so we define

$$p\bar{p} = (t + \mathbf{x})(t - \mathbf{x}) = t^2 - x^2 \in \mathfrak{R},$$

which is exactly the desired form of invariant interval required in the special theory of relativity. Recall that Clifford involution is combination of *grade* involution and *reverse* involution, so we can try to interpret it geometrically in  $\mathfrak{R}^3$ , namely, grade involution means space inversion, while for the reverse involution we have seen that it is related to the fact that the Pauli matrices are Hermitian.

To be clear, if we specify paravector  $\alpha + \beta e_1$ , with  $e_1^2 = 1$  we have a natural „hyperbolic unit“. It follows

$$(\alpha + \beta e_1)^2 = \alpha^2 + \beta^2 + \alpha\beta e_1,$$

so, we have a paravector again, with the same direction of vector, but

$$(\alpha + \beta e_1)(\alpha - \beta e_1) = \alpha^2 - \beta^2 \in \mathfrak{R}.$$

Notice that with the Clifford involution there is no need for negative signature (Minkowski). According to the Minkowski formulation of SR we can define unit vector „in time direction“  $e_0$ ,  $e_0^2 = 1$  and three space vectors  $e_i$ ,  $e_i^2 = -1$ , which means that we have a negative signature (1, -1, -1, -1). Such an approach is possible in geometric algebra, too, we have STA (space-time algebra, *Hestenes*). But, everything you can do with STA you can in *Cl3* also, without the negative signature (*Sobczyk, Baylis*). Those who argue that the negative signatures are necessary in SR are maybe wrong. Some authors write sentences like: „*The principle of relativity force us to consider the scalar product with negative square of vectors*“, forgetting that their definition of norm of elements prejudice such result (*Witte: Classical Physics with geometric algebra*). Yet, it is possible to describe geometry in one space using formalism of higher space, so we can say that Minkowski geometry formulation of SR is a 3D problem described in 4D. But in *Cl3*, all we need are three orthonormal vectors and one involution. Time is not a fourth dimension any more, it is a real parameter (as is in quantum mechanics). If there is a fourth dimension of time how it is that we cannot move through the time as we move through the space? There are other interesting arguments in favor of the 3D space, for example, gravitational and electrostatic forces depend on the square of the distance. And what about definition of velocity (we use it also in the theory of relativity):  $dx/dt$ ? If there is a time dimension then time is vector, which means that the speed is naturally bivector, like magnetic field, not a vector. It does not matter if we use proper time to define the four-velocity vector, the space velocity is still defined by the previous formula, up to a factor. Minkowski gave us nice mathematical theory, but his conclusion about fourth time dimension was pure mathematical abstraction, widely accepted among physicist. At that time, the geometric ideas of Grassmann, Hamilton and Clifford were largely suppressed. This begs us to question what would Einstein choose if he knew that? At the beginning of the 20th century another important theory was developing, quantum mechanics, and Pauli introduces his matrices to formulate the half spin, we already commented it. Dirac's matrices are also representation of one Clifford algebra, and again, Dirac's theory has nice formulation in *Cl3* (*Baylis*), as minimal standard model in *Cl7* (*Baylis*) ... It is not without grounds to question the merits of introducing time as a fourth dimension. Usual argument is one that Minkowski gave, in fact, this is not an argument, it is just the observation that in special theory of relativity invariant interval is not  $dt^2 + dx^2$  but  $dt^2 - dx^2$ . But we see that the invariant interval  $dt^2 - dx^2$  is easy to get in *Cl3*, with completely natural requirements for multiplication of vectors. Minkowski has introduced a fourth dimension *ad hoc*. If his formalism was undoubtedly the **only possible** to formulate the special theory of relativity then there would be a solid base to believe that indeed there must be a fourth dimension of time. Thus, without that condition, with the knowledge that there is a natural way to formulate the theory without the fourth

dimension, it is difficult to avoid the impression that this widely accepted mantra of fourth dimension does not have a solid foundation. According to some authors, one of the stumbling blocks in the theory of quantum gravity is probably the existence of a fourth dimension of time in formalism. Here we develop formalism using paravectors which define the 4D linear space, but time is identified as a real scalar, we say that time is a real parameter. It would be interesting to investigate whether there is any experiment that would unambiguously prove the existence of a fourth dimension of time. Probably, there is no such an experiment. Therefore, it is difficult to avoid the impression how physicists are binding ritual cat during meditation. But the future will show, perhaps the time dimension does exist, maybe more of them (if time exists). In any case, it is not true that the Minkowski space is the only correct framework for the formulation of SR. Especially, it is **not true** that in SR we **must** introduce the vectors whose square is negative.

We'll pick a system of physical units in which is  $c=1$ . In geometric algebra we are combining different geometric objects which may have different physical units. Therefore we always choose the system of units such that all is reduced to the same physical unit (usually length). So we study the geometric relationships, and that is the goal here. In the application to a particular situation (experiment) physical units are converted (analysis of physical units), so that there is no problem here.

Starting from the invariant interval in SR  $t^2 - x^2 = \tau^2$ , where  $\tau$  is the invariant *proper time* in the particle rest frame, it follows

$$t^2 - x^2 = t^2(1 - v^2) = \tau^2 \Rightarrow t^2 / \tau^2 = 1 / (1 - v^2) = \gamma^2,$$

where  $\gamma$  is well known relativistic factor. Now, instead of the four-velocity vector, we define the *proper velocity* (paravector)  $u \equiv \gamma(1 + \mathbf{v})$  which is simply  $u_0 = 1$  in the rest frame. Notice that proper velocity is not a list of coordinates, like four-velocity vector, but plays the same role. Obviously,  $u\bar{u} = 1$ . Let us imagine that a body initially at rest we want to analyze in the new reference frame in which the body has a velocity  $\mathbf{v}$  (boost). Recipe is very simple: just make geometric product of two proper velocities  $u_0 \rightarrow u_0 u = u$ . For the series of boosts we have a series of transformations

$$u_0 \rightarrow u_0 u_1 \rightarrow u_0 u_1 u_2 = u_1 u_2.$$

Notice that this is really easy to calculate, and that from the form of the proper velocity paravector we immediately see the relativistic factor  $\gamma$  and 3D velocity vector  $\mathbf{v}$ . For example, let's specify that all velocity vectors are parallel to  $e_1$ , then

$$\gamma_1(1 + v_1 e_1) \gamma_2(1 + v_2 e_1) = \gamma_1 \gamma_2 (1 + v_1 v_2 + (v_1 + v_2) e_1) = \gamma_1 \gamma_2 (1 + v_1 v_2) \left( 1 + \frac{v_1 + v_2}{1 + v_1 v_2} e_1 \right),$$

so, from the form of the paravector (parts are colored in red) we immediately see that

$$\gamma = \gamma_1 \gamma_2 (1 + v_1 v_2), \quad \mathbf{v} = \frac{v_1 + v_2}{1 + v_1 v_2} e_1,$$

known results of the special theory of relativity (relativistic velocity addition). Notice how the geometric product makes derivation of formulas easy, and, as stated earlier, obtained formulas are just special cases of general formulas in  $Cl_3$ . So, from the polar form of general multivector

$$M = \rho(\cosh \varphi + \mathbf{f} \sinh \varphi) = \rho \gamma (1 + \mathcal{G} \mathbf{f}), \quad \gamma^{-1} = \sqrt{1 - \mathcal{G}^2},$$

reducing to the real part of multivector (paravector) we have

$$\gamma = \cosh \varphi, \quad \gamma \mathbf{v} = \sinh \varphi, \quad u = \cosh \varphi + \hat{\mathbf{v}} \sinh \varphi = \cosh \varphi (1 + \hat{\mathbf{v}} \tanh \varphi) = \exp(\varphi \hat{\mathbf{v}}).$$



Using the spectral decomposition we have

$$\gamma(1 + v\hat{v}) = k_+ u_+ + k_- u_- \Rightarrow k_{\pm} = \gamma(1 \pm v) = \cosh \varphi \pm \sinh \varphi,$$

defining implicitly factor  $k$  (*Bondi factor*)  $\varphi \equiv \ln k$ , and recalling definitions of hyperbolic sine and cosine we get

$$k^{\pm 1} = \cosh \varphi \pm \sinh \varphi, \quad k = \sqrt{(1+v)/(1-v)}, \quad u = k u_+ + k^{-1} u_-.$$

Our earlier example with two "boosts" parallel to  $e_1$  now has the form

$$u_1 u_2 = (k_1 u_+ + u_- / k_1)(k_2 u_+ + u_- / k_2) = k_1 k_2 u_+ + u_- / (k_1 k_2),$$

i.e. relativistic velocity addition rule is equivalent to the multiplication of the Bondi factors:  $k = k_1 k_2$ .

**Example:** In the referent frame  $S_1$  starship has velocity  $v$ , in the referent frame of starship another starship has velocity  $v$  and so on, all in the same direction. Find the velocity  $v_n$  of the  $n$ -th starship in  $S_1$ . Discuss solution for  $n \rightarrow \infty$ ?

*Solution:*

Let  $k_1 = \sqrt{(1+v)/(1-v)}$ , then  $k_n = \sqrt{(1+v_n)/(1-v_n)} = k_1^n = \left(\sqrt{(1+v)/(1-v)}\right)^n$ , whence we find the required velocity  $v_n$ .

If the velocity vectors does not lie in the same direction, in expressions appears versor  $\mathbf{v}_1 \mathbf{v}_2$ , which may seem like a complication, but actually provides new opportunities for elegant research, for example, it is rather easy to get Thomas precession (see [14]), for some time unnoticed, but the scope of this text seeks to stop here.

## Lorentz transformations

We are now ready to comment on restricted Lorentz transformations (LT). Generally, LT consists of „boosts“  $B$  and rotors  $R$ . We can write (see [22]), quite generally  $L = BR$ ,  $L\bar{L} = 1$  (*unimodularity condition*). Here we can regard it as the *definition* of Lorentz transformations, which is well researched and justified. If we define (see above)

$$B = \cosh(\varphi/2) + \hat{v} \sinh(\varphi/2) = e^{\varphi\hat{v}/2}, \quad R = \cos(\theta/2) - j\hat{w} \sin(\theta/2) = e^{-j\hat{w}\theta/2},$$

(unit vector  $\hat{w}$  defines the rotation axis) we can write LT of some element, say vector, as

$$p' = LpL^\dagger = BRpR^\dagger B^\dagger.$$

There is a possibility to write  $L$  as

$$L = e^{\varphi\hat{v}/2 - j\hat{w}\theta/2} \neq e^{\varphi\hat{v}/2} e^{-j\hat{w}\theta/2},$$

where we have to be careful due to a general non-commutativity of vectors in the exponent (see [19]). However, it is always possible to find (using logarithms) the vectors  $\hat{v}'$  and  $\hat{w}'$  that satisfy

$$L = e^{\varphi\hat{v}'/2 - j\hat{w}'\theta/2} = e^{\varphi\hat{v}'/2} e^{-j\hat{w}'\theta/2}.$$

It is convenient in applications to resolve the element to be transformed to components parallel and orthogonal to  $\hat{v}$  or  $\hat{w}$  and take the advantage of the commutation properties. For further details see Baylis articles about APS (algebra of physical space, *CIB*). In [22] you can find nice chapter about the special theory of relativity.

We see that rotations are natural part of LT, so, geometric algebra formalism can provide a lot of opportunities because of powerful rotor techniques. Later in the text we will discuss some powerful techniques with spinors (*eigenspinors*).

### Extended Lorentz transformations. Speed limit?

This chapter is **speculative**, with interesting consequences (**new preserved quantities and change of the speed limit in nature**). Those faint hearted can take this as just a mathematical exercise.

Earlier we defined MA as

$$M\bar{M} = |M|^2 = t^2 - x^2 + n^2 - b^2 + 2j(tb - \mathbf{x} \cdot \mathbf{n}) \in \mathbb{C},$$

and showed its properties. Now we look for general bilinear transformation  $M' = XMY$  that preserves MA (see [11]):

$$M' = XMY \Rightarrow M'\bar{M}' = XMY\bar{Y}\bar{M}\bar{X} = |M|^2 |X|^2 |Y|^2,$$

so we have the possibilities

$$|X|^2 = |Y|^2 = \pm 1,$$

which gives

$$X = e^{Z+F} \Rightarrow |X|^2 = e^{Z+F} e^{Z-F} = e^{2Z} = \pm 1,$$

and we will choose (for now) possibility 1 and  $Z = 0$ , although we could consider  $Z = j\pi/2$ , too. Now the general transformation is given with

$$M' = XMY = e^{\bar{p}+j\bar{q}} M e^{\bar{r}+j\bar{s}},$$

so we have 12 parameters from 4 vectors in exponents.

The question is what the motive for the consideration of such transformations we have. Elements of geometric algebra are linear combinations of unit blades of Clifford basis, each of which actually defines the subspace. If we limit ourselves to the real part of the multivectors only (paravectors) we put in a privileged position space of real numbers (grade 0) and vectors (grade 1). The idea is that all subspaces we treat equally. In fact, this whole structure is based on a new multiplication of vectors, so, manipulating multivectors we actually manipulate subspaces. Addition of vectors and bivectors is actually an operation that relates subspaces and it is important to understand it well. If subspaces are treated equally, then we must consider all possible transformations of subspaces and all possible symmetries and they are more than what classical (restricted) Lorentz transformations imply. The reader should be able to stop a little and think carefully about this. Remember that symmetries in the flow of time give the law of conservation of energy, translational invariance gives the law of conservation of impulse, etc. Where we to stop, and why? If we truly accept the naturalness of the new multiplication of vectors we must accept the consequences of such a multiplication, too, and they reveal an unusually rich structure of our good old 3D Euclidean space. But true, the final judgement will be given by experiments.

Considering invariant MA expressed in two reference frames we can compare real and imaginary part

$$t^2 - x^2 + n^2 - b^2 = t'^2 - x'^2 + n'^2 - b'^2,$$

$$tb - \mathbf{x} \cdot \mathbf{n} = t'b' - \mathbf{x}' \cdot \mathbf{n}' .$$

Differential of multivector

$$dX = dt + d\mathbf{x} + j d\mathbf{n} + j db ,$$

gives MA

$$|dX|^2 = dt^2 - d\mathbf{x}^2 + d\mathbf{n}^2 - db^2 + 2j(dbdt - d\mathbf{x} \cdot d\mathbf{n}) ,$$

and we can try to find the conditions for existence of the **real** proper time. There are many reasons for defining the real proper time, for example, makes it easy to define a generalized velocity. Typically, in the special theory of relativity we will chose the rest frame. Here, due to the additional elements (except velocity), it will not be enough, because we want ( $\tau$  is a proper time)

$$|dX|^2 = dt^2 - d\mathbf{x}^2 + d\mathbf{n}^2 - db^2 + 2j(dbdt - d\mathbf{x} \cdot d\mathbf{n}) = d\tau^2 \in \mathfrak{R} .$$

The first condition, if we want the real proper time, is certainly the disappearance of the imaginary part of the MA in each system of reference (recall that the MA is invariant to our transformations, and cannot have an imaginary part in one reference frame and not in the other). This means that in every reference frame must be valid

$$dbdt - d\mathbf{x} \cdot d\mathbf{n} = dt^2 (d\dot{b} - d\dot{\mathbf{x}} \cdot d\dot{\mathbf{n}}) = dt^2 (h - d\dot{\mathbf{x}} \cdot d\dot{\mathbf{n}}) = 0 \Rightarrow h = d\dot{\mathbf{x}} \cdot d\dot{\mathbf{n}}, \quad d\dot{b} \equiv h ,$$

with a common designation  $d\mathbf{x}/dt \equiv \dot{\mathbf{x}}$ , which implies  $h' = d\dot{\mathbf{x}}' \cdot d\dot{\mathbf{n}}'$ . If we define  $d\dot{\mathbf{x}} \equiv \mathbf{v}$ ,  $d\dot{\mathbf{n}} \equiv \mathbf{w}$ , it follows  $h = \mathbf{w} \cdot \mathbf{v}$ . Vector  $\mathbf{w}$  comes from the bivector part of the multivector, so we expect it to be related to angular momentum-like quantities, then  $h$  could be a flow of such a quantity, much like flow is defined for the flowing of liquid through the tube. The difference is that here bivectors do not transform as surfaces (see [11]).

Considering the invariance of MA and proper time as an real number we have

$$\begin{aligned} |dX|^2 &= |dX'|^2 = d\tau^2 = dt^2 - d\mathbf{x}^2 + d\mathbf{n}^2 - db^2 \Rightarrow \\ 1 &= \frac{dt^2}{d\tau^2} \left( 1 - \frac{d\mathbf{x}^2}{dt^2} + \frac{d\mathbf{n}^2}{dt^2} - \frac{db^2}{dt^2} \right) = \gamma^2 (1 - v^2 + w^2 - h^2) , \\ \gamma &= 1 / \sqrt{1 - v^2 + w^2 - (\mathbf{w} \cdot \mathbf{v})^2} = 1 / \sqrt{1 - v^2 + w^2 - w^2 v^2 \cos^2 \alpha} . \end{aligned}$$

Note that now our relativistic factor  $\gamma$  has contributions from all subspaces. It would be naturally to require that „rest frame“ (with the condition  $v = 0$ ) be replaced by  $\gamma = 1$ , which would mean that there is no resting particles, but

$$-v^2 + w^2 - w^2 v^2 \cos^2 \alpha = 0 \Rightarrow v = w / \sqrt{1 + w^2 \cos^2 \alpha} .$$

It is not so difficult to accept this, because what if the velocity of the particle may not be zero? For example, how to reconcile the principles of quantum mechanics and the idea of completely peaceful electrons? Including all subspaces and all quantities related to them it follows that a „rest frame“ becomes something like „center of energy-impulse-angular momentum, etc. frame“.

Relativistic factor  $\gamma$  is defined as the ratio of two real times, so it must be a real number, which gives us condition

$$1 - v^2 + w^2 - w^2 v^2 \cos^2 \alpha > 0 \Rightarrow v_{\max} < \sqrt{\frac{1 + w^2}{1 + w^2 \cos^2 \alpha}} .$$

This is a completely new result: limit speed is 1 for  $w = 0$  or  $\cos \alpha = \pm 1$ , otherwise, it is greater than 1. This result is not new in geometric algebra (*Pavšić*, using C-algebras, but the author got

this result independently, in comment on article by [12]). What could be the physical meaning of this? Consider an electron, it certainly isn't a „small ball“ (recall the great Ruđer Bošković and his points as the source of forces), namely, has the spin, and spin is just „like angular momentum“ quantity. Can we treat the relativistic electron as an Einstein's relativistic train? Probably not! In the eyes of geometry it is hard to accept that electron is just a very little ball with the spin packed in it, like train with passengers in it. Relativistic formulas for train do not depend on the type of cargo in it. But spin is probably not just a “cargo”, rather, it is geometric element, so it should be a part of transformations. If just derived formulas are applicable to the electron then its (limit) speed would depend on the orientation of the spin relative to the velocity vector. And more, speed for a given energy would depend on the orientation of the spin, so electrons with the same energy are supposed to arrive at the target with different times. Maybe someone in the future carry out such an experiment, a positive result would certainly significantly change our current understanding of relativity. Especially interesting would be to see how electrons behave in quantum tunneling, because there are suggestions of some authors that the electron might be moving with speeds exceeding 1. This sometime is formulated by introducing complex numbers, making the whole philosophy about it, although it is likely the matter of inappropriate mathematics. But, it is hard to be sure.

Now that we have defined (invariant, real) proper time we can define a multivector of generalized velocity

$$V = \frac{dX}{d\tau} = \frac{dt}{d\tau} + \frac{d\mathbf{x}}{dt} \frac{dt}{d\tau} + j \frac{d\mathbf{n}}{dt} \frac{dt}{d\tau} + j \frac{d\mathbf{b}}{dt} \frac{dt}{d\tau} = \gamma(1 + \mathbf{v} + j\mathbf{w} + j\mathbf{h}),$$

with the invariant amplitude

$$V\bar{V} = 1 \Rightarrow \frac{d(V\bar{V})}{d\tau} = \frac{dV}{d\tau}\bar{V} + V \frac{d\bar{V}}{d\tau} = 0,$$

which is a kind of expression of orthogonality of generalized velocity and generalized acceleration. Multiplying generalized velocity with the mass we get generalized momentum

$$P = mV = E + \mathbf{p} + j\mathbf{l} + jH, \quad P\bar{P} = E^2 - p^2 + l^2 - H^2 = m^2.$$

This is very different from the usual formula for the energy-momentum  $E^2 - p^2 = m^2$ . Two additional conserved quantities appear, the last of which ( $H$ ) is brand new ([11]). Under the terms of our derivation must be  $H = \gamma m h = \mathbf{l} \cdot \mathbf{v}$ , so the new conserved quantity has a form of flow, and we have finally

$$P\bar{P} = E^2 - p^2 + l^2 - (\mathbf{l} \cdot \mathbf{v})^2 = m^2.$$

If this is physical, motion of particles with spin should satisfy the law of conservation of flow, an idea that had been already presented by some authors (unfortunately, I forgot the source!). Speed of particle will be generally higher with  $\mathbf{w}$  than without it for a given energy, which can be deduced from the previous formula: adding positive term to the negative square of momentum gives possibility to increase the speed. Let's make sure about it directly

$$\gamma = E / m = 1 / \sqrt{1 - v^2 + w^2 - (\mathbf{w} \cdot \mathbf{v})^2} \Rightarrow$$

$$v = \sqrt{\frac{1 + w^2 - m^2 / E^2}{1 + w^2 \cos^2 \alpha}} = \sqrt{\frac{1 + (l/E)^2 - m^2 / E^2}{1 + (l/E)^2 \cos^2 \alpha}} \geq \sqrt{1 - m^2 / E^2}.$$

We should note at the end that just discussed formalism reduces to the usual special theory of relativity, it is sufficient to reject the imaginary part of multivectors and keep the real one, i.e.

paravectors. All “strange” implications just disappear. Of course, there is a plenty of possibilities to treat time differently, but we will stop here (see ([11] i ([12] for some interesting discussions).

## Electromagnetic field in geometric algebra

Here we will not describe the entire EM theory in  $Cl_3$  (see *Hestenes, Baylis, Chappell, Jancewicz*), we only comment on a few ideas. In the geometric algebra formalism, for electromagnetic (EM) wave in vacuum we define

$$E = B, \mathbf{E} \perp \mathbf{B}, \mathbf{F} = \mathbf{E} + j\mathbf{B} \Rightarrow \mathbf{F}^2 = 0, \quad c = 1,$$

so complex vector  $\mathbf{F}$  is a *nilpotent*. Note that the term with magnetic field is a bivector. It is useful to expand the magnetic field bivector in orthonormal basis, so, if we start with the magnetic field vector

$$\mathbf{B} = B_1 e_1 + B_2 e_2 + B_3 e_3$$

we get the bivector

$$j\mathbf{B} = j(B_1 e_1 + B_2 e_2 + B_3 e_3) = B_3 e_1 e_2 + B_2 e_3 e_1 + B_1 e_2 e_3.$$

The reader should check this simple expression and try to create a mental image of it. Also, we can represent bivectors using parallelograms, it is straightforward to see how to add them graphically (figure next to main title). [GAViewer can help here](#). Although this may seem like "just a neat trick", here we are going to try to show that the bivector, as a geometric object, is fully adequate for the description of the magnetic field, actually, physical properties of the magnetic field **require** to be treated as a bivector. In any formalism that does not imply the existence of bivectors (as for the Gibbs vector formalism in the language of the scalar and cross products) problematic situations must necessarily occur. Here we will discuss the issue of Maxwell's theory and mirror symmetry, as an example. If we use a coordinate approach then in 3D we can define a richer structure by introducing tensors. Let's look at a quote from the article by Jancewicz: *A system of units compatible with geometry*, 2004:

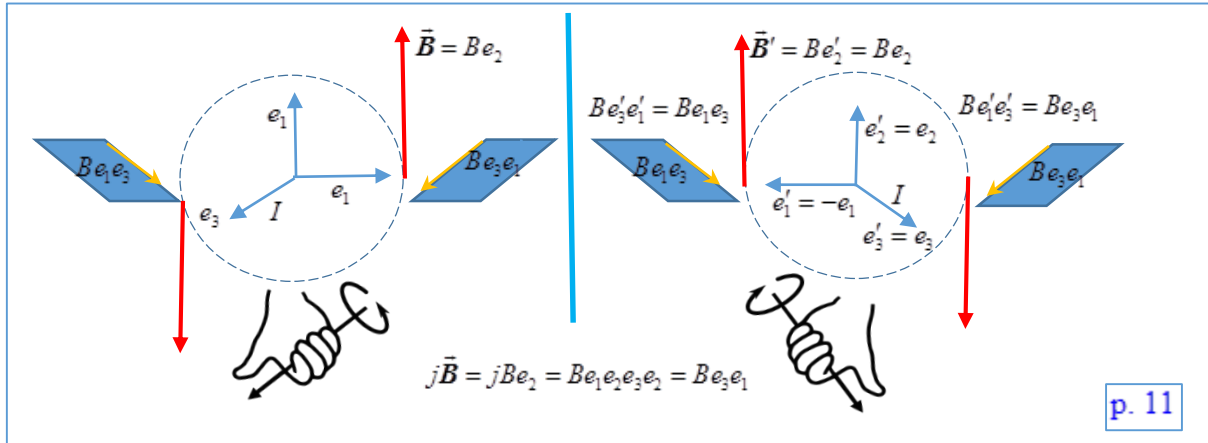
*„For the three-dimensional description, antisymmetric contravariant tensors are needed of ranks from zero to three (they are known as multivectors) as well as antisymmetric covariant tensors of ranks from zero to three (called exterior forms). It makes altogether eight types of directed quantities.“*

So, for example, axial vectors (like the cross product) become antisymmetric tensor of the second rank. This whole geometric structure becomes simple and intuitive in geometric algebra, without the need for introducing any coordinates (here we often introduce a basis, but it is solely for the purpose of easier understanding of the text and it is not necessary). Due to the independence of the basis, ceases to be important if we work in the right or left coordinate system, for example, geometric product takes care of everything. For two vectors we could have expressions like  $\mathbf{ab} \pm \mathbf{ba}$  and we do not need to use any basis to conclude that it is a scalar or bivector, in any dimension.

Maxwell's electromagnetic theory is the first physical theory which initially met the postulate(s) of special theory of relativity. It is therefore no wonder that both theories fit perfectly in  $Cl_3$ . Let's look at some interesting facts related to the theory in the language of geometric algebra. For example, we can visualize solutions for electromagnetic wave in vacuum ([22]) by a simple and interesting picture, namely, a wave vector  $\mathbf{k}$  is parallel to the direction vector of the nilpotent  $\mathbf{F}$ , so solution can be written (for  $\mathbf{k} \parallel \mathbf{x}$ ) as

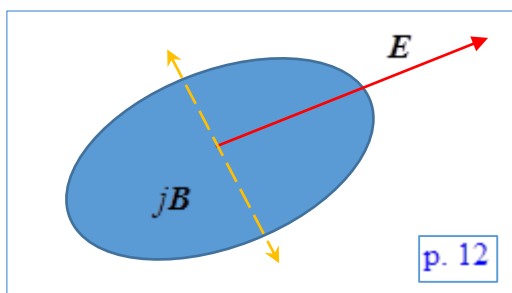
$$\mathbf{F}_0 e^{\pm j\mathbf{k}\mathbf{x}} e^{j\omega t}, \quad \mathbf{F}_0 = \mathbf{E}_0 + j\mathbf{B}_0.$$

We can imagine the spatial part of the wave  $F_0 e^{\pm jkx}$  as spatial periodic **steady** spiral which extends along the direction of wave propagation. This "spiral" is the nilpotent here because it is proportional to  $F_0$ . It turns out (see below) that rotation of nilpotent around his direction can be achieved by



multiplying it by a complex phase, like  $e^{j\omega t}$ , so, we have a spiral in space which rotates around the direction of propagation of the wave. Bivector part  $j\mathbf{B}$  defines the plane orthogonal to  $\mathbf{B}$ , so vector  $\mathbf{E}$  belongs to that plane. Bivector  $j\mathbf{B}$  provides an opportunity for consideration of electromagnetic phenomena more complete and more elegant than (axial) vector  $\mathbf{B}$ .

Let's look at some more properties of EM fields in vacuum. Maxwell's equations are completely mirror-symmetrical in the language of geometric algebra, as well as their solutions. When we use the cross product we immediately need to introduce the right hand rule, and we see that we have the left hand rule in the mirror. If we set the figures (the original and the one in the mirror, p. 11) one over another they do not match, vectors are standing wrong. However, if vectors (axial) of the magnetic field are replaced by bivectors, images exactly match. And, of course, we don't need the right hand rule, as stated, geometric product takes care of everything. It is clear that for those who have thought for a long time in the language "vector-right hand rule" will be difficult to accept a new paradigm. We're taught to imagine arrows, so we have yet to develop an intuition for objects that are not vectors or scalars. Geometry is the language of physics more than we dreamed of.



Vector  $\mathbf{E}$  belongs to the plane defined by bivector  $j\mathbf{B}$ . Area of the circle which presents bivector is  $B$ , so its radius must be  $\sqrt{B/\pi} \approx 0.56\sqrt{B}$ . Direction and possible orientations of wave propagation are plotted in p. 12.

This image is rotated about the wave propagation axis by an angle dependent on the position ( $\mathbf{k} \cdot \mathbf{x}$ ), which gives a static image in space. It has been said, this whole spatial image rotates in time, depending on the frequency of the wave ( $\omega$ ). [This last assertion reader can check itself if he takes a simple nilpotent  \$e\_1 + je\_2\$  and multiply it by a complex phase  \$e^{j\theta}\$ . Immediately we get the matrix of the](#)

rotation around the z-axis. We can rotate elegantly the whole picture, so this particular example is not special in any way.

For any complex vector  $\mathbf{F} = \mathbf{v} + j\mathbf{w}$  in  $Cl3$  we have

$$\mathbf{F}\mathbf{F}^\dagger = (\mathbf{v} + j\mathbf{w})(\mathbf{v} - j\mathbf{w}) = v^2 + w^2 - 2j\mathbf{v} \wedge \mathbf{w} = v^2 + w^2 + 2\mathbf{v} \times \mathbf{w},$$

so if we use complex vector of electromagnetic field in vacuum (nilpotent)  $\mathbf{F} = \mathbf{E} + jc\mathbf{B}$ , where we use SI system of units for a moment, we get

$$\mathbf{F}\mathbf{F}^\dagger = E^2 + c^2B^2 + 2c\mathbf{E} \times \mathbf{B}.$$

Now we have

$$\frac{1}{2} \varepsilon_0 c \mathbf{F}\mathbf{F}^\dagger = c\xi + \mathbf{S},$$

with

$$\xi = \frac{1}{2} \varepsilon_0 (E^2 + c^2B^2),$$

$$\mathbf{S} = \mathbf{E} \times \mathbf{B} / \mu_0.$$

Here  $\xi$  is the energy density and  $\mathbf{S}$  is the energy-current density (energy flow), known as *Poynting vector*. So, Poynting vector is proportional to nilpotent direction vector.

## Eigenspinors

Let's look at one more elegant and powerful way to describe the motion of relativistic particles ([5], [6], [7]). Imagine the laboratory reference frame and the frame that is fixed to the particle in motion, for example, under the influence of electromagnetic field. We shall consider here paravectors and restricted Lorentz transformations only. At any moment we can find a transformation which transforms elements from the lab frame to the inertial frame of reference that coincides with the particle movement (*commoving frame*). The proper velocity in lab frame is  $u_0 \equiv e_0 = 1$ , so, for that very instant of time we can get the proper velocity of the particle as

$$u = \Lambda e_0 \Lambda^\dagger,$$

where  $\Lambda$  is the Lorentz transformation, named *eigenspinor* due to the special choice of the reference frame. Let us mention passing that applying such a transformation to orthonormal basis vectors  $u_\mu = \Lambda e_\mu \Lambda^\dagger$ ,  $\mu = 0, 1, 2, 3$  we get so-called *Frenet tetrad*. Recall, for the Lorentz transformations we have  $\Lambda \bar{\Lambda} = 1$  (unimodularity). If an eigenspinor  $\Lambda$  is known at very instant of time we have all information needed to describe the particle movement. Eigenspinors are changing in time, so we need the first time derivative

$$\dot{\Lambda} = \dot{\Lambda} \bar{\Lambda} \Lambda \equiv \mathbf{\Omega} \Lambda / 2, \quad \mathbf{\Omega} = 2 \dot{\Lambda} \bar{\Lambda}.$$

This all seems like a trivial relation, but it is not. We have

$$\frac{d}{dt}(\Lambda \bar{\Lambda}) = 0 = \dot{\Lambda} \bar{\Lambda} + \Lambda \dot{\bar{\Lambda}},$$

and using  $\overline{\dot{\Lambda}\bar{\Lambda}} = \Lambda\bar{\dot{\Lambda}} = -\dot{\Lambda}\bar{\Lambda}$  we see that  $\mathbf{\Omega}$  is a complex vector (so it is in bold format). For the first time derivative of the proper velocity we have

$$\dot{u} = \dot{\Lambda}e_0\Lambda^\dagger + \Lambda e_0\dot{\Lambda}^\dagger = \dot{\Lambda}\bar{\Lambda}\Lambda e_0\Lambda^\dagger + (\dot{\Lambda}\bar{\Lambda}\Lambda e_0\Lambda^\dagger)^\dagger = \frac{\mathbf{\Omega}u + (\mathbf{\Omega}u)^\dagger}{2} = \langle \mathbf{\Omega}u \rangle_R,$$

which is a paravector. The Lorentz force (see [8]) is now

$$\dot{p} = m\dot{u} = e\langle \mathbf{F}u \rangle_R, \quad \mathbf{F} = \mathbf{E} + j\mathbf{B},$$

so we see how just defined  $\mathbf{\Omega}$  gets physical meaning: it is proportional to the complex vector of the electromagnetic field  $\mathbf{F}$ . It is surprising how the electromagnetic theory simply and naturally formulates in  $Cl_3$ . And this is not an isolated example. Geometric product makes the geometry of our 3D world a natural framework for physics. Someone who knows well the geometric algebra, but did not know anything about the electromagnetism, could probably discover the electromagnetic field as a purely geometric object. Gibbs scalar and cross products, and then the whole apparatus of theoretical physics with coordinates, matrices, tensors ... blurred the whole picture, a lot.

This brief review on the eigenspinors should point out on a powerful and elegant technique that is widely applicable, except electromagnetism, for example, in quantum mechanics.

## Spinorial equations

Having a radius-vector in 2D we can write

$$\mathbf{r} = e_1x + e_2y = e_1r\left(\frac{x}{r} + \frac{y}{r}e_1e_2\right) = e_1r \exp(\varphi e_1e_2),$$

We have seen how this expression cannot be generalized to higher dimensions, but we can do that in the "sandwich" form

$$\mathbf{r} = Ue_1U^\dagger.$$

What we wrote? In 2D,  $U$  is a complex number with the complex unit  $e_1e_2$ , but it generally can be treated as a *spinor* (for the definition of spinors see literature, it is enough here to use a term „spinor“ as element of even part of the  $Cl_3$ , or just rotor with a dilatation). Note that starting with the unit vector  $e_1$  we can get any vector in the plane defined by bivector  $e_1e_2$ , or any bivector (and thus any plane). These relations are easy to generalize to higher dimensions. If spinor  $U$  depends on time we have all dynamics contained in the spinor. This is a really powerful technique to describe various types of movement, that we will see below, informative only. It turned out that equations, as a rule, are much easier to solve if they are expressed in terms of  $U$  instead of  $\mathbf{r}$ .

Note that for a complex number  $U$  we get a complex conjugate as  $U^\dagger$ . Module of  $\mathbf{r}$  is just  $r = \sqrt{\mathbf{r}^2}$ , i.e.  $r = UU^\dagger$ . The time rate of vector  $\mathbf{r}$  is the first derivation of  $\mathbf{r} = Ue_1U^\dagger$ , what is generally the correct approach. In 2D, for simplicity, we are to take derivative of  $U^2e_1$ , i.e.

$$\dot{\mathbf{r}} = 2\dot{U}Ue_1 \Rightarrow \dot{\mathbf{r}}e_1 = 2\dot{U}U \Rightarrow \dot{\mathbf{r}}e_1U^\dagger = 2r\dot{U}.$$

Introducing a new variable



$$\frac{d}{ds} = r \frac{d}{dt}, \quad \frac{dt}{ds} = r,$$

and using  $\frac{dU}{ds} = U'$ , we get a new equation for  $U$

$$2U' = \dot{\mathbf{r}} U e_1,$$

or, deriving once more

$$2U'' = \ddot{\mathbf{r}} U e_1 + \dot{\mathbf{r}} U' e_1 = U (\ddot{\mathbf{r}} + \dot{\mathbf{r}}^2 / 2).$$

For the particular problem, let's look at the motion of the body under the action of central force (Kepler's problem)

$$\mu \ddot{\mathbf{r}} = -k \mathbf{r} r^{-3},$$

where  $\mu$  is a reduced mass and  $k$  is a constant. The equation for  $U$  now becomes

$$U'' = \frac{1}{2\mu} U \left( \frac{\mu \dot{\mathbf{r}}^2}{2} - \frac{k}{r} \right) = \frac{E}{2\mu} U \Rightarrow$$

$$U'' = \kappa U, \quad \kappa = \text{const},$$

where we have introduced the total energy  $E$ . This is a well-known and relatively simple equation, which for bound states ( $E < 0$ ) takes the form of the equation for the harmonic oscillator. The advantages of this approach are numerous, like ease of solving the specific equations, better stability of solutions (no singularities for  $r = 0$ ), and observe that the equation is linear, which has a great advantages in the perturbation approach.

### C/3 and quantum mechanics

We have already shown that orthonormal basis vectors in C/3 could be represented by Pauli matrices. Now we are to develop this idea a little further in order to get a sense of how quantum mechanics fits nicely in the formalism of geometric algebra.

In "standard" quantum mechanics formulation the wave function of electron has the form

$$|\psi\rangle = \alpha |\uparrow\rangle + \beta |\downarrow\rangle, \quad \alpha, \beta \in \mathbb{C},$$

so that such a quantum state is usually given in the form of spinor (see [18])

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

If we set the direction of z-axis in the direction of the state  $|\uparrow\rangle$  we get the spin operators in the form

$$\hat{s}_k = \frac{1}{2} \hbar \hat{\sigma}_k,$$

where  $\hat{\sigma}_k$  are earlier defined Pauli matrices. We can now look for observables in form

$$s_k = \frac{1}{2} \hbar n_k = \langle \psi | \hat{s}_k | \psi \rangle, \quad n_k = \langle \psi | \hat{\sigma}_k | \psi \rangle,$$

where the components are given as

$$n_1 = \alpha\beta^* + \alpha^*\beta, \quad n_2 = i(\alpha\beta^* - \alpha^*\beta), \quad n_3 = \alpha\alpha^* - \beta\beta^*.$$

We have

$$|\mathbf{n}|^2 = \langle \psi | \psi \rangle^2 = (|\alpha|^2 + |\beta|^2)^2,$$

so we can take advantage of the previous relation and normalize the vector  $\mathbf{n}$  to be  $|\mathbf{n}|^2 = 1$ . By introducing the spherical coordinates we can write

$$n_1 = \sin \theta \cos \varphi, \quad n_2 = \cos \theta \sin \varphi, \quad n_3 = \cos \theta,$$

or

$$\alpha = \cos(\theta/2)e^{i\gamma}, \quad \beta = \sin(\theta/2)e^{i\delta}, \quad \delta - \gamma = \varphi,$$

which gives for the spinor

$$|\psi\rangle = \begin{pmatrix} \cos(\theta/2)e^{-i\varphi/2} \\ \sin(\theta/2)e^{i\varphi/2} \end{pmatrix} e^{i(\gamma+\delta)/2}.$$

We can neglect the overall phase  $\exp(i(\gamma+\delta)/2)$ . We see the dependence on the half angles, suggesting a link to the rotors in  $Cl3$ . Let us introduce now a common designation in  $Cl3$ :  $e_i \rightarrow \sigma_i$ , it follows relation to rotors

$$\mathbf{n} = \sum_{k=1}^3 n_k \sigma_k = \sin \theta (\sigma_1 \cos \varphi + \sigma_2 \sin \varphi) + \sigma_3 \cos \theta \equiv R \sigma_3 R^\dagger, \quad R = e^{-j\varphi\sigma_3/2} e^{-j\theta\sigma_2/2}.$$

If we introduce now a spinor in  $Cl3$ , which by analogy we denote by  $\psi$ , we will seek a general form for the new object by comparison

$$|\psi\rangle = \begin{pmatrix} a^0 + ia^3 \\ -a^2 + ia^1 \end{pmatrix} \leftrightarrow \psi = a^0 + a^k j\sigma_k,$$

where summation over  $k$  is understood. We see immediately that  $|\uparrow\rangle \leftrightarrow 1$ ,  $|\downarrow\rangle \leftrightarrow -j\sigma_2$ ,

and that the appropriate vectors of observable have components  $(0, 0, \pm 1)$ . For operators we can find relationship

$$\hat{\sigma}_k |\psi\rangle \leftrightarrow \sigma_k \psi \sigma_3,$$

where  $\sigma_3$  is included to ensure belonging to the even part of the algebra. This choice is, of course, a consequence of initial selection of z-axis and does not affect the generality of the expression. The choice of the z-axis usually has a physical background, for example, the direction of the external magnetic field. [What we get if we multiply all three Pauli matrices?](#) We can establish an analogy with the multiplication by the imaginary unit as

$$i|\psi\rangle \leftrightarrow \psi j\sigma_3.$$

Suggestive is that we have a multiplication by the bivector  $j\sigma_3$ , for it is to expect. Namely, we get the vectors of observables just by rotation of the vector  $\sigma_3$  and they are invariant to the rotations in the

$j\sigma_3$  plane, which gives a geometric picture of the phase invariance. Notice here the role of pseudoscalars and bivectors, which, unlike ordinary imaginary unit, immediately give a clear geometric meaning to quantities in the theory. This is definitely a good motivation for the study of quantum mechanics in this new language. Instead of non-intuitive matrices over complex numbers we now have elements of geometric algebra, which always introduce clarity. And we were just a little peek into the area ...

Now let's look at observables in the Pauli theory. We will assume that we can separate the spatial and spin components. The *inner* product in quantum mechanics is defined as

$$\langle \psi | \phi \rangle = (\psi_1^*, \psi_2^*) \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \psi_1^* \phi_1 + \psi_2^* \phi_2 .$$

The real part can now be found as

$$\text{Re} \langle \psi | \phi \rangle \leftrightarrow \langle \psi^\dagger \phi \rangle ,$$

for example

$$\text{Re} \langle \psi | \psi \rangle \leftrightarrow \langle \psi^\dagger \psi \rangle = \langle (a^0 - a^j j \sigma_j) (a^0 + a^j j \sigma_j) \rangle = \sum_{k=0}^3 a^k a^k .$$

We have

$$\langle \psi | \phi \rangle = \text{Re} \langle \psi | \phi \rangle - i \text{Re} \langle \psi | i \phi \rangle$$

so we can find the analogy (do not confuse  $\langle a | b \rangle$  with  $\langle ab \rangle$  - grade 0)

$$\langle \psi | \phi \rangle \leftrightarrow \langle \psi^\dagger \phi \rangle - \langle \psi^\dagger \phi j \sigma_3 \rangle j \sigma_3 .$$

Here we have  $\langle \psi^\dagger \phi \rangle$ , grade 0 of the product  $\psi^\dagger \phi$ , and  $\langle \psi^\dagger \phi j \sigma_3 \rangle j \sigma_3$ , the projection of the product  $\psi^\dagger \phi$  to the plane  $j\sigma_3$ . To such a projection for the arbitrary multivector  $A$  we can give a special designation, say  $A_{\dagger} \equiv \langle A \rangle - \langle A j \sigma_3 \rangle j \sigma_3$ , and for even multivectors it becomes ([show that](#))

$$A_{\dagger} = (A + \sigma_3 A \sigma_3) / 2, \quad A \text{ is even,}$$

so if we recall reflections we can conceive a geometric meaning. [It is clear that the real part of  \$A\$  is not to change, while for the bivector part we leave to the reader to explore the possibilities.](#)

Let's look for the expected value of the spin  $\langle \psi | \hat{s}_k | \psi \rangle$ . We demand

$$\langle \psi | \hat{\sigma}_k | \psi \rangle \leftrightarrow \langle \psi^\dagger \sigma_k \psi \sigma_3 \rangle - \langle \psi^\dagger \sigma_k \psi j \rangle j \sigma_3 .$$

If we take advantage of the reverse involution it follows

$$(\psi^\dagger \sigma_k \psi j)^\dagger = j^\dagger \psi^\dagger \sigma_k \psi = -\psi^\dagger \sigma_k \psi j ,$$

which means that there is no grades 0 or 1, so must be  $\langle \psi^\dagger \sigma_k \psi j \rangle = 0$ , what we expect because  $\hat{\sigma}_k$  are Hermitian operators. Element  $\psi^\dagger \sigma_k \psi$  has odd grades only and is equal to its reverse, so, it is a vector. Using that we define the *spin vector* as

$$s \equiv \frac{1}{2} \hbar \psi \sigma_3 \psi^\dagger .$$

The expected value is now

$$\langle \psi | \hat{s}_k | \psi \rangle = \frac{1}{2} \hbar \langle \sigma_k \psi \sigma_3 \psi^\dagger \rangle = \sigma_k \cdot s .$$

This expression is different from what we are accustomed in quantum mechanics. Instead of calculating the expected value of the operator here we have a simple projection of the spin vector on the desired direction in space. This immediately raises the question of co-existence of all three components of the spin vector. The problem does not really exist, reader is referred to the article *Doran et al, 1996b*.

We can use our form of spinors and define the scalar  $\rho = \psi \psi^\dagger$ , then if we define

$$R = \rho^{-1/2} \psi$$

we see that  $RR^\dagger = 1$ , so we have a rotor. According to that, spinors here are just rotors with dilatation, and spin vector is

$$s = \frac{1}{2} \hbar \rho R \sigma_3 R^\dagger .$$

It follows that the form of the expected value is just instruction for the rotation of the fixed vector  $\sigma_3$  in the direction of the spin vector, followed by its dilatation. Note again a clear geometric meaning, which is not so easy to achieve in the quantum theory as is usually formulated.

Let us imagine now that we want to rotate the spin vector, so let's introduce a transformation  $s \rightarrow R_0 s R_0^\dagger$ . In doing so, the spinor must transform as  $\psi \rightarrow R_0 \psi$  ([show that](#)), what is often taken as a way to identify an object as a spinor. Similar property we have for already mentioned *eigenspinors* in the special theory of relativity, which under the general Lorentz transformation transform as  $\Lambda \rightarrow L \Lambda$ , i.e. not "sandwich" form, but „spinor“ form of transformation. [We leave to the reader that, taking into account just shown property of transformation, show that spinors change sign after rotation by  \$2\pi\$](#) . This is a result that is also clear in "ordinary" quantum theory, but here we see that there is nothing "quantum" in this phenomenon, it's actually the property of our 3D space. It is certainly not insignificant conclusion, one could say that we have good reason to re-examine the fundamentals (and philosophy, if you like) of quantum theory. And again, all that just due to the new product of vectors. So, if you want, this can also be support to grounds for the new multiplication of vectors.

## Differentiation and integration

Here we will only briefly comment on this area, reader is referred to the literature. Geometric algebra contains a powerful apparatus of differential and integral calculus. It should be no surprise that here we also have a significant improvement over the traditional approach. In particular, there is a *fundamental theorem of calculus* which combines and expands many well-known theorems of classical integral calculus. In addition, all elements of the algebra (including a full multivectors) can be on equal footing included in the calculus, so we can derive in the "direction" of the multivector, bivector for example. It is, in fact, very nice feature! What essentially distinguishes the classical theory of geometric algebra is reflected in the several elements.

First, in the differential calculus we use various "operators" containing elements of algebra, so due to the properties of non-commutativity provide us new opportunities. Let's look at an example of such an "operator"

$$\nabla A = e^k \partial_k A, \quad \partial_k A \equiv \frac{\partial}{\partial x^k} A,$$

Einstein summation convention is understood. Here we introduced again vectors of reciprocal basis, but in the orthonormal systems they are equal to base vectors, here it is convenient for the Einstein summation convention and for possible generalization. Notice that  $\nabla \equiv e^k \partial_k$  has grade 1, i.e. acts as a vector. In *Cl3* is often defined the „operator“

$$\partial = \partial_t - \nabla,$$

which has the form of paravector. We can derive from left and from right, however, in this we have a geometric product of basis vectors and it is not commutative. Operators of derivation, as elements of algebra, can have inverse. For example, Maxwell's equations can without much effort be written in the form

$$\partial F = J,$$

and this makes it possible to find the inverse of "operator"  $\partial$  using Green's functions. In this way, this simple mathematical form of the equation is not just a "neat trick" but actually provides features that without the geometric product would not exist (or it would be difficult to achieve). Note again that the "operators" are also elements of algebra, therefore here we do not consider them as operators. As interesting examples of the power of geometric algebra in electromagnetism see [10], or [2].

Second, in the integral calculus we encounter with the *measure*, objects like  $dx dy$ . In geometric algebra such objects have the orientation (like blades), which gives many possibilities. For example, unification of all the important theorems of classical integral calculus (Stokes, Gauss, Green, Cauchy ...) in one is great and inspiring achievement of geometric algebra. We refer the reader ready to learn this beautiful, but nontrivial topic, to the literature [18], [20] (on the internet you can search for the phrase "geometric calculus").

## Geometric models

It is known that the geometry of the 3D space can be formulated inserting 3D vector space in the space of higher dimension. Geometric algebra is an ideal framework for the investigation of such models. Here we will only briefly discuss one of them, conformal (developed and patented by *Hestenes*). The idea is that the  $n$ -dimensional Euclidean vector space examines in  $n+2$ -dimensional Minkowski space. For  $n=3$  apart from the usual unit vectors let's introduce two more:  $e$ ,  $e^2=1$  and  $\bar{e}$ ,  $\bar{e}^2=-1$ , so we have the basis

$$\{e, e_1, e_2, e_3, \bar{e}\}.$$

This is an orthonormal basis of 5D vector space  $\mathfrak{R}^{4,1}$ . Two added unit vectors define 2D subspace  $\mathfrak{R}^{1,1}$  in which we will introduce a new basis  $\{o, \infty\}$  (character  $\infty$  we use here as the designation of the vector that represents the point at infinity, while character  $o$  represents the origin)

$$o = (e + \bar{e})/2, \quad \infty = \bar{e} - e.$$

Factor  $\frac{1}{2}$  does not play an important role, may be dropped, but then the rest of formulas have a little different form. **Show that  $o$  and  $\infty$  are nilpotents and that  $e \wedge \bar{e} = o \wedge \infty$ ,  $o \cdot \infty = \infty \cdot o = -1$ ,  $o \cdot o = \infty \cdot \infty = 0$ .** Now we have a new basis

$$\{o, e_1, e_2, e_3, \infty\}$$

in which geometric elements such as lines or circles have a simple form.

If we have two points in  $\mathfrak{R}^3$  and two vectors  $\mathbf{p}$  and  $\mathbf{q}$  coming out of the common origin and end up in our points, square distance of points is given as  $(\mathbf{p}-\mathbf{q}) \cdot (\mathbf{p}-\mathbf{q})$ . The idea is to find vectors  $\mathbf{p}$  and  $\mathbf{q}$  in this algebra which inner product will give us the distance of 3D points (up to factor), i.e.  $\mathbf{p} \cdot \mathbf{q} \approx (\mathbf{p}-\mathbf{q}) \cdot (\mathbf{p}-\mathbf{q})$ . In this case, it should be  $\mathbf{p} \cdot \mathbf{p} = 0$ , because the distance is zero, so such vectors are called *null-vectors*. Accordingly, the points are represented by null-vectors in this model, so it can be shown that for 3D vector  $\mathbf{p}$  corresponding null vector is given by

$$\mathbf{p} = o + \mathbf{p} + \mathbf{p}^2 \infty / 2,$$

where  $\mathbf{p} \cdot \mathbf{p} = 0$  ([check it](#)). Find  $\mathbf{p} \cdot \mathbf{q}$ . The conformal model points can have *weight*, but here we will not deal with it, except for note that weight has a geometric meaning, for example, the weight can show the way in which straight line and plane are intersecting. The vectors of the model that are not points (are not the null-vectors) can represent a variety of geometric elements. Take the vector  $\boldsymbol{\pi} = \mathbf{n} + \lambda \infty$  as an example, and if we want to find all the points  $\mathbf{x}$  that would belong to such an object we have to write the condition  $\mathbf{x} \cdot \boldsymbol{\pi} = 0$ , which means that the distance between the point represented by  $\mathbf{x}$  and the point represented by  $\boldsymbol{\pi}$  is zero. We have

$$\mathbf{x} \cdot \boldsymbol{\pi} = (o + \mathbf{x} + \mathbf{x}^2 \infty / 2) \cdot (\mathbf{n} + \lambda \infty) = \mathbf{x} \cdot \mathbf{n} - \lambda = 0,$$

so we see that we have the equation of the plane perpendicular to the vector  $\mathbf{n}$ , with the distance from the origin  $\lambda/n$ . If we recall that circle in 3D is defined by three points we could appreciate the fact that a circle in this model we get easily: make the outer product of their null-vectors. If one of the points is  $\infty$  we get the straight line. It cannot be easier than that. It is particularly important that transformations of elements can be implemented using a single formalism, thus, for example, the same formalism operate rotations and translations. Interested reader can find beautifully exposed the theory in [19], where you can take advantage of the software that accompanies the book: *GAViewer*. Everything is available free on the Internet.

## Appendix

### A1. Some properties of Pauli matrices

Let's look at Pauli matrices and some of their properties. For linear combinations of Pauli matrices we have

$$a = \sum_{i=1}^3 a_i \hat{\sigma}_i \quad \text{and} \quad b = \sum_{i=1}^3 b_i \hat{\sigma}_i, \quad a_i, b_i \in \mathfrak{R}.$$

It follows

$$a^2 = aa = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \sum_{i=1}^3 a_i^2,$$

so  $a$  behaves like a vector. Also we have

$$\frac{ab + ba}{2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \sum_{i=1}^3 a_i b_i,$$

Which means that Pauli matrices could be interpreted as unit vectors (we have a scalar product of vectors  $\vec{a} = \sum_{i=1}^3 a_i e_i$  and  $\vec{b} = \sum_{i=1}^3 b_i e_i$ ). Of course, it means that products of unit vectors  $e_i$  should be non-commutative (as for matrices). If we find antisymmetric part

$$\frac{ab - ba}{2}$$

and multiply it by matrix

$$\begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix}.$$

we obtain coefficients of the cross product ([show that](#)). From  $\hat{\sigma}_i \hat{\sigma}_j = -\hat{\sigma}_j \hat{\sigma}_i$  follows

$$(\hat{\sigma}_i \hat{\sigma}_j)^2 = \hat{\sigma}_i \hat{\sigma}_j \hat{\sigma}_i \hat{\sigma}_j = -\hat{\sigma}_i \hat{\sigma}_i \hat{\sigma}_j \hat{\sigma}_j = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

So, we have objects with the negative square, they are not vectors, obviously. It means that matrix  $\hat{\sigma}_i \hat{\sigma}_j$  does not represent a vector. Here we have a problem of geometric interpretation. But, with unit vectors we have  $e_i e_j$  which clearly gives us geometric meaning (oriented parallelogram), defining a

plane along the way. Similarly,  $\hat{\sigma}_1 \hat{\sigma}_2 \hat{\sigma}_3$  is just product of the unit matrix and imaginary unit, but  $e_1 e_2 e_3$  is an oriented volume that squares to -1 and commutes with all vectors, which means that we have again an „imaginary unit“, but this time with a clear geometric interpretation. Finally, if we seek for the 2D matrix representation of  $Cl_3$  we get Pauli matrices as a solution. The very existence of the matrix representation proves that  $Cl_3$  is an well-defined algebra.

## A2. Everything is a “boost”

For the complex vector  $F = \vec{v} + j\vec{w}$  we have  $W = \sqrt{F^2} \in \mathbb{C}$  or  $\sqrt{F^2} = F$ , so for  $F \neq N$ ,  $N^2 = 0$  we define  $F/W = f$ ,  $f^2 = 1$  and  $F/\sqrt{-F^2} = I = -jf$ ,  $I^2 = -1$ . Suppose we have the exponential form  $\exp(\varphi f)$ , defining  $\tanh \varphi = W$ ,  $\Gamma = 1/\sqrt{1-W^2}$ ,  $\kappa = \sqrt{(1+W)/(1-W)} = \Gamma(1+W)$  (generalized *Bondi factor*,  $\varphi = \ln \kappa$ ) and the idempotents  $f_{\pm} = (1 \pm f)/2$ ,  $f_+ f_- = 0$  we have

$$e^{\varphi f} = \cosh \varphi + f \sinh \varphi = \Gamma(1+Wf) = \kappa f_+ + \kappa^{-1} f_-.$$

Now we can read “speed” as ratio  $W/1$  and it is easy to find the successive “boosts” as

$$e^{\varphi_1 f} e^{\varphi_2 f} = e^{(\varphi_1 + \varphi_2) f} = \Gamma_1 \Gamma_2 (1+W_1 f)(1+W_2 f) = \Gamma_1 \Gamma_2 (1+W_1 W_2) \left( 1 + \frac{W_1 + W_2}{1+W_1 W_2} f \right),$$

$$\Gamma = \Gamma_1 \Gamma_2 (1+W_1 W_2), \quad W = \frac{W_1 + W_2}{1+W_1 W_2},$$

or

$$e^{\varphi_1 f} e^{\varphi_2 f} = (\kappa_1 f_+ + \kappa_1^{-1} f_-)(\kappa_2 f_+ + \kappa_2^{-1} f_-) = \kappa_1 \kappa_2 f_+ + \kappa_1^{-1} \kappa_2^{-1} f_- \Rightarrow \kappa = \kappa_1 \kappa_2.$$

Generally we have a complex scalar  $\varphi = \ln \kappa = \varphi_R + j\varphi_I$  (explicit formulae for  $\varphi_R$  and  $\varphi_I$  are rather cumbersome, one can use *Mathematica* and  $j \rightarrow i$ , where  $i$  is the ordinary imaginary unit) which leads to  $\exp(\varphi f) = \exp(\varphi_R f) \exp(\varphi_I jf)$ .

From  $F = \vec{v} + j\vec{w}$ ,  $W = \sqrt{(\vec{v} + j\vec{w})^2} = \sqrt{v^2 - w^2 + 2j\vec{v} \cdot \vec{w}}$ , for  $\vec{w} = 0$  we have the well-known relations for the boosts in restricted special relativity.



From  $F = j\vec{w}$  we have  $W = \sqrt{(j\vec{w})^2} = \sqrt{-w^2} = jw$ ,  $f = j\vec{w} / jw = \hat{w}$ ,  $\Gamma = 1/\sqrt{1+w^2}$ ,  $\kappa = \sqrt{(1+jw)/(1-jw)}$ ,  $\varphi = \log \kappa = j \arctan w$ ,  $\exp(\varphi\hat{w}) = \Gamma(1+jw\hat{w})$  and for the successive transformations we have

$$\Gamma = \Gamma_1 \Gamma_2 (1 - w_1 w_2), \quad w = (w_1 + w_2) / (1 - w_1 w_2).$$

It is an interesting possibility to interpret such transformations like “boosts”, defining new rotating frame of reference with time  $t = \Gamma \tau$  ( $\tau$  is a “proper time”), introducing thus a rotating frames as “inertial”. Regarding the invariance of MA instigates to reexamine paradigm “inertial frame of reference”.

For the well known pure rotations  $\exp(\theta j\hat{n})$  we have  $\theta j\hat{n} = j\theta(j\theta\hat{n}) / j\theta$ ,  $\varphi = j\theta$ ,  $W = \tanh(j\theta) = j \tan \theta$ ,  $\Gamma = 1/\sqrt{1+\tan^2 \theta}$ ,  $\kappa = \sqrt{(1+j \tan \theta)/(1-j \tan \theta)}$ ,  $f_{\pm} = (1 \pm \hat{n}) / 2$  and so

$$\begin{aligned} e^{\theta j\hat{n}} &= \Gamma(1 + j\hat{n} \tan \theta), \\ \Gamma &= \Gamma_1 \Gamma_2 (1 - \tan \theta_1 \tan \theta_2), \\ \tan \theta &= (\tan \theta_1 + \tan \theta_2) / (1 - \tan \theta_1 \tan \theta_2) = \tan(\theta_1 + \theta_2). \end{aligned}$$

## Acknowledgement

I would like to express gratitude to Eckhard Hitzer for kind words and support.

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Yao, Yuan: *New relativistic solutions for classical charges in an electromagnetic field*, thesis, University of Windsor

Small amount of literature is offered intentionally, but, due to current possibilities of obtaining information, we will give a list of names of people active in this area, so the reader can easily obtain the necessary articles, books, etc.

## 3D geometry

[http://geocalc.clas.asu.edu/GA\\_Primer/GA\\_Primer/index.html](http://geocalc.clas.asu.edu/GA_Primer/GA_Primer/index.html)

## Some web resources

<https://gaupdate.wordpress.com>

<http://geocalc.clas.asu.edu>

<https://staff.science.uva.nl/l.dorst/clifford/>

## Software

Cinderella

<https://www.cinderella.de/tiki-index.php>

clifford.m, Mathematica package, <https://arxiv.org/abs/0810.2412>

<http://www.fata.unam.mx/investigacion/departamentos/nanotec/aragon/software/clifford.m>

CLIFFORD

<http://math.tntech.edu/rafal/>

Clifford algebra for CAS Maxima

<https://github.com/dprodanov/clifford>

Clifford Multivector Toolbox for MATLAB

<http://clifford-multivector-toolbox.sourceforge.net/>

CLUCalc/CLUViz

<http://www.clucalc.info/>

GA20 and GA30 (Formerly called pauliGA)

<https://github.com/peeterjoot/gapauli>

Gaalet

<https://sourceforge.net/projects/gaalet/>

Gaalop

<http://www.gaalop.de/>

GABLE

<https://staff.fnwi.uva.nl/l.dorst/GABLE/index.html>

Gaigen

<https://sourceforge.net/projects/g25/>

GAlgebra

<https://github.com/brombo/galgebra>

GA Sandbox

<https://sourceforge.net/projects/gasandbox/>

GAViewer, <http://www.geometricalgebra.net/downloads.html>, this nice tool is recommended with the text. You can manipulate the images to some extent.

GluCat

<https://sourceforge.net/projects/glucats/>

GMac

<https://gacomputing.info/gmac-info/>

SpaceGroupVisualizer

<http://spacegroup.info/>

The GrassmannAlgebra package

<https://sites.google.com/site/grassmannalgebra/thegrassmannalgebrapackage>

Versor

<http://versor.mat.ucsb.edu/>

Some important details can be found at <https://gacomputing.info/ga-software/>.

## List of names of people in the field of geometric algebra

(in alphabetical order)

Those I did not remember, or those who I do not know for, forgive me, list will be expanding.

Abbott, Derek	<b>Clifford, William Kingdon</b>
Ablamowicz, Rafal	Colapinto, Pablo
Aharonov, Yakir	Conte, Elio
Almeida, José	Cory, David G.
Aragón-Camarasa, Gerardo	Dargys, Adolfas
Aragón, José Luis	Denker, John
Aristidou, Andreas	Dixon, Geoffrey
Artin, Emil	Doran, Chris
Arthur, John	Dorst, Leo
Artūras, Acus	Dresden, Max
Augello, Agnese	Dress, Andreas
Babalic, Elena-Mirela	Eid, Ahmad Hosny
Barbosa, Afonso Manuel	Falcón, Luis Eduardo
Bajguz, Wieslaw	Farach, Horacio A.
Bartsch, Thomas	Farouki, Rida T.
Baugh, James	Fernandes, Leandro
Baylis, William	Fernandez, Virginia
Bayro-Corrochano, Eduardo	Figuerola-O'Farrill, José
Benger, Werner	Finkelstein, David Ritz
Bengtsson, Ingemar	Fontijne, Daniel
Blanchfield, Kate	Franchini, Silvia
Bouma, Timaeus	Francis, Matthew
Brackx, Freddy	Galiautdinov, Andrei
Brini, Andrea	Gentile, Antonio
Bromborsky, Alan	Goldman, Ronald
Browne, John	<b>Grassmann, Hermann</b> (first ideas)
Buchholz, Sven	Gull, Stephen
Cameron, Jonathan	Gunn, Charles
Campbell, Earl	Gunn, Lachlan
Castro, Carlos	Halma, Arvid
Challinor, Anthony	Havel, Timothy F.
Chapell, James	Henselder, Peter
Chisholm, John Stephen Roy	<b>Hestenes, David (big dog)</b>
Chisolm, Eric	Hilley, Basil
Cibura, Carsten	

Hildenbrand, Dietmar  
 Hitzer, Eckhard  
 Horn, Martin Erik  
 Howard, Mark  
 Ichikawa, Daisuke  
 Iqbal, Azhar  
 Ivezić, Tomislav  
 Jancewicz, Bernard  
 Jones, George Liewellyn  
 Joot, Peeter  
 Kanatani, Kenichi  
 Klawitter , Daniel  
 Kocik, Jerzy  
 Kosokowsky, David E.  
 Kumar, Datta Bidyut  
 Kuroe, Yasuaki  
 Kwasniewski, Andrzej Krzysztof  
 Lachièze-Rey, Marc  
 Lasenby, Anthony  
 Lasenby, Joan  
 Laurinolli, Teuvo  
 Lazaroiu, Calin Luliu  
 Leopardi, Paul  
 Lewis, Antony  
 Li, Hongbo  
 Lounesto, Pertti  
 Luca, Redaelli  
 Lundholm, Douglas  
Lynn, Ben  
 Mann, Stephen  
 Matos, Sérgio  
 Matzke, Douglas  
 Mawardi, Bahri  
 Macdonald, Alan  
 Meinrenken, Eckhar  
 Miller, Richard Alan  
 Mornev, Oleg  
 Moya, Antonio Manuel  
 Naeve, Ambjorn  
 Najfeld, Igor  
 Nesterov, Michael M.  
 Nitta, Tohru  
 Oliveira, Manuel  
 Paiva, Carlos  
 Parkin, Spencer T.  
 Parra, Josep  
 Pavšič, Matej  
 Perwass, Christian  
 Pesonen, Janne  
 Pezzaglia, William  
 Poole, Charles P. Jr  
 Porteous, Ian R.  
 Pozo, Jose Maria  
 Regonati, Francesco  
 Renaud, Peter  
 Riesz, Marcel  
 Rocha, Marlene Lucete Matias  
 Rockwood, Alyn  
 Rodrigues, Waldyr  
 Salingaros, Nikos  
 Setiawan, Sandi  
 Seybold, Florian  
 Schönberg, Mário  
 Shirokov, Dmitry  
 Snygg, John  
**Sobczyk, Garrett Eugene**  
 Soiguine, Alexander  
 Somaroo, Shyamal  
 Sommen, Frank  
 Sommer, Gerald  
 Sorbello, Filippo  
 Sugon, Quirino  
 Suter, Jaap  
 Svenson, Lars  
 Tachibana, Kanta  
 Tarkhanov, Victor



Teolis, Antonio  
Tisza, Laszlo  
Trayling, Greg  
Vargas, José  
Varlamov, Vadim  
Vassallo, Giorgio  
Velebil, Jirka  
Vugt, Floris T. van  
Zamora, Julio  
Zhang, Hua

Zhaoyuan, Yu  
Yao , Yuan  
Wang, Dongming  
Wareham, Rich  
Wei, Jiansu  
Wei, Lu  
Witte, Frank  
Wu, Dimin

