Closed forms, (including Gelfond ' s Constant), using MKB constant like integrals.

Marvin Ray Burns
(part – time undergraduate student and number harvester as of 2016)

Abstract

First we will consider

$\int_1^\infty \text{Cos}[\text{Pi}*I*x]\, x^{1/x}\, dx$ **and** $\int_1^\infty \text{Sin}[\text{Pi}*I*x]\, x^{1/x}\, dx,$

where it appears that the limit of the ratio of a **to** $a-1$, **as** a **goes to infinity, in integral(Sin[Pi** $*I*x$**]** x**^(1/**x**), {**x**, 1,** a**})/integral(Sin[Pi** $*I*x$**]** x**^(1/**x**), {**x**, 1,** $a-1$**}) and integral(Cos[Pi** $*I*x$**]** x**^(1/**x**), {**x**, 1,** a**})/integral(Cos[Pi** $*I*x$**]** x**^(1/**x**), {**x**, 1,** $a-1$**}) is Gelfond ' s Constant, (**e**^pi). We will consider that the hypothesis and provide hints for** a **proof using** L**' Hospital' s Rule, since we have indeterminate forms as** a **goes to infinity.**

We shall compare the ratios of

$\int_1^\infty \text{Cos}[\text{Pi}*I*x]\, x^{1/x}\, dx$ **and** $\int_1^\infty \text{Sin}[\text{Pi}*I*x]\, x^{1/x}\, dx$ **with** $\int_1^\infty \text{Cos}[\text{Pi}*x]\, x^{1/x}\, dx$ **and** $\int_1^\infty \text{Sin}[\text{Pi}*x]\, x^{1/x}\, dx.$

In later parts of this paper we will look at the ratio of three deceptively similar looking integrals which have the terms of MKB for integrands. (For reference we call these the "later integrals.")
Finally after finding no closed form for the ratios of the later integrals, we will settle for closed forms of their differences.

Preliminaries

Below, and throughout this paper, we will show the best known (to the author) Mathematica 11 options, found to give the desired results. Many of the computations took several minutes and produced a few warning messages which are not displayed.
In 1999[1], the constant referred to at https://oeis.org/A037077, Limit[Sum[(-1)^n n^(1/n),{n,1,2x}],x->Infinity] [2], was named the MRB constant, (after its original investigator), by Simon Plouffe[3]. Then on Feb 23, 2009, Marvin Ray Burns named

https://oeis.org/A157852 “MKB constant” (MKB) after his wife at the time. Technically, A157852 is the integer sequence of the digits of MKB, (the integral analog of the MRB constant[4]). Hence MKB was named after one with a close relationship to the person the MRB constant was named after.

MKB= $\left|\lim_{n\to\infty}\int_1^{2n} e^{i\pi x} x^{1/x}\,dx\right|$.

It appears that

$\lim_{n\to a}\int_1^{2n} e^{i\pi x} x^{1/x}\,dx \simeq \lim_{n\to a}\int_1^{2n}\text{Cos}[\text{Pi}*x]\; x^{1/x}\,dx + I*\lim_{n\to a}\int_1^{2n}\text{Sin}[\text{Pi}*x]\; x^{1/x}\,dx$ and $\lim_{n\to a}\int_1^{2n-1} e^{i\pi x} x^{1/x}\,dx \simeq \lim_{n\to a}\int_1^{2n-1}\text{Cos}[\text{Pi}*x]\; x^{1/x}\,dx + I*\lim_{n\to a}\int_1^{2n-1}\text{Sin}[\text{Pi}*x]\; x^{1/x}\,dx,\ a\in\mathbb{N}$.

They are indicated to be true in the result to the following Mathematica code.

In[37]:=
```
d := 20;
Table[NIntegrate[x^(1/x) Exp[I*Pi*x], {x, 1, 2 n}, WorkingPrecision → d] -
  (NIntegrate[x^(1/x) Cos[Pi*x], {x, 1, 2 n}, WorkingPrecision → d] +
    I NIntegrate[x^(1/x) Sin[Pi*x], {x, 1, 2 n}, WorkingPrecision → d]), {n, 1, 21}]
```

Out[37]= $\{0.\times10^{-21}+0.\times10^{-20}\,i,\ 0.\times10^{-21}+0.\times10^{-20}\,i,\ 0.\times10^{-21}+0.\times10^{-20}\,i,\ 0.\times10^{-21}+0.\times10^{-20}\,i,\ 0.\times10^{-21}+0.\times10^{-20}\,i,\ 0.\times10^{-21}+0.\times10^{-20}\,i,\ 0.\times10^{-21}+0.\times10^{-20}\,i,\ -3.105217\times10^{-14}+0.\times10^{-20}\,i,\ 0.\times10^{-21}+0.\times10^{-20}\,i,\ 0.\times10^{-21}+0.\times10^{-20}\,i,\ 0.\times10^{-21}+0.\times10^{-20}\,i,\ 0.\times10^{-21}+0.\times10^{-20}\,i,\ 0.\times10^{-21}+0.\times10^{-20}\,i,\ 0.\times10^{-21}+0.\times10^{-20}\,i,\ -3.478646\times10^{-14}+0.\times10^{-20}\,i,\ -2.783919\times10^{-14}+0.\times10^{-20}\,i,\ -5.530932\times10^{-14}+0.\times10^{-20}\,i,\ 0.\times10^{-21}+0.\times10^{-20}\,i,\ 0.\times10^{-21}+0.\times10^{-20}\,i,\ 0.\times10^{-21}+0.\times10^{-20}\,i,\ 0.\times10^{-21}+0.\times10^{-20}\,i\}$

In[38]:=
```
d := 20;
Table[NIntegrate[x^(1/x) Exp[I*Pi*x], {x, 1, 2 n - 1}, WorkingPrecision → d] -
  (NIntegrate[x^(1/x) Cos[Pi*x], {x, 1, 2 n - 1}, WorkingPrecision → d] +
    I NIntegrate[x^(1/x) Sin[Pi*x], {x, 1, 2 n - 1}, WorkingPrecision → d]), {n, 1, 21}]
```

Out[38]= $\{0,\ 0.\times10^{-21}+0.\times10^{-21}\,i,\ 0.\times10^{-21}+0.\times10^{-21}\,i,\ 0.\times10^{-21}+0.\times10^{-21}\,i,\ 0.\times10^{-21}+0.\times10^{-22}\,i,\ 0.\times10^{-21}+0.\times10^{-22}\,i,\ 0.\times10^{-21}+0.\times10^{-22}\,i,\ 3.521728\times10^{-14}-2.7706332\times10^{-14}\,i,\ 0.\times10^{-21}+0.\times10^{-22}\,i,\ 0.\times10^{-21}+0.\times10^{-22}\,i,\ 0.\times10^{-21}+0.\times10^{-23}\,i,\ 0.\times10^{-21}+1.3585779\times10^{-15}\,i,\ 0.\times10^{-21}+0.\times10^{-23}\,i,\ 0.\times10^{-21}-9.210306\times10^{-15}\,i,\ 3.521728\times10^{-14}-2.7706332\times10^{-14}\,i,\ 0.\times10^{-21}+0.\times10^{-22}\,i,\ 0.\times10^{-21}+0.\times10^{-22}\,i,\ 0.\times10^{-21}+0.\times10^{-22}\,i,\ 0.\times10^{-21}+0.\times10^{-22}\,i,\ 0.\times10^{-21}+0.\times10^{-22}\,i,\ 0.\times10^{-21}+0.\times10^{-22}\,i\}$

Also, it can be proved that

$\lim_{n\to\infty}\int_1^{2n} e^{i\pi x} x^{1/x}\,dx == \lim_{n\to\infty}\int_1^{2n}\text{Cos}[\text{Pi}*x]\; x^{1/x}\,dx + I*\lim_{n\to\infty}\int_1^{2n}\text{Sin}[\text{Pi}*x]\; x^{1/x}\,dx$.

This is shown to be true up to 23 digits of precision in the result to the following Mathematica code. (Some unknown [to this author] form of regularization is used.) We will need the["NumericalCalculus`"] package.

```
Needs["NumericalCalculus`"]
```

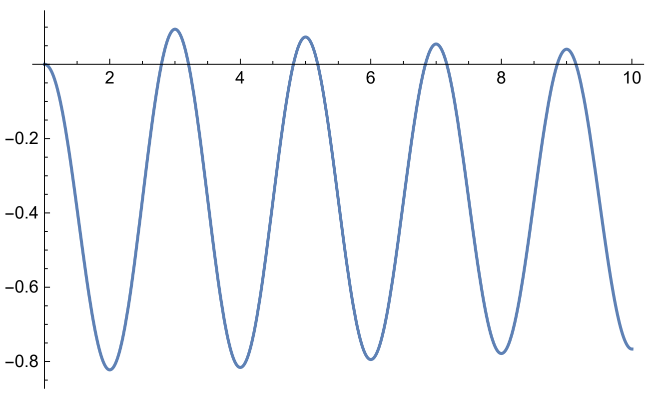
```
digits = 50;
Rationalize[NLimit[NIntegrate[Exp[Pi * I * x] x^ (1 / x), {x, 1, 2 a},
    WorkingPrecision → 50, PrecisionGoal → 50, MaxRecursion → 50], a → Infinity,
   WorkingPrecision → 50, Terms → 15, Method → SequenceLimit, WynnDegree → 6], 0] -
 (NIntegrate[x^ (1 / x) * Cos[Pi * x], {x, 1, Infinity}, WorkingPrecision → 2 * digits] +
   I NIntegrate[x^ (1 / x) Sin[Pi * x], {x, 1, Infinity}, WorkingPrecision → 2 * digits] - I / Pi)
```

$9.799242913445728747553048568494203547251606464399860365576612315013382138315720 \times 10^{-23} - 3.422998270568720522742953726075580777060144547611096469893522817755464792 44989 \times 10^{-23}\, i$

The following plots are hard to interpret, but here are what NIntegrate[x^(1/x)*Cos[Pi*x],{x,1,a}] and NIntegrate[x^(1/x)*Cos[Pi*x],{x,1,a}] for a from 1 to 10 look like. At least we notice that the cosine plot is shifted on both the x and f(x) axes, and they have the same, or about the same period of about 2.

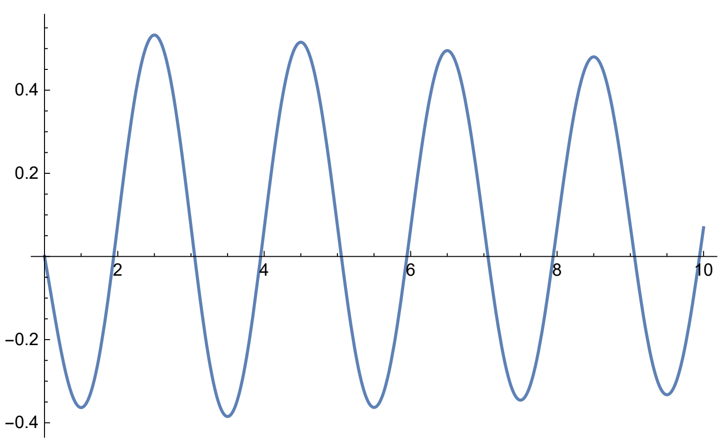
```
In[30] := Plot[Integrate[Sin[Pi * x] x^ (1 / x), {x, 1, a}], {a, 1, 10}]
```

Out[30] =




```
In[25] := Plot[Integrate[Cos[Pi * x] x^ (1 / x), {x, 1, a}], {a, 1, 10}]
```

Out[25] =



For more precise measurements see Mathar's work[5], where Mathar also used many methods to compute MKB. He called it M1, published many error-bounds of methods, and compared M1 to what he called the MRB constant, M. R. Burns' constant, and M.

Tools

We will use many integration and limit options available in Mathematica 11.0. We also use a Intel 6 core 3.5 GH extreme edition desktop for the computations.

Operations and Results

Realizing his mistake of confusing integral(Sin[Pi*I*x]x^(1/x),{x,1,a}) with integral(Sin[Pi*x]x^(1/x),{x,1,a}), through the following operations, on Monday, Aug 8, 2016 at 2:00PM, Marvin Ray Burns began to see that the limit of the ratio of a to a-1, as a goes to infinity, in integral(Sin[Pi*I*x]x^(1/x),{x,1,a})/ integral(Sin[Pi*I*x]x^(1/x),{x,1,a-1}) and integral(-Cos[Pi*I*x]x^(1/x),{x,1,a})/integral(Cos[Pi*I*x]x^(1/x),{x,1,a-1}) is Gelfond's Constant, (e^pi).

.

```
i1 = Table[
  NIntegrate[Sin[Pi * I * x] x^(1/x), {x, 1, a}, WorkingPrecision → 20], {a, 9990, 10001}]
```

{$2.0989175549269083147\times10^{13629}$ ⅈ, $4.8570402004948497061\times10^{13630}$ ⅈ, $1.1239526514136629501\times10^{13632}$ ⅈ, $2.6009040701590741249\times10^{13633}$ ⅈ, $6.0186716707026764644\times10^{13634}$ ⅈ, $1.3927621974178786927\times10^{13636}$ ⅈ, $3.2229479272454941452\times10^{13637}$ ⅈ, $7.4581241228073160071\times10^{13638}$ ⅈ, $1.7258614376558644163\times10^{13640}$ ⅈ, $3.9937625775425609291\times10^{13641}$ ⅈ, $9.2418424666174747439\times10^{13642}$ ⅈ, $2.1386261832231106579\times10^{13644}$ ⅈ}

```
i2 = Ratios[i1]
```

{23.140690729341148011, 23.140690729698949373, 23.140690730056647960, 23.140690730414243812, 23.140690730771736967, 23.140690731129127467, 23.140690731486415350, 23.140690731843600657, 23.140690732200683426, 23.140690732557663699, 23.140690732914541513}

```
N[E^Pi, 20]
```

23.140692632779269006

```
N[i2 - E^Pi, 20]
```

{$-1.903438120995\times10^{-6}$, $-1.903080319633\times10^{-6}$, $-1.902722621046\times10^{-6}$, $-1.902365025194\times10^{-6}$, $-1.902007532038\times10^{-6}$, $-1.901650141539\times10^{-6}$, $-1.901292853656\times10^{-6}$, $-1.900935668349\times10^{-6}$, $-1.900578585579\times10^{-6}$, $-1.900221605307\times10^{-6}$, $-1.899864727493\times10^{-6}$}

Larger a: (Both sine and cosine in the this operation give Gelfond's Constant.)

```
i11 = Table[NIntegrate[Cos[Pi * I * x] x^(1/x), {x, 1, a}, WorkingPrecision → 30],
  {a, 999 990, 1 000 000}]
```

{8.16402704299180404095259449988 × 10^1 364 361,
1.88921240445149992403409339252 × 10^1 364 363, 4.37176835688857590343446623243 × 10^1 364 364,
1.01165747807172488243183607690 × 10^1 364 366, 2.34104547494103980098507246551 × 10^1 364 367,
5.41734137742750968168005476577 × 10^1 364 368, 1.25361031700280490130583526513 × 10^1 364 370,
2.90094110266711182266711546873 × 10^1 364 371, 6.71297864017550964458508807654 × 10^1 364 372,
1.55342975360723204911512762411 × 10^1 364 374, 3.59474404543783042332075392032 × 10^1 364 375}

```
i12 = Ratios[i11]
```

{23.1406926324827036186672961841, 23.1406926324827041886616877912,
23.1406926324827047586544156856, 23.1406926324827053286454798737,
23.1406926324827058986348803621, 23.1406926324827064686226171574,
23.1406926324827070386086902659, 23.1406926324827076085930996944,
23.1406926324827081785758454491, 23.1406926324827087485569275368}

```
N[E^Pi, 20]
```

23.140692632779269006

```
N[i12 - E^Pi, 20]
```

{−2.9656538706179018 39 × 10^-10, −2.965648170673985768 × 10^-10,
−2.965642470746706824 × 10^-10, −2.965636770836064943 × 10^-10, −2.965631070942060058 × 10^-10,
−2.965625371064692106 × 10^-10, −2.965619671203961020 × 10^-10, −2.965613971359866736 × 10^-10,
−2.965608271532409188 × 10^-10, −2.965602571721588312 × 10^-10}

Even larger a,(about as big as Mathematica will tolerate)!

**i21 = Table[NIntegrate[Cos[Pi * I * x] x^(1 / x), {x, 1, a}, WorkingPrecision → 40],
{a, 9 999 990, 10 000 008}]**

{$1.248750590784479766789088244974362158446 \times 10^{13\,643\,749}$,
$2.889695359634080199319377313714578172209 \times 10^{13\,643\,750}$,
$6.686955211965069049415827804642155765176 \times 10^{13\,643\,751}$,
$1.547407752092216146387935592164153432162 \times 10^{13\,643\,753}$,
$3.580808716874046215569412546222015226236 \times 10^{13\,643\,754}$,
$8.286239389394650135109617259085898574115 \times 10^{13\,643\,755}$,
$1.917493187915811799795191512425441918062 \times 10^{13\,643\,757}$,
$4.437212048700115233728772579760159563694 \times 10^{13\,643\,758}$,
$1.026801601654186402948091558294664519330 \times 10^{13\,643\,760}$,
$2.376090025872139277296579342903098022129 \times 10^{13\,643\,761}$,
$5.498436895651140346939155692172340668487 \times 10^{13\,643\,762}$,
$1.272376381629768218143854086101177504926 \times 10^{13\,643\,764}$,
$2.944367076049786971400355256642342436912 \times 10^{13\,643\,765}$,
$6.813469350493284237431892945563446297757 \times 10^{13\,643\,766}$,
$1.576684000026034571600891047822655334778 \times 10^{13\,643\,768}$,
$3.648555982361789107489894116297179355389 \times 10^{13\,643\,769}$,
$8.443011254130941753836957370586199018495 \times 10^{13\,643\,770}$,
$1.953771283269108031257718779721784119022 \times 10^{13\,643\,772}$,
$4.521162074087431137089658317298126596488 \times 10^{13\,643\,773}$}

i22 = Ratios[i21]

{23.140692632775770567040023720954839938866, 23.14069263277577056771657128578128293096,
23.14069263277577056839311865227146209701, 23.14069263277577056906966582042545538291,
23.14069263277577056974621279024334073474, 23.14069263277577057042275956172519609852,
23.14069263277577057109930613487109942026, 23.14069263277577057177585250968112864590,
23.14069263277577057245239868615536172137, 23.14069263277577057312894466429387659253,
23.14069263277577057380549044409675120524, 23.14069263277577057448203602556406350528,
23.14069263277577057515858140869589143843, 23.14069263277577057583512659349231295041,
23.14069263277577057651167157995340598690, 23.14069263277577057718821636807924849356,
23.14069263277577057786476095786991841598, 23.14069263277577057854130534932549369975}

N[E^Pi, 20]

23.140692632779269006

N[i22 – E^Pi, 20]

{$-3.4984386890626469937 \times 10^{-12}$, $-3.4984380125150821673 \times 10^{-12}$,
$-3.4984373359677156771 \times 10^{-12}$, $-3.4984366594205475231 \times 10^{-12}$,
$-3.4984359828735777052 \times 10^{-12}$, $-3.4984353063268062234 \times 10^{-12}$,
$-3.4984346297802330774 \times 10^{-12}$, $-3.4984339532338582674 \times 10^{-12}$,
$-3.4984332766876817932 \times 10^{-12}$, $-3.4984326001417036547 \times 10^{-12}$,
$-3.4984319235959238518 \times 10^{-12}$, $-3.4984312470503423845 \times 10^{-12}$,
$-3.4984305705049592527 \times 10^{-12}$, $-3.4984298939597744562 \times 10^{-12}$,
$-3.4984292174147879951 \times 10^{-12}$, $-3.4984285408699998693 \times 10^{-12}$,
$-3.4984278643254100786 \times 10^{-12}$, $-3.4984271877810186231 \times 10^{-12}$}

When we switch forms of the operations, Mathematica can give at least 42 digits of accuracy here. Again we will need the [“NumericalCalculus`”] package.

```
Needs["NumericalCalculus`"]

N[E^Pi -
   Rationalize[NLimit[NIntegrate[Cos[Pi * I * x] x^(1/x), {x, 1, a}, WorkingPrecision → 50,
       PrecisionGoal → 50, MaxRecursion → 50] / NIntegrate[Cos[Pi * I * x] x^(1/x), {x, 1,
        a - 1}, WorkingPrecision → 50, PrecisionGoal → 50, MaxRecursion → 50], a → Infinity,
     WorkingPrecision → 50, Terms → 15, Method → SequenceLimit, WynnDegree → 6], 0], 50]
```

$-5.97027082450873174295192689200169137001548244048 79 \times 10^{-42}$

```
N[E^Pi -
   Rationalize[NLimit[NIntegrate[Sin[Pi * I * x] x^(1/x), {x, 1, a}, WorkingPrecision → 50,
       PrecisionGoal → 50, MaxRecursion → 50] / NIntegrate[Sin[Pi * I * x] x^(1/x), {x, 1,
        a - 1}, WorkingPrecision → 50, PrecisionGoal → 50, MaxRecursion → 50], a → Infinity,
     WorkingPrecision → 50, Terms → 15, Method → SequenceLimit, WynnDegree → 6], 0], 50]
```

$-5.97027082450873174295192689200169137001548244048 79 \times 10^{-42}$

The following should help in a proof of the hypothesis: Cos[Pi*I*x] == Cosh[Pi*x], Sin[Pi*I*x] == I Sin-h[Pi*x], and Limit[x^(1/x),x->Infinity]==1.
Using L’Hospital’s Rule, we have the following:

$$e^{\pi} - \left(\cosh(\pi) - \lim_{a\to\infty}\left(\cos(\pi\, i) - \frac{x^{1/x}\cos(\pi\, i\, x)\,/.\, x \to a}{x^{1/x}\cos(\pi\, i\, x)\,/.\, x \to a-1}\right)\right)$$

0

$$e^{\pi} - \left(i\sin(1) - \lim_{a\to\infty}\left(-\frac{x^{1/x}\sinh(\pi\, x)\,/.\, x \to a}{x^{1/x}\sinh(\pi\, x)\,/.\, x \to a-1} + \sinh(i)\right)\right)$$

0

If we perform the same operations on only the integrals without the "I," (the later integrals), we get the following results, (proof not provided, since there are no **known** closed form solutions for two of the three ratios of the later integrals). Nonetheless, we will make a passing note to a few approximations.

```
Needs["NumericalCalculus`"]
```

```
NLimit[NIntegrate[Sin[Pi * x] x^(1/x), {x, 1, a}, WorkingPrecision → 50,
    PrecisionGoal → 50, MaxRecursion → 50] / NIntegrate[Sin[Pi * x] x^(1/x),
    {x, 1, a - 1}, WorkingPrecision → 50, PrecisionGoal → 50, MaxRecursion → 50],
  a → Infinity, WorkingPrecision → 50, Terms → 15]
```

14.436291291052278530533116235683502718

That is close to the following:

$$\left(137\,250\,424\,206\,313\, e + 2\,085\,856\,139\,684\,320\, e^2\right) / 1\,093\,465\,977\,254\,055$$

and

$$\left(70\,064\,463\,976\,981\,915\, e^{\pi/2} + 2\,940\,546\,676\,813\,034\, e^{\pi}\right) / 28\,060\,517\,609\,691\,197$$

And for smaller coefficients it is also close the following:

$$\frac{1}{28}\left(-70\, e - 97\, e^2 - 151\, e^3 - 3\, e^4 - 58\, e^5 - 21\, e^6 - 93\, e^7 + 24\, e^8 - 12\, e^9 - 49\, e^{10} - 37\, e^{11} + 19\, e^{12} - 48\, e^{13} + 141\, e^{14} - 87\, e^{15} - 11\, e^{16} - 58\, e^{17} - 46\, e^{18} - 31\, e^{19} + 21\, e^{20}\right)$$

and

$$\frac{1}{38}\left(91\, e^{\pi/20} - 168\, e^{\pi/19} + 40\, e^{\pi/18} + 80\, e^{\pi/17} - 6\, e^{\pi/16} + 8\, e^{\pi/15} + 121\, e^{\pi/14} + 12\, e^{\pi/13} - 8\, e^{\pi/12} + 21\, e^{\pi/11} + 69\, e^{\pi/10} - 11\, e^{\pi/9} + 81\, e^{\pi/8} + 72\, e^{\pi/7} + 38\, e^{\pi/6} + 56\, e^{\pi/5} - 22\, e^{\pi/4} + 38\, e^{\pi/3} - 24\, e^{\pi/2} - 5\, e^{\pi}\right)$$

```
Needs["NumericalCalculus`"]

NLimit[NIntegrate[Cos[Pi * x] x^(1/x), {x, 1, a}, WorkingPrecision → 70,
    PrecisionGoal → 70, MaxRecursion → 60] / NIntegrate[Cos[Pi * x] x^(1/x),
    {x, 1, a - 1}, WorkingPrecision → 70, PrecisionGoal → 70, MaxRecursion → 60],
  a → Infinity, WorkingPrecision → 70, Terms → 15]
```

0.999999999999999999999999999681834364367520582249

We arive at about 1. The random digits could be some hard to avoid round of error, but we make no claim of certianty here.

Next, we will perform our operations on the MKB constant, mentioned at the top of this paper. For reference we will call its integrand "e."

```
Needs["NumericalCalculus`"]
```

```
NLimit[NIntegrate[Exp[Pi * I * x] x^(1/x), {x, 1, a}, WorkingPrecision → 50, PrecisionGoal →
 50, MaxRecursion → 50] / NIntegrate[Exp[Pi * I * x] x^(1/x), {x, 1, a - 1}, WorkingPrecision →
 50, PrecisionGoal → 50, MaxRecursion → 50], a → Infinity, WorkingPrecision → 50, Terms → 15]
```

```
5.158082192458413431339787028600589211 - 6.211244321438292793271496484947123567 ⅈ
```

That is close to the following:

$$\left(3\,635\,966\,283\,803\,821\, e + 902\,615\,612\,958\,241\, e^{2}\right) / 3\,209\,149\,808\,096\,290 - \left(\left(1\,211\,008\,478\,422\,557\, e + 27\,332\,794\,326\,472\, e^{2}\right) / 562\,500\,154\,689\,724\right) I$$

and

$$\left(-1\,515\,494\,137\,444\,387\, e^{\pi/2} + 389\,348\,082\,242\,624\, e^{\pi}\right) / 333\,366\,930\,773\,403 - \left(\left(437\,182\,811\,555\,465\, e^{\pi/2} - 2\,053\,701\,405\,670\, e^{\pi}\right) / 330\,937\,546\,000\,493\right) I$$

And for smaller coefficients it is also close the following:

$$\frac{1}{2}\left(41\, e^{\pi/24} + 16\, e^{\pi/23} + 70\, e^{\pi/22} - 30\, e^{\pi/21} - 55\, e^{\pi/20} - 12\, e^{\pi/19} + 42\, e^{\pi/18} - 32\, e^{\pi/17} + 32\, e^{\pi/16} - 17\, e^{\pi/15} - 26\, e^{\pi/14} + 16\, e^{\pi/13} - 37\, e^{\pi/12} + e^{\pi/11} + 23\, e^{\pi/10} - 19\, e^{\pi/9} + 37\, e^{\pi/8} + 77\, e^{\pi/7} + 30\, e^{\pi/6} - e^{\pi/5} - 44\, e^{\pi/4} + 6\, e^{\pi/3} - 54\, e^{\pi/2} + 5\, e^{\pi}\right) - \frac{i}{6}\left(-5\, e^{\pi/27} - 30\, e^{\pi/26} + 11\, e^{\pi/25} - 6\, e^{\pi/24} - 34\, e^{\pi/23} - 25\, e^{\pi/22} + 20\, e^{\pi/21} + 50\, e^{\pi/20} + 31\, e^{\pi/19} + 8\, e^{\pi/18} + 14\, e^{\pi/17} - 59\, e^{\pi/16} + 17\, e^{\pi/15} - 37\, e^{\pi/14} + 61\, e^{\pi/13} + 4\, e^{\pi/12} - 12\, e^{\pi/11} + 51\, e^{\pi/10} + 38\, e^{\pi/9} - 41\, e^{\pi/8} + 17\, e^{\pi/7} - e^{\pi/6} + 14\, e^{\pi/5} - 27\, e^{\pi/4} - 9\, e^{\pi/3} - e^{\pi/2}\right)$$

and

$$\frac{1}{3}\left(-11\, e - 12\, e^{2} - 26\, e^{3} - 34\, e^{4} + 19\, e^{5} + 13\, e^{6} - 13\, e^{7} + 19\, e^{8} - 52\, e^{9} + 26\, e^{10} + 36\, e^{11} - 30\, e^{12} + 14\, e^{13} + 6\, e^{14} - e^{16} - 45\, e^{17} - 8\, e^{18} + 28\, e^{19} - 28\, e^{20} + 18\, e^{21} + 8\, e^{22} + 12\, e^{23} + 34\, e^{24} - 12\, e^{25} - e^{26}\right) - \frac{i}{4}\left(-18\, e + 49\, e^{2} - 23\, e^{3} + 31\, e^{4} + 4\, e^{5} + 78\, e^{6} + 27\, e^{7} - 23\, e^{8} + 43\, e^{9} - 22\, e^{10} - 6\, e^{11} - 19\, e^{12} - 51\, e^{13} + 2\, e^{14} + 22\, e^{15} - 6\, e^{16} - 44\, e^{17} - 5\, e^{18} - 58\, e^{19} + 7\, e^{20} - 15\, e^{21} - 41\, e^{22} + 5\, e^{23} - 28\, e^{24} - 26\, e^{25} + 14\, e^{26}\right)$$

Using Mathematica, the limit of the **differences** of the later integrals appear to go by the following rule: cosine->0; sine-> -2/Pi, and e-> -2/Pi * I; (reaching as far as 20 to 28 digits of precision).

```
Needs["NumericalCalculus`"]
```

```
NLimit[NIntegrate[Cos[Pi * x] x^ (1 / x), {x, 1, a}, WorkingPrecision → 50,
   PrecisionGoal → 50, MaxRecursion → 50] - NIntegrate[Cos[Pi * x] x^ (1 / x),
   {x, 1, a - 1}, WorkingPrecision → 50, PrecisionGoal → 50, MaxRecursion → 50],
 a → Infinity, WorkingPrecision → 50, Terms → 15]
```

$0. \times 10^{-28}$

```
Needs["NumericalCalculus`"]
```

```
-2 / Pi - NLimit[
  NIntegrate[Sin[Pi * x] x^ (1 / x), {x, 1, a}, WorkingPrecision → 50, PrecisionGoal → 50,
    MaxRecursion → 50] - NIntegrate[Sin[Pi * x] x^ (1 / x), {x, 1, a - 1}, WorkingPrecision → 50,
    PrecisionGoal → 50, MaxRecursion → 50], a → Infinity, WorkingPrecision → 50, Terms → 30]
```

1.6163×10^{-20}

```
Needs["NumericalCalculus`"]
```

```
-2 / Pi I - NLimit[NIntegrate[Exp[Pi * I * x] x^ (1 / x), {x, 1, a}, WorkingPrecision → 50,
    PrecisionGoal → 50, MaxRecursion → 50] - NIntegrate[Exp[Pi * I * x] x^ (1 / x),
    {x, 1, a - 1}, WorkingPrecision → 50, PrecisionGoal → 50, MaxRecursion → 50],
  a → Infinity, WorkingPrecision → 50, Terms → 30]
```

$0. \times 10^{-25} + 1.616 \times 10^{-20}\, i$

Open Question

Do there exist closed form solutions for the ratios of the later integrals, specifically those with the integrands sine and e? And maybe we might ask about cosine too.

Sumary

The limits of ratios of the hyperbolic integrals and the limits of limits of the differences of the later integrals had exact solutions that were easy to recognize, while we have not found exact values for the limits of ratios of the later integrals, except, perhaps cosine which seems to give 1. The limits of differences of the later integrals have known closed forms.

References

1 S. R. Finch, Mathematical Constants, Cambridge, 2003, pp 450, 452

2 http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.695.5959&rep=rep1&type=pdf page 28.

3 http://www.plouffe.fr/simon/articles/Tableofconstants.pdf

4 www.people.fas.harvard.edu/~sfinch/csolve/erradd.pdf p. 64.

5 http://arxiv.org/abs/0912.3844