

Closed forms, (including Gelfond's Constant), using MKB constant like integrals.

Marvin Ray Burns
(part – time undergraduate student and number harvester as of 2016)

Abstract

First we will consider

$$\int_1^{\infty} \text{Cos}[\text{Pi} * I * x] x^{1/x} dx \text{ and } \int_1^{\infty} \text{Sin}[\text{Pi} * I * x] x^{1/x} dx,$$

where it appears that the limit of the ratio of a to $a - 1$, as a goes to infinity,

in $\text{integral}(\text{Sin}[\text{Pi} * I * x] x^{1/x}, \{x, 1, a\}) / \text{integral}(\text{Sin}[\text{Pi} * I * x] x^{1/x}, \{x, 1, a - 1\})$ and
 $\text{integral}(\text{Cos}[\text{Pi} * I * x] x^{1/x}, \{x, 1, a\}) / \text{integral}(\text{Cos}[\text{Pi} * I * x] x^{1/x}, \{x, 1, a - 1\})$ is Gelfond's
Constant, $(e^{\wedge} \text{pi})$. We will consider that the hypothesis and provide hints for a proof using L' Hospital's
Rule, since we have indeterminate forms as a goes to infinity.

We shall compare the ratios of

$$\int_1^{\infty} \text{Cos}[\text{Pi} * I * x] x^{1/x} dx \text{ and } \int_1^{\infty} \text{Sin}[\text{Pi} * I * x] x^{1/x} dx \text{ with } \int_1^{\infty} \text{Cos}[\text{Pi} * x] x^{1/x} dx \text{ and } \int_1^{\infty} \text{Sin}[\text{Pi} * x] x^{1/x} dx.$$

In later parts of this paper we will look at the ratio of three deceptively similar looking integrals

which have the terms of MKB for integrands. (For reference we call these the "later integrals.")

Finally after finding no closed form for the ratios of the later integrals,

we will settle for closed forms of their differences.

Preliminaries

Below, and throughout this paper, we will show the best known (to the author) Mathematica 11 options, found to give the desired results. Many of the computations took several minutes and produced a few warning messages which are not displayed.

In 1999¹, the constant referred to at <https://oeis.org/A037077>,

$\text{Limit}[\text{Sum}[(-1)^n n^{1/n}, \{n, 1, 2x\}], x \rightarrow \text{Infinity}]^2$, was named the MRB constant, (after its original investigator), by Simon Plouffe³. Then on Feb 23, 2009, Marvin Ray Burns named

<https://oeis.org/A157852> "MKB constant" (MKB) after his wife at the time. Technically, A157852 is the integer sequence of the digits of MKB, (the integral analog of the MRB constant⁴). Hence MKB was named after one with a close relationship to the person the MRB constant was named after.

$$\text{MKB} = \left| \lim_{n \rightarrow \infty} \int_1^{2^n} e^{i \pi x} x^{1/x} dx \right|.$$

It appears that

$$\begin{aligned} \lim_{n \rightarrow a} \int_1^{2^n} e^{i \pi x} x^{1/x} dx &\approx \lim_{n \rightarrow a} \int_1^{2^n} \text{Cos}[\text{Pi} * x] x^{1/x} dx + \\ &I * \lim_{n \rightarrow a} \int_1^{2^n} \text{Sin}[\text{Pi} * x] x^{1/x} dx \text{ and } \lim_{n \rightarrow a} \int_1^{2^{n-1}} e^{i \pi x} x^{1/x} dx \approx \lim_{n \rightarrow a} \int_1^{2^{n-1}} \text{Cos}[\text{Pi} * x] x^{1/x} dx + \\ &I * \lim_{n \rightarrow a} \int_1^{2^{n-1}} \text{Sin}[\text{Pi} * x] x^{1/x} dx, a \in \mathbb{N}. \end{aligned}$$

They are indicated to be true in the result to the following Mathematica code.

```
d := 20;
Table[NIntegrate[x^(1/x) Exp[I * Pi * x], {x, 1, 2 n}, WorkingPrecision -> d] -
(NIntegrate[x^(1/x) Cos[Pi * x], {x, 1, 2 n}, WorkingPrecision -> d] +
I NIntegrate[x^(1/x) Sin[Pi * x], {x, 1, 2 n}, WorkingPrecision -> d]), {n, 1, 21}]
{0. * 10^-21 + 0. * 10^-20 i, 0. * 10^-21 + 0. * 10^-20 i, 0. * 10^-21 + 0. * 10^-20 i,
0. * 10^-21 + 0. * 10^-20 i, 0. * 10^-21 + 0. * 10^-20 i, 0. * 10^-21 + 0. * 10^-20 i,
-3.105217 * 10^-14 + 0. * 10^-20 i, 0. * 10^-21 + 0. * 10^-20 i, 0. * 10^-21 + 0. * 10^-20 i,
0. * 10^-21 + 0. * 10^-20 i, 0. * 10^-21 + 0. * 10^-20 i, 0. * 10^-21 + 0. * 10^-20 i,
-3.478646 * 10^-14 + 0. * 10^-20 i, -2.783919 * 10^-14 + 0. * 10^-20 i, -5.530932 * 10^-14 + 0. * 10^-20 i,
0. * 10^-21 + 0. * 10^-20 i, 0. * 10^-21 + 0. * 10^-20 i, 0. * 10^-21 + 0. * 10^-20 i, 0. * 10^-21 + 0. * 10^-20 i}
```

```
d := 20;
Table[NIntegrate[x^(1/x) Exp[I * Pi * x], {x, 1, 2 n - 1}, WorkingPrecision -> d] -
(NIntegrate[x^(1/x) Cos[Pi * x], {x, 1, 2 n - 1}, WorkingPrecision -> d] +
I NIntegrate[x^(1/x) Sin[Pi * x], {x, 1, 2 n - 1}, WorkingPrecision -> d]), {n, 1, 21}]
{0, 0. * 10^-21 + 0. * 10^-21 i, 0. * 10^-21 + 0. * 10^-21 i, 0. * 10^-21 + 0. * 10^-21 i, 0. * 10^-21 + 0. * 10^-22 i,
0. * 10^-21 + 0. * 10^-22 i, 0. * 10^-21 + 0. * 10^-22 i, 3.521728 * 10^-14 - 2.7706332 * 10^-14 i,
0. * 10^-21 + 0. * 10^-22 i, 0. * 10^-21 + 0. * 10^-22 i, 0. * 10^-21 + 0. * 10^-23 i,
0. * 10^-21 + 1.3585779 * 10^-15 i, 0. * 10^-21 + 0. * 10^-23 i, 0. * 10^-21 - 9.210306 * 10^-15 i,
3.521728 * 10^-14 - 2.7706332 * 10^-14 i, 0. * 10^-21 + 0. * 10^-22 i, 0. * 10^-21 + 0. * 10^-22 i,
0. * 10^-21 + 0. * 10^-22 i, 0. * 10^-21 + 0. * 10^-22 i, 0. * 10^-21 + 0. * 10^-22 i, 0. * 10^-21 + 0. * 10^-22 i}
```

Also, it can be proved that

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_1^{2^n} e^{i \pi x} x^{1/x} dx &== \lim_{n \rightarrow \infty} \int_1^{2^n} \text{Cos}[\text{Pi} * x] x^{1/x} dx + \\ &I * \lim_{n \rightarrow \infty} \int_1^{2^n} \text{Sin}[\text{Pi} * x] x^{1/x} dx. \end{aligned}$$

This is shown to be true up to 23 digits of precision in the result to the following Mathematica code. (Some unknown [to this author] form of regularization is used.) We will need the["NumericalCalculus`"] package.

```
Needs ["NumericalCalculus`"]
```

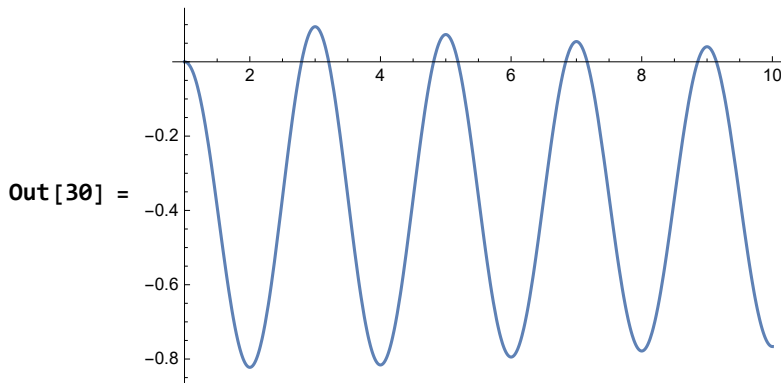
```

digits = 50;
Rationalize[NLimit[NIntegrate[Exp[Pi * I * x] x^(1/x), {x, 1, 2 a},
  WorkingPrecision -> 50, PrecisionGoal -> 50, MaxRecursion -> 50], a -> Infinity,
  WorkingPrecision -> 50, Terms -> 15, Method -> SequenceLimit, WynnDegree -> 6], 0] -
(NIntegrate[x^(1/x) * Cos[Pi * x], {x, 1, Infinity}, WorkingPrecision -> 2 * digits] +
  I NIntegrate[x^(1/x) Sin[Pi * x], {x, 1, Infinity}, WorkingPrecision -> 2 * digits] - I/Pi)
9.799242913445728747553048568494203547251606464399860365576612315013382138315720 × 10-23 -
  3.42299827056872052274295372607558077706014454761109646989352281775546479244989 × 10-23 i

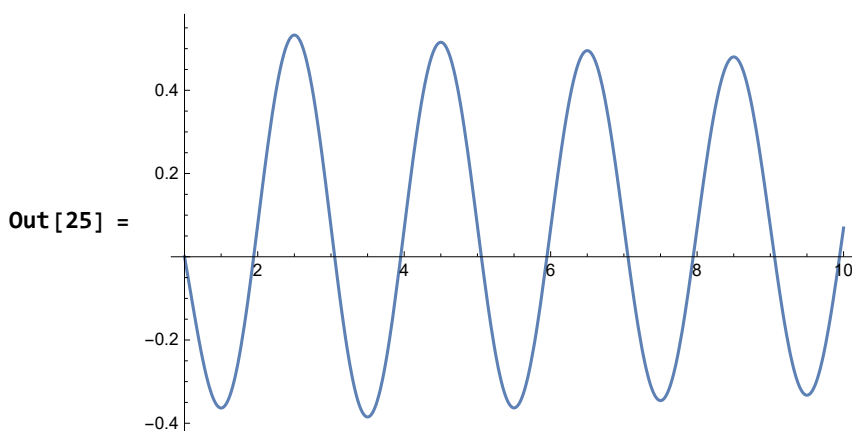
```

The following plots are hard to interpret, but here are what $\text{NIntegrate}[x^{1/x} \cos[\pi x], \{x, 1, a\}]$ and $\text{NIntegrate}[x^{1/x} \sin[\pi x], \{x, 1, a\}]$ for a from 1 to 10 look like. At least we notice that the cosine plot is shifted on both the x and $f(x)$ axes, and they have the same, or about the same period of about 2.

```
In[30] := Plot[Integrate[Sin[Pi * x] x^(1/x), {x, 1, a}], {a, 1, 10}]
```



```
In[25] := Plot[Integrate[Cos[Pi * x] x^(1/x), {x, 1, a}], {a, 1, 10}]
```



For more precise measurements see Mathar's work⁵, where Mathar also used many methods to compute MKB. He called it M1, published many error-bounds of methods, and compared M1 to what he called the MRB constant, M. R. Burns' constant, and M.

Tools

We will use many integration and limit options available in Mathematica 11.0. We also use a Intel 6 core 3.5 GH extreme edition desktop for the computations.

Operations and Results

Realizing his mistake of confusing $\text{integral}(\text{Sin}[\text{Pi}^I * x] x^{(1/x)}, \{x, 1, a\})$ with $\text{integral}(\text{Sin}[\text{Pi} * x] x^{(1/x)}, \{x, 1, a\})$, through the following operations, on Monday, Aug 8, 2016 at 2:00PM, Marvin Ray Burns began to see that the limit of the ratio of a to a-1, as a goes to infinity, in $\text{integral}(\text{Sin}[\text{Pi}^I * x] x^{(1/x)}, \{x, 1, a\}) / \text{integral}(\text{Sin}[\text{Pi}^I * x] x^{(1/x)}, \{x, 1, a-1\})$ and $\text{integral}(-\text{Cos}[\text{Pi}^I * x] x^{(1/x)}, \{x, 1, a\}) / \text{integral}(-\text{Cos}[\text{Pi}^I * x] x^{(1/x)}, \{x, 1, a-1\})$ is Gelfond's Constant, (e^π) .

```
i1 = Table[
  NIntegrate[Sin[Pi * I * x] x^(1/x), {x, 1, a}, WorkingPrecision -> 20], {a, 9990, 10001}]
```

```
{2.0989175549269083147 × 1013 629 i, 4.8570402004948497061 × 1013 630 i,
 1.1239526514136629501 × 1013 632 i, 2.6009040701590741249 × 1013 633 i,
 6.0186716707026764644 × 1013 634 i, 1.3927621974178786927 × 1013 636 i,
 3.2229479272454941452 × 1013 637 i, 7.4581241228073160071 × 1013 638 i,
 1.7258614376558644163 × 1013 640 i, 3.9937625775425609291 × 1013 641 i,
 9.2418424666174747439 × 1013 642 i, 2.1386261832231106579 × 1013 644 i}
```

```
i2 = Ratios[i1]
```

```
{23.140690729341148011, 23.140690729698949373,
 23.140690730056647960, 23.140690730414243812, 23.140690730771736967,
 23.140690731129127467, 23.140690731486415350, 23.140690731843600657,
 23.140690732200683426, 23.140690732557663699, 23.140690732914541513}
```

```
N[E^Pi, 20]
```

```
23.140692632779269006
```

```
N[i2 - E^Pi, 20]
```

```
{-1.903438120995 × 10-6, -1.903080319633 × 10-6, -1.902722621046 × 10-6, -1.902365025194 × 10-6,
 -1.902007532038 × 10-6, -1.901650141539 × 10-6, -1.901292853656 × 10-6, -1.900935668349 × 10-6,
 -1.900578585579 × 10-6, -1.900221605307 × 10-6, -1.899864727493 × 10-6}
```

Larger a: (Both sine and cosine in the this operation give Gelfond's Constant.)

```

i11 = Table[NIntegrate[Cos[Pi * I * x] x^(1/x), {x, 1, a}, WorkingPrecision -> 30],
  {a, 999990, 1000000}]
{8.16402704299180404095259449988 × 101364361,
  1.88921240445149992403409339252 × 101364363, 4.37176835688857590343446623243 × 101364364,
  1.01165747807172488243183607690 × 101364366, 2.34104547494103980098507246551 × 101364367,
  5.41734137742750968168005476577 × 101364368, 1.25361031700280490130583526513 × 101364370,
  2.90094110266711182266711546873 × 101364371, 6.71297864017550964458508807654 × 101364372,
  1.55342975360723204911512762411 × 101364374, 3.59474404543783042332075392032 × 101364375}

```

```

i12 = Ratios[i11]

```

```

{23.1406926324827036186672961841, 23.1406926324827041886616877912,
  23.1406926324827047586544156856, 23.1406926324827053286454798737,
  23.1406926324827058986348803621, 23.1406926324827064686226171574,
  23.1406926324827070386086902659, 23.1406926324827076085930996944,
  23.1406926324827081785758454491, 23.1406926324827087485569275368}

```

```

N[E^Pi, 20]

```

```

23.140692632779269006

```

```

N[i12 - E^Pi, 20]

```

```

{-2.965653870617901839 × 10-10, -2.965648170673985768 × 10-10,
  -2.965642470746706824 × 10-10, -2.965636770836064943 × 10-10, -2.965631070942060058 × 10-10,
  -2.965625371064692106 × 10-10, -2.965619671203961020 × 10-10, -2.965613971359866736 × 10-10,
  -2.965608271532409188 × 10-10, -2.965602571721588312 × 10-10}

```

Even larger a,(about as big as Mathematica will tolerate)!

```
i21 = Table[NIntegrate[Cos[Pi * I * x] x^(1/x), {x, 1, a}, WorkingPrecision -> 40],
  {a, 9999990, 10000008}]
```

```
{1.248750590784479766789088244974362158446 × 1013 643 749,
  2.889695359634080199319377313714578172209 × 1013 643 750,
  6.686955211965069049415827804642155765176 × 1013 643 751,
  1.547407752092216146387935592164153432162 × 1013 643 753,
  3.580808716874046215569412546222015226236 × 1013 643 754,
  8.286239389394650135109617259085898574115 × 1013 643 755,
  1.917493187915811799795191512425441918062 × 1013 643 757,
  4.437212048700115233728772579760159563694 × 1013 643 758,
  1.026801601654186402948091558294664519330 × 1013 643 760,
  2.376090025872139277296579342903098022129 × 1013 643 761,
  5.498436895651140346939155692172340668487 × 1013 643 762,
  1.272376381629768218143854086101177504926 × 1013 643 764,
  2.944367076049786971400355256642342436912 × 1013 643 765,
  6.813469350493284237431892945563446297757 × 1013 643 766,
  1.57668400026034571600891047822655334778 × 1013 643 768,
  3.648555982361789107489894116297179355389 × 1013 643 769,
  8.443011254130941753836957370586199018495 × 1013 643 770,
  1.953771283269108031257718779721784119022 × 1013 643 772,
  4.521162074087431137089658317298126596488 × 1013 643 773}
```

```
i22 = Ratios[i21]
```

```
{23.14069263277577056704002372095483993866, 23.14069263277577056771657128578128293096,
  23.14069263277577056839311865227146209701, 23.14069263277577056906966582042545538291,
  23.14069263277577056974621279024334073474, 23.14069263277577057042275956172519609852,
  23.14069263277577057109930613487109942026, 23.14069263277577057177585250968112864590,
  23.14069263277577057245239868615536172137, 23.14069263277577057312894466429387659253,
  23.14069263277577057380549044409675120524, 23.14069263277577057448203602556406350528,
  23.14069263277577057515858140869589143843, 23.14069263277577057583512659349231295041,
  23.14069263277577057651167157995340598690, 23.14069263277577057718821636807924849356,
  23.14069263277577057786476095786991841598, 23.14069263277577057854130534932549369975}
```

```
N[E^Pi, 20]
```

```
23.140692632779269006
```

```
N[i22 - E^Pi, 20]
```

```
{-3.4984386890626469937 × 10-12, -3.4984380125150821673 × 10-12,
  -3.4984373359677156771 × 10-12, -3.4984366594205475231 × 10-12,
  -3.4984359828735777052 × 10-12, -3.4984353063268062234 × 10-12,
  -3.4984346297802330774 × 10-12, -3.4984339532338582674 × 10-12,
  -3.4984332766876817932 × 10-12, -3.4984326001417036547 × 10-12,
  -3.4984319235959238518 × 10-12, -3.4984312470503423845 × 10-12,
  -3.4984305705049592527 × 10-12, -3.4984298939597744562 × 10-12,
  -3.4984292174147879951 × 10-12, -3.4984285408699998693 × 10-12,
  -3.4984278643254100786 × 10-12, -3.4984271877810186231 × 10-12}
```

When we switch forms of the operations, Mathematica can give at least 42 digits of accuracy here. Again we will need the ["NumericalCalculus`"] package.

Needs ["NumericalCalculus`"]

```
N[E^Pi -
  Rationalize[NLimit[NIntegrate[Cos[Pi * I * x] x^(1/x), {x, 1, a}, WorkingPrecision -> 50,
    PrecisionGoal -> 50, MaxRecursion -> 50] / NIntegrate[Cos[Pi * I * x] x^(1/x), {x, 1,
    a - 1}, WorkingPrecision -> 50, PrecisionGoal -> 50, MaxRecursion -> 50], a -> Infinity,
    WorkingPrecision -> 50, Terms -> 15, Method -> SequenceLimit, WynnDegree -> 6], 0], 50]
- 5.9702708245087317429519268920016913700154824404879 x 10^-42
```

```
N[E^Pi -
  Rationalize[NLimit[NIntegrate[Sin[Pi * I * x] x^(1/x), {x, 1, a}, WorkingPrecision -> 50,
    PrecisionGoal -> 50, MaxRecursion -> 50] / NIntegrate[Sin[Pi * I * x] x^(1/x), {x, 1,
    a - 1}, WorkingPrecision -> 50, PrecisionGoal -> 50, MaxRecursion -> 50], a -> Infinity,
    WorkingPrecision -> 50, Terms -> 15, Method -> SequenceLimit, WynnDegree -> 6], 0], 50]
- 5.9702708245087317429519268920016913700154824404879 x 10^-42
```

The following should help in a proof of the hypothesis: $\text{Cos}[Pi*I*x] == \text{Cosh}[Pi*x]$, $\text{Sin}[Pi*I*x] == I \text{Sin}h[Pi*x]$, and $\text{Limit}[x^(1/x), x \rightarrow \text{Infinity}] == 1$.

Using L'Hospital's Rule, we have the following:

$$e^\pi - \left(\cosh(\pi) - \lim_{a \rightarrow \infty} \left(\cos(\pi i) - \frac{x^{1/x} \cos(\pi i x) / . x \rightarrow a}{x^{1/x} \cos(\pi i x) / . x \rightarrow a - 1} \right) \right)$$

0

$$e^\pi - \left(i \sin(1) - \lim_{a \rightarrow \infty} \left(-\frac{x^{1/x} \sinh(\pi x) / . x \rightarrow a}{x^{1/x} \sinh(\pi x) / . x \rightarrow a - 1} + \sinh(i) \right) \right)$$

0

If we perform the same operations on only the integrals without the "I," (the later integrals), we get the following results, (proof not provided, since there are no **known** closed form solutions for two of the three ratios of the later integrals). Nonetheless, we will make a passing note to a few approximations.

Needs ["NumericalCalculus`"]

That is close to the following:

$$c - \left(\frac{3\,635\,966\,283\,803\,821\,e + 902\,615\,612\,958\,241\,e^2}{3\,209\,149\,808\,096\,290} - \frac{1\,211\,008\,478\,422\,557\,e + 27\,332\,794\,326\,472\,e^2}{562\,500\,154\,689\,724} I \right)$$

$$0. \times 10^{-37} + 0. \times 10^{-37} i$$

and

$$c - \left(\frac{(-1\,515\,494\,137\,444\,387\,e^{\pi/2} + 389\,348\,082\,242\,624\,e^{\pi})}{333\,366\,930\,773\,403} - \frac{(437\,182\,811\,555\,465\,e^{\pi/2} - 2\,053\,701\,405\,670\,e^{\pi})}{330\,937\,546\,000\,493} I \right)$$

$$0. \times 10^{-37} + 0. \times 10^{-37} i$$

And for smaller coefficients it is also close the following:

$$c - \left(1 + \frac{1}{2} \left(41\,e^{\pi/24} + 16\,e^{\pi/23} + 70\,e^{\pi/22} - 30\,e^{\pi/21} - 55\,e^{\pi/20} - 12\,e^{\pi/19} + 42\,e^{\pi/18} - \right. \right.$$

$$\left. 32\,e^{\pi/17} + 32\,e^{\pi/16} - 17\,e^{\pi/15} - 26\,e^{\pi/14} + 16\,e^{\pi/13} - 37\,e^{\pi/12} + e^{\pi/11} + 23\,e^{\pi/10} - \right.$$

$$\left. 19\,e^{\pi/9} + 37\,e^{\pi/8} + 77\,e^{\pi/7} + 30\,e^{\pi/6} - e^{\pi/5} - 44\,e^{\pi/4} + 6\,e^{\pi/3} - 54\,e^{\pi/2} + 5\,e^{\pi} \right) -$$

$$\frac{i}{6} \left(-5\,e^{\pi/27} - 30\,e^{\pi/26} + 11\,e^{\pi/25} - 6\,e^{\pi/24} - 34\,e^{\pi/23} - 25\,e^{\pi/22} + 20\,e^{\pi/21} + 50\,e^{\pi/20} + \right.$$

$$\left. 31\,e^{\pi/19} + 8\,e^{\pi/18} + 14\,e^{\pi/17} - 59\,e^{\pi/16} + 17\,e^{\pi/15} - 37\,e^{\pi/14} + 61\,e^{\pi/13} + 4\,e^{\pi/12} - \right.$$

$$\left. 12\,e^{\pi/11} + 51\,e^{\pi/10} + 38\,e^{\pi/9} - 41\,e^{\pi/8} + 17\,e^{\pi/7} - e^{\pi/6} + 14\,e^{\pi/5} - 27\,e^{\pi/4} - 9\,e^{\pi/3} - e^{\pi/2} \right)$$

$$0. \times 10^{-37} + 0. \times 10^{-37} i$$

and

$$c - \left(\frac{1}{3} \left(-11\,e - 12\,e^2 - 26\,e^3 - 34\,e^4 + 19\,e^5 + 13\,e^6 - 13\,e^7 + 19\,e^8 - 52\,e^9 + 26\,e^{10} + 36\,e^{11} - 30\,e^{12} + 14\,e^{13} + \right. \right.$$

$$\left. 6\,e^{14} - e^{16} - 45\,e^{17} - 8\,e^{18} + 28\,e^{19} - 28\,e^{20} + 18\,e^{21} + 8\,e^{22} + 12\,e^{23} + 34\,e^{24} - 12\,e^{25} - e^{26} \right) -$$

$$\frac{i}{4} \left(-18\,e + 49\,e^2 - 23\,e^3 + 31\,e^4 + 4\,e^5 + 78\,e^6 + 27\,e^7 - 23\,e^8 + 43\,e^9 - 22\,e^{10} - \right.$$

$$\left. 6\,e^{11} - 19\,e^{12} - 51\,e^{13} + 2\,e^{14} + 22\,e^{15} - 6\,e^{16} - 44\,e^{17} - 5\,e^{18} - \right.$$

$$\left. 58\,e^{19} + 7\,e^{20} - 15\,e^{21} - 41\,e^{22} + 5\,e^{23} - 28\,e^{24} - 26\,e^{25} + 14\,e^{26} \right)$$

$$- 5.0862071 \times 10^{-29} - 2.91865007 \times 10^{-28} i$$

Using Mathematica, the limit of the **differences** of the later integrals appear to go by the following rule: cosine->0; sine-> -2/Pi, and e-> -2/Pi * I; (reaching as far as 20 to 28 digits of precision).

Needs["NumericalCalculus`"]

```
NLimit[NIntegrate[Cos[Pi * x] x^(1/x), {x, 1, a}, WorkingPrecision -> 50,
  PrecisionGoal -> 50, MaxRecursion -> 50] - NIntegrate[Cos[Pi * x] x^(1/x),
  {x, 1, a - 1}, WorkingPrecision -> 50, PrecisionGoal -> 50, MaxRecursion -> 50],
  a -> Infinity, WorkingPrecision -> 50, Terms -> 15]
```

$$0. \times 10^{-28}$$

Needs["NumericalCalculus`"]

```
-2/Pi - NLimit[
  NIntegrate[Sin[Pi * x] x^(1/x), {x, 1, a}, WorkingPrecision -> 50, PrecisionGoal -> 50,
  MaxRecursion -> 50] - NIntegrate[Sin[Pi * x] x^(1/x), {x, 1, a - 1}, WorkingPrecision -> 50,
  PrecisionGoal -> 50, MaxRecursion -> 50], a -> Infinity, WorkingPrecision -> 50, Terms -> 30]
```

$$1.6163 \times 10^{-20}$$

Needs["NumericalCalculus`"]

```
-2/Pi I - NLimit[NIntegrate[Exp[Pi * I * x] x^(1/x), {x, 1, a}, WorkingPrecision -> 50,
  PrecisionGoal -> 50, MaxRecursion -> 50] - NIntegrate[Exp[Pi * I * x] x^(1/x),
  {x, 1, a - 1}, WorkingPrecision -> 50, PrecisionGoal -> 50, MaxRecursion -> 50],
  a -> Infinity, WorkingPrecision -> 50, Terms -> 30]
```

$$0. \times 10^{-25} + 1.616 \times 10^{-20} i$$

Open Question

Do there exist closed form solutions for the ratios of the later integrals, specifically those with the integrands sine and e? And maybe we might ask about cosine too.

Summary

The limits of ratios of the hyperbolic integrals and the limits of limits of the differences of the later integrals had exact solutions that were easy to recognize, while we have not found exact values for the limits of ratios of the later integrals, except, perhaps cosine which seems to give 1. The limits of differences of the later integrals have known closed forms.

References

- 1 S. R. Finch, *Mathematical Constants*, Cambridge, 2003, pp 450, 452
- 2 <http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.695.5959&rep=rep1&type=pdf> page 28.
- 3 <http://www.plouffe.fr/simon/articles/Tableofconstants.pdf>

4 www.people.fas.harvard.edu/~sfinch/csolve/erradd.pdf p. 64.

5 <http://arxiv.org/abs/0912.3844>