Closed forms, (including Gelfond ' s Constant), using MKB constant like integrals.

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Abstract

First we will consider

$$\int_1^\infty \text{Cos}[\text{Pi}*I*x]\, x^{1/x}\, dx \text{ and } \int_1^\infty \text{Sin}[\text{Pi}*I*x]\, x^{1/x}\, dx,$$

where our numerical analysis makes it appear that the limit of the ratio of a to $a-1$, as a goes to infinity, in integral(Sin[Pi$*I*x$] x^(1/x), {x, 1, a})/integral(Sin[Pi$*I*x$] x^(1/x), {x, 1, $a-1$}) and integral(Cos[Pi$*I*x$] x^(1/x), {x, 1, a})/integral(Cos[Pi$*I*x$] x^(1/x), {x, 1, $a-1$}) is Gelfond ' s Constant, (e^pi). We will consider that the hypothesis and provide hints for a proof using L' Hospital' s Rule, since we have indeterminate forms as a goes to infinity.

We shall struggle with numeric evaluation to compare the ratios of

$$\int_1^\infty \text{Cos}[\text{Pi}*I*x]\, x^{1/x}\, dx \text{ and } \int_1^\infty \text{Sin}[\text{Pi}*I*x]\, x^{1/x}\, dx \text{ with } \int_1^\infty \text{Cos}[\text{Pi}*x]\, x^{1/x}\, dx \text{ and } \int_1^\infty \text{Sin}[\text{Pi}*x]\, x^{1/x}\, dx.$$

In later parts of this paper we will look at the ratio of three deceptively similar looking integrals which have the terms of MKB for integrands. (For reference we call these the "later integrals.")
Finally, after finding no closed form for the ratios of the later integrals,
we will settle for closed forms of their differences.

Preliminaries

Below, and throughout this paper, we will show the best known (to the author) Mathematica 11 options, found to give the desired results. Many of the computations took several minutes and produced a few warning messages which are not displayed.
In 1999[1], the constant referred to at https://oeis.org/A037077,
Limit[Sum[(-1)^n n^(1/n),{n,1,2x}],x->Infinity] [2], was named the MRB constant, (after its original investigator), by Simon Plouffe[3]. Then on Feb 23, 2009, Marvin Ray Burns named

https://oeis.org/A157852 “MKB constant” (MKB) after his wife at the time. Technically, A157852 is the integer sequence of the digits of MKB, (the integral analog of the MRB constant[4]). Hence MKB was named after one with a close relationship to the person the MRB constant was named after.

MKB= $\left|\lim_{n\to\infty} \int_1^{2n} e^{i \pi x} x^{1/x} dx\right|$.

It appears that

$\lim_{n\to a} \int_1^{2n} e^{i\pi x} x^{1/x} dx \simeq \lim_{n\to a} \int_1^{2n} \text{Cos}[\text{Pi}*x]\ x^{1/x} dx + I*\lim_{n\to a} \int_1^{2n} \text{Sin}[\text{Pi}*x]\ x^{1/x} dx$ and $\lim_{n\to a} \int_1^{2n-1} e^{i\pi x} x^{1/x} dx \simeq \lim_{n\to a} \int_1^{2n-1} \text{Cos}[\text{Pi}*x]\ x^{1/x} dx + I*\lim_{n\to a} \int_1^{2n-1} \text{Sin}[\text{Pi}*x]\ x^{1/x} dx$, $a \in \mathbb{N}$.

They are indicated to be true in the result to the following Mathematica code.

```
d := 20;
Table[NIntegrate[x^(1/x) Exp[I*Pi*x], {x, 1, 2 n}, WorkingPrecision → d] -
  (NIntegrate[x^(1/x) Cos[Pi*x], {x, 1, 2 n}, WorkingPrecision → d] +
   I NIntegrate[x^(1/x) Sin[Pi*x], {x, 1, 2 n}, WorkingPrecision → d]), {n, 1, 21}]
```

$\{0.\times10^{-21}+0.\times10^{-20}\,i,\ 0.\times10^{-21}+0.\times10^{-20}\,i,\ 0.\times10^{-21}+0.\times10^{-20}\,i,$
$0.\times10^{-21}+0.\times10^{-20}\,i,\ 0.\times10^{-21}+0.\times10^{-20}\,i,\ 0.\times10^{-21}+0.\times10^{-20}\,i,\ 0.\times10^{-21}+0.\times10^{-20}\,i,$
$-3.105217\times10^{-14}+0.\times10^{-20}\,i,\ 0.\times10^{-21}+0.\times10^{-20}\,i,\ 0.\times10^{-21}+0.\times10^{-20}\,i,$
$0.\times10^{-21}+0.\times10^{-20}\,i,\ 0.\times10^{-21}+0.\times10^{-20}\,i,\ 0.\times10^{-21}+0.\times10^{-20}\,i,\ 0.\times10^{-21}+0.\times10^{-20}\,i,$
$-3.478646\times10^{-14}+0.\times10^{-20}\,i,\ -2.783919\times10^{-14}+0.\times10^{-20}\,i,\ -5.530932\times10^{-14}+0.\times10^{-20}\,i,$
$0.\times10^{-21}+0.\times10^{-20}\,i,\ 0.\times10^{-21}+0.\times10^{-20}\,i,\ 0.\times10^{-21}+0.\times10^{-20}\,i,\ 0.\times10^{-21}+0.\times10^{-20}\,i\}$

```
d := 20;
Table[NIntegrate[x^(1/x) Exp[I*Pi*x], {x, 1, 2 n - 1}, WorkingPrecision → d] -
  (NIntegrate[x^(1/x) Cos[Pi*x], {x, 1, 2 n - 1}, WorkingPrecision → d] +
   I NIntegrate[x^(1/x) Sin[Pi*x], {x, 1, 2 n - 1}, WorkingPrecision → d]), {n, 1, 21}]
```

$\{0,\ 0.\times10^{-21}+0.\times10^{-21}\,i,\ 0.\times10^{-21}+0.\times10^{-21}\,i,\ 0.\times10^{-21}+0.\times10^{-21}\,i,\ 0.\times10^{-21}+0.\times10^{-22}\,i,$
$0.\times10^{-21}+0.\times10^{-22}\,i,\ 0.\times10^{-21}+0.\times10^{-22}\,i,\ 3.521728\times10^{-14}-2.7706332\times10^{-14}\,i,$
$0.\times10^{-21}+0.\times10^{-22}\,i,\ 0.\times10^{-21}+0.\times10^{-22}\,i,\ 0.\times10^{-21}+0.\times10^{-23}\,i,$
$0.\times10^{-21}+1.3585779\times10^{-15}\,i,\ 0.\times10^{-21}+0.\times10^{-23}\,i,\ 0.\times10^{-21}-9.210306\times10^{-15}\,i,$
$3.521728\times10^{-14}-2.7706332\times10^{-14}\,i,\ 0.\times10^{-21}+0.\times10^{-22}\,i,\ 0.\times10^{-21}+0.\times10^{-22}\,i,$
$0.\times10^{-21}+0.\times10^{-22}\,i,\ 0.\times10^{-21}+0.\times10^{-22}\,i,\ 0.\times10^{-21}+0.\times10^{-22}\,i,\ 0.\times10^{-21}+0.\times10^{-22}\,i\}$

Also, it can be proved that

$\lim_{n\to\infty} \int_1^{2n} e^{i\pi x} x^{1/x} dx == \lim_{n\to\infty} \int_1^{2n} \text{Cos}[\text{Pi}*x]\ x^{1/x} dx + I*\lim_{n\to\infty} \int_1^{2n} \text{Sin}[\text{Pi}*x]\ x^{1/x} dx$.

This is shown to be true up to 23 digits of precision in the result to the following Mathematica code. (Some unknown [to this author] form of regularization is used, [at least for the operation with sine in it. See work below as to why the cosine operation might involve convergent integrals.])

We will need the["NumericalCalculus`"] package.

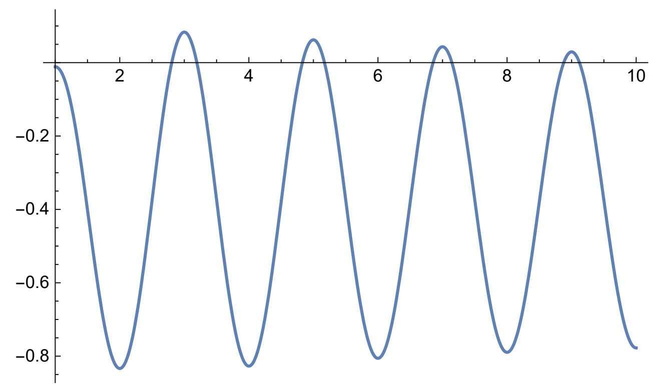
```
Needs["NumericalCalculus`"]

digits = 50;
Rationalize[NLimit[NIntegrate[Exp[Pi * I * x] x^(1/x), {x, 1, 2 a},
     WorkingPrecision → 50, PrecisionGoal → 50, MaxRecursion → 50], a → Infinity,
    WorkingPrecision → 50, Terms → 15, Method → SequenceLimit, WynnDegree → 6], 0] -
  (NIntegrate[x^(1/x) * Cos[Pi * x], {x, 1, Infinity}, WorkingPrecision → 2 * digits] +
    I NIntegrate[x^(1/x) Sin[Pi * x], {x, 1, Infinity}, WorkingPrecision → 2 * digits] - I/Pi)
9.7992429134457287475530485684942035472516064643998603655766123150133821383157 20 × 10^-23 -
  3.4229982705687205227429537260755807770601445476110964698935228177554647924498 9 × 10^-23 i
```

For what we seek below, the following plots are hard to interpret in terms of
$\int_1^a \text{Cos}[\text{Pi}*I*x]\, x^{1/x}\, dx \Big/ \int_1^{a-1} \text{Cos}[\text{Pi}*I*x]\, x^{1/x}\, dx$
and $\int_1^a \text{Sin}[\text{Pi}*x]\, x^{1/x}\, dx \Big/ \int_1^{a-1} \text{Sin}[\text{Pi}*x]\, x^{1/x}\, dx$,
and equally hard for this author to interpret in terms of
$\int_1^a \text{Cos}[\text{Pi}*I*x]\, x^{1/x}\, dx - \int_1^{a-1} \text{Cos}[\text{Pi}*I*x]\, x^{1/x}\, dx$
and $\int_1^a \text{Sin}[\text{Pi}*x]\, x^{1/x}\, dx - \int_1^{a-1} \text{Sin}[\text{Pi}*x]\, x^{1/x}\, dx$.

When it come to the infinite limits we fight for below, this author believes there has to be some regularization involved to give the results shown. For the results appear to be in absolute value and singular, (where the literal functions appear to be continuously oscillating). Nonetheless, here are what $\int_1^a \text{Cos}[\text{Pi} * I * x]\, x^{1/x}$ and $\int_1^a \text{Sin}[\text{Pi} * I * x]\, x^{1/x}$ for a from 1 to 10 look like. At least we notice that the cosine plot seems to lose magnitude quicker than the sine plot, and they have the same, or about the same period of about 2.

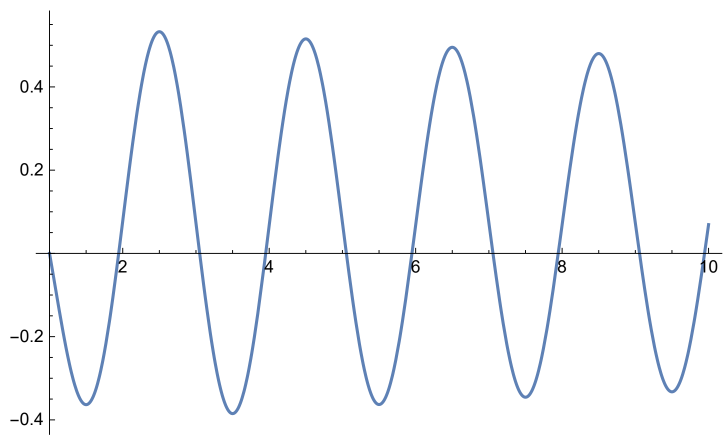
```
In[30] := Plot[Integrate[Sin[Pi * x] x^(1/x), {x, 1, a}], {a, 1, 10}]
```

Out[30] =



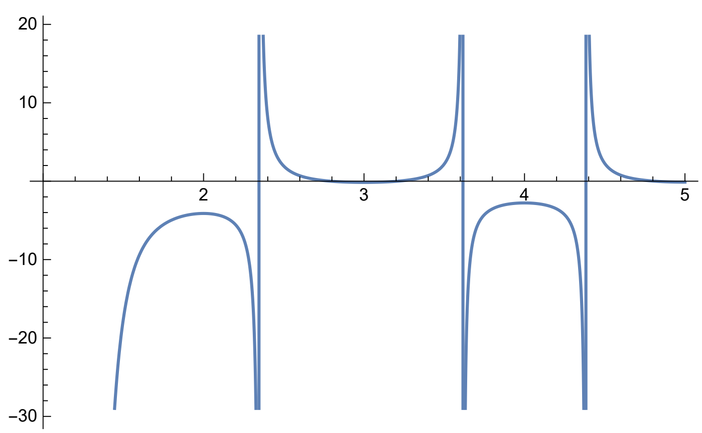
```
In[25] := Plot[Integrate[Cos[Pi * x] x^(1/x), {x, 1, a}], {a, 1, 10}]
```

Out[25] =



As an example of the quandary this author is in, here is a look at $\int_1^a \mathrm{Sin}[\mathrm{Pi} * x]\, x^{1/x}\, dx \Big/ \int_1^{a-1} \mathrm{Sin}[\mathrm{Pi} * x]\, x^{1/x}\, dx$. How does one say, take the limit as a->infinity, here?

```
Plot[Integrate[Sin[Pi * x] x^(1/x), {x, 1, a}] /
  Integrate[Sin[Pi * x] x^(1/x), {x, 0, a - 1}], {a, 1, 5}]
```



For more precise measurements and a general scheme for calculation of the digits of MKB, by flattening out the oscillatory integral, see Mathar's work[5], where Mathar also used many integration methods to compute MKB. He called it M1, published many error-bounds of methods, and compared M1 to what he called the MRB constant, M. R. Burns' constant, and M.

Tools

We will use many integration and limit options available in Mathematica 11.0. We also use a Intel 6 core 3.5 GH extreme edition desktop for the computations.

Operations and Results

Realizing his mistake of confusing integral(Sin[Pi*I*x]x^(1/x),{x,1,a}) with integral(Sin[Pi*x]x^(1/x),{x,1,a}), through the following operations, on Monday, Aug 8, 2016 at 2:00PM, Marvin Ray Burns began to see that the limit of the ratio of a to a-1, as a goes to infinity, in integral(Sin[Pi*I*x]x^(1/x),{x,1,a})/ integral(Sin[Pi*I*x]x^(1/x),{x,1,a-1}) and integral(-Cos[Pi*I*x]x^(1/x),{x,1,a})/integral(Cos[Pi*I*x]x^(1/x),{x,1,a-1}) is Gelfond's Constant, (e^pi).

.

```
i1 = Table[
  NIntegrate[Sin[Pi * I * x] x^(1/x), {x, 1, a}, WorkingPrecision → 20], {a, 9990, 10001}]
```

$\{2.0989175549269083147 \times 10^{13629}\, i,\ 4.8570402004948497061 \times 10^{13630}\, i,$
$1.1239526514136629501 \times 10^{13632}\, i,\ 2.6009040701590741249 \times 10^{13633}\, i,$
$6.0186716707026764644 \times 10^{13634}\, i,\ 1.3927621974178786927 \times 10^{13636}\, i,$
$3.2229479272454941452 \times 10^{13637}\, i,\ 7.4581241228073160071 \times 10^{13638}\, i,$
$1.7258614376558644163 \times 10^{13640}\, i,\ 3.9937625775425609291 \times 10^{13641}\, i,$
$9.2418424666174747439 \times 10^{13642}\, i,\ 2.1386261832231106579 \times 10^{13644}\, i\}$

```
i2 = Ratios[i1]
```

```
{23.140690729341148011, 23.140690729698949373,
 23.140690730056647960, 23.140690730414243812, 23.140690730771736967,
 23.140690731129127467, 23.140690731486415350, 23.140690731843600657,
 23.140690732200683426, 23.140690732557663699, 23.140690732914541513}
```

```
N[E^Pi, 20]
```

```
23.140692632779269006
```

```
N[i2 - E^Pi, 20]
```

$\{-1.903438120995 \times 10^{-6},\ -1.903080319633 \times 10^{-6},\ -1.902722621046 \times 10^{-6},\ -1.902365025194 \times 10^{-6},$
$-1.902007532038 \times 10^{-6},\ -1.901650141539 \times 10^{-6},\ -1.901292853656 \times 10^{-6},\ -1.900935668349 \times 10^{-6},$
$-1.900578585579 \times 10^{-6},\ -1.900221605307 \times 10^{-6},\ -1.899864727493 \times 10^{-6}\}$

Larger a: (Both sine and cosine in the this operation give Gelfond's Constant.)

```
i11 = Table[NIntegrate[Cos[Pi * I * x] x^(1/x), {x, 1, a}, WorkingPrecision → 30],
  {a, 999 990, 1 000 000}]
```

{$8.16402704299180404095259449988 \times 10^{1364361}$,
$1.88921240445149992403409339252 \times 10^{1364363}$, $4.37176835688857590343446623243 \times 10^{1364364}$,
$1.01165747807172488243183607690 \times 10^{1364366}$, $2.34104547494103980098507246551 \times 10^{1364367}$,
$5.41734137742750968168005476577 \times 10^{1364368}$, $1.25361031700280490130583526513 \times 10^{1364370}$,
$2.90094110266711182266711546873 \times 10^{1364371}$, $6.71297864017550964458508807654 \times 10^{1364372}$,
$1.55342975360723204911512762411 \times 10^{1364374}$, $3.59474404543783042332075392032 \times 10^{1364375}$}

```
i12 = Ratios[i11]
```

{23.1406926324827036186672961841, 23.1406926324827041886616877912,
23.1406926324827047586544156856, 23.1406926324827053286454798737,
23.1406926324827058986348803621, 23.1406926324827064686226171574,
23.1406926324827070386086902659, 23.1406926324827076085930996944,
23.1406926324827081785758454491, 23.1406926324827087485569275368}

```
N[E^Pi, 20]
```

23.140692632779269006

```
N[i12 - E^Pi, 20]
```

{$-2.9656538706179018 39 \times 10^{-10}$, $-2.965648170673985768 \times 10^{-10}$,
$-2.965642470746706824 \times 10^{-10}$, $-2.965636770836064943 \times 10^{-10}$, $-2.965631070942060058 \times 10^{-10}$,
$-2.965625371064692106 \times 10^{-10}$, $-2.965619671203961020 \times 10^{-10}$, $-2.965613971359866736 \times 10^{-10}$,
$-2.965608271532409188 \times 10^{-10}$, $-2.965602571721588312 \times 10^{-10}$}

Even larger a,(about as big as Mathematica will tolerate)!

```
i21 = Table[NIntegrate[Cos[Pi * I * x] x^(1/x), {x, 1, a}, WorkingPrecision → 40],
  {a, 9999990, 10000008}]
```

$\{1.248750590784479766789088244974362158446 \times 10^{13643749}$,
$2.889695359634080199319377313714578172209 \times 10^{13643750}$,
$6.686955211965069049415827804642155765176 \times 10^{13643751}$,
$1.547407752092216146387935592164153432162 \times 10^{13643753}$,
$3.580808716874046215569412546222015226236 \times 10^{13643754}$,
$8.286239389394650135109617259085898574115 \times 10^{13643755}$,
$1.917493187915811799795191512425441918062 \times 10^{13643757}$,
$4.437212048700115233728772579760159563694 \times 10^{13643758}$,
$1.026801601654186402948091558294664519330 \times 10^{13643760}$,
$2.376090025872139277296579342903098022129 \times 10^{13643761}$,
$5.498436895651140346939155692172340668487 \times 10^{13643762}$,
$1.272376381629768218143854086101177504926 \times 10^{13643764}$,
$2.944367076049786971400355256642342436912 \times 10^{13643765}$,
$6.813469350493284237431892945563446297757 \times 10^{13643766}$,
$1.576684000026034571600891047822655334778 \times 10^{13643768}$,
$3.648555982361789107489894116297179355389 \times 10^{13643769}$,
$8.443011254130941753836957370586199018495 \times 10^{13643770}$,
$1.953771283269108031257718779721784119022 \times 10^{13643772}$,
$4.521162074087431137089658317298126596488 \times 10^{13643773}\}$

```
i22 = Ratios[i21]
```

{23.14069263277577056704002372095483993866, 23.14069263277577056771657128578128293096,
23.14069263277577056839311865227146209701, 23.14069263277577056906966582042545538291,
23.14069263277577056974621279024334073474, 23.14069263277577057042275956172519609852,
23.14069263277577057109930613487109942026, 23.14069263277577057177585250968112864590,
23.14069263277577057245239868615536172137, 23.14069263277577057312894466429387659253,
23.14069263277577057380549044409675120524, 23.14069263277577057448203602556406350528,
23.14069263277577057515858140869589143843, 23.14069263277577057583512659349231295041,
23.14069263277577057651167157995340598690, 23.14069263277577057718821636807924849356,
23.14069263277577057786476095786991841598, 23.14069263277577057854130534932549369975}

```
N[E^Pi, 20]
```

23.140692632779269006

```
N[i22 - E^Pi, 20]
```

$\{-3.4984386890626469937 \times 10^{-12}$, $-3.4984380125150821673 \times 10^{-12}$,
$-3.4984373359677156771 \times 10^{-12}$, $-3.4984366594205475231 \times 10^{-12}$,
$-3.4984359828735777052 \times 10^{-12}$, $-3.4984353063268062234 \times 10^{-12}$,
$-3.4984346297802330774 \times 10^{-12}$, $-3.4984339532338582674 \times 10^{-12}$,
$-3.4984332766876817932 \times 10^{-12}$, $-3.4984326001417036547 \times 10^{-12}$,
$-3.4984319235959238518 \times 10^{-12}$, $-3.4984312470503423845 \times 10^{-12}$,
$-3.4984305705049592527 \times 10^{-12}$, $-3.4984298939597744562 \times 10^{-12}$,
$-3.4984292174147879951 \times 10^{-12}$, $-3.4984285408699998693 \times 10^{-12}$,
$-3.4984278643254100786 \times 10^{-12}$, $-3.4984271877810186231 \times 10^{-12}\}$

When we switch forms of the operations, Mathematica can give at least 42 digits of accuracy here. Again we will need the [“NumericalCalculus`”] package.

```
Needs["NumericalCalculus`"]
```

```
N[E^Pi -
  Rationalize[NLimit[NIntegrate[Cos[Pi * I * x] x^(1/x), {x, 1, a}, WorkingPrecision → 50,
      PrecisionGoal → 50, MaxRecursion → 50] / NIntegrate[Cos[Pi * I * x] x^(1/x), {x, 1,
       a - 1}, WorkingPrecision → 50, PrecisionGoal → 50, MaxRecursion → 50], a → Infinity,
    WorkingPrecision → 50, Terms → 15, Method → SequenceLimit, WynnDegree → 6], 0], 50]
```

$-5.9702708245087317429519268920016913700154824404879 \times 10^{-42}$

```
N[E^Pi -
  Rationalize[NLimit[NIntegrate[Sin[Pi * I * x] x^(1/x), {x, 1, a}, WorkingPrecision → 50,
      PrecisionGoal → 50, MaxRecursion → 50] / NIntegrate[Sin[Pi * I * x] x^(1/x), {x, 1,
       a - 1}, WorkingPrecision → 50, PrecisionGoal → 50, MaxRecursion → 50], a → Infinity,
    WorkingPrecision → 50, Terms → 15, Method → SequenceLimit, WynnDegree → 6], 0], 50]
```

$-5.9702708245087317429519268920016913700154824404879 \times 10^{-42}$

The following should help in a proof of the hypothesis: Cos[Pi*I*x] == Cosh[Pi*x], Sin[Pi*I*x] == I Sinh[Pi*x], and Limit[x^(1/x),x->Infinity]==1.
Using L’Hospital’s Rule, we have the following:

$$e^{\pi} - \left(\cosh(\pi) - \lim_{a\to\infty} \left(\cos(\pi\, i) - \frac{x^{1/x}\cos(\pi\, i\, x) \,/.\, x \to a}{x^{1/x}\cos(\pi\, i\, x) \,/.\, x \to a-1}\right)\right)$$

0

$$e^{\pi} - \left(i\sin(1) - \lim_{a\to\infty} \left(-\frac{x^{1/x}\sinh(\pi\, x) \,/.\, x \to a}{x^{1/x}\sinh(\pi\, x) \,/.\, x \to a-1} + \sinh(i)\right)\right)$$

0

If we perform the same operations on only the integrals without the "I," (the later integrals), we get the following results, (proof not provided, since there are no **known** closed form solutions for two of the three ratios of the later integrals). Nonetheless, we will make a passing note to a few approximations.

```
Needs["NumericalCalculus`"]
```

```
NLimit[NIntegrate[Sin[Pi * x] x^(1/x), {x, 1, a}, WorkingPrecision → 50,
    PrecisionGoal → 50, MaxRecursion → 50] / NIntegrate[Sin[Pi * x] x^(1/x),
    {x, 1, a - 1}, WorkingPrecision → 50, PrecisionGoal → 50, MaxRecursion → 50],
  a → Infinity, WorkingPrecision → 50, Terms → 15]
```

14.436291291052278530533116235683502718

Increasing the number of NLimit terms here to 20 causes Mathematica to give an unevaluated result.

```
a = 14.436291291052278530533116235683502718;
```

a Is close to the following:

$$a - \frac{137\,250\,424\,206\,313\, e + 2\,085\,856\,139\,684\,320\, e^2}{1\,093\,465\,977\,254\,055}$$

$0. \times 10^{-36}$

and

$$a - \left(70\,064\,463\,976\,981\,915\, e^{\pi/2} + 2\,940\,546\,676\,813\,034\, e^{\pi}\right) / 28\,060\,517\,609\,691\,197$$

$0. \times 10^{-36}$

And for smaller coefficients it is also close the following:

$$a - \frac{1}{28}\left(-70\, e - 97\, e^2 - 151\, e^3 - 3\, e^4 - 58\, e^5 - 21\, e^6 - 93\, e^7 + 24\, e^8 - 12\, e^9 - 49\, e^{10} - 37\, e^{11} + 19\, e^{12} - 48\, e^{13} + 141\, e^{14} - 87\, e^{15} - 11\, e^{16} - 58\, e^{17} - 46\, e^{18} - 31\, e^{19} + 21\, e^{20}\right)$$

1.3543×10^{-31}

and

$$a - \frac{1}{38}\left(91\, e^{\pi/20} - 168\, e^{\pi/19} + 40\, e^{\pi/18} + 80\, e^{\pi/17} - 6\, e^{\pi/16} + 8\, e^{\pi/15} + 121\, e^{\pi/14} + 12\, e^{\pi/13} - 8\, e^{\pi/12} + 21\, e^{\pi/11} + 69\, e^{\pi/10} - 11\, e^{\pi/9} + 81\, e^{\pi/8} + 72\, e^{\pi/7} + 38\, e^{\pi/6} + 56\, e^{\pi/5} - 22\, e^{\pi/4} + 38\, e^{\pi/3} - 24\, e^{\pi/2} - 5\, e^{\pi}\right)$$

$0. \times 10^{-36}$

```
Needs["NumericalCalculus`"]
```

```
NLimit[NIntegrate[Cos[Pi * x] x^(1/x), {x, 1, a}, WorkingPrecision → 70,
    PrecisionGoal → 70, MaxRecursion → 60] / NIntegrate[Cos[Pi * x] x^(1/x),
    {x, 1, a - 1}, WorkingPrecision → 70, PrecisionGoal → 70, MaxRecursion → 60],
  a → Infinity, WorkingPrecision → 70, Terms → 15]
```

```
0.999999999999999999999999999681834364367520582249
```

We arive at about 1. The random digits could be some hard to avoid round of error, but we make no claim of certianty, yet. If those digits are just round off error so that the limit of the ratio is 1, then as far as this author knows, $\int_1^\infty$ **Cos[Pi * x] x^(1/x)** ⅆ**x** could be convergent.
Like with the operation with the sine integrand, increasing the number of NLimit terms here to 20 causes Mathematica to give an unevaluated result.
Just now noticed, is that if you add the following options to NLimit, you get precisely 1.

```
NLimit[NIntegrate[Cos[Pi * x] x^(1/x), {x, 1, a}, WorkingPrecision → 70,
    PrecisionGoal → 70, MaxRecursion → 60] / NIntegrate[Cos[Pi * x] x^(1/x), {x, 1, a - 1},
    WorkingPrecision → 70, PrecisionGoal → 70, MaxRecursion → 60], a → Infinity,
  WorkingPrecision → 70, Terms → 30, Method → SequenceLimit, WynnDegree → 6]

1.000000000000000000000000000000000000000000
```

Next, we will perform our operations on the MKB constant as mentioned at the top of this paper. For reference we will call its integrand "e."

```
Needs["NumericalCalculus`"]

NLimit[NIntegrate[Exp[Pi * I * x] x^(1/x), {x, 1, a}, WorkingPrecision → 50, PrecisionGoal →
  50, MaxRecursion → 50] / NIntegrate[Exp[Pi * I * x] x^(1/x), {x, 1, a - 1}, WorkingPrecision →
  50, PrecisionGoal → 50, MaxRecursion → 50], a → Infinity, WorkingPrecision → 50, Terms → 15]

In[530]:= c = 5.158082192458413431339787028600589211289794187814179827291`36.73666664191001 -
    6.2112443214382927932714964849471235666902076502552021666062`36.8643846581354 ⅈ

Out[530]= 5.158082192458413431339787028600589211 - 6.211244321438292793271496484947123567 ⅈ
```

Finding approxiamtions for c might be very useless because changing the number of terms used by NLimit, even a little bit, has a significant effect on c:

```
NLimit[NIntegrate[Exp[Pi * I * x] x^(1/x), {x, 1, a}, WorkingPrecision → 64,
    PrecisionGoal → 64, MaxRecursion → 64] / NIntegrate[Exp[Pi * I * x] x^(1/x),
    {x, 1, a - 1}, WorkingPrecision → 64, PrecisionGoal → 64, MaxRecursion → 64],
  a → Infinity, WorkingPrecision → 64, Terms → 20]

5.158082192508406740997736430180682591658604305 -
  6.211244321292326821487544300461476145882682464 I

In[531]:= c - (5.158082192508406740997736430180682591658604305 -
     6.211244321292326821487544300461476145882682464 I)

Out[531]= -4.9993309657949401580093380 × 10^-11 - 1.45965971783952184485647421 × 10^-10 ⅈ
```

Using Mathematica, the limit of the **differences** of the later integrals appear to go by the following rule: cosine->0, sine-> -2/Pi, and e-> -2/Pi * I; (reaching as far as 20 to 28 digits of precision). We will try a few improvements in accuracy, but won't get very far.

First we try cosine.

```
Needs["NumericalCalculus`"]

NLimit[NIntegrate[Cos[Pi * x] x^(1/x), {x, 1, a}, WorkingPrecision → 50,
   PrecisionGoal → 50, MaxRecursion → 50] - NIntegrate[Cos[Pi * x] x^(1/x),
   {x, 1, a - 1}, WorkingPrecision → 50, PrecisionGoal → 50, MaxRecursion → 50],
 a → Infinity, WorkingPrecision → 50, Terms → 15]
```

$0. \times 10^{-28}$

```
Needs["NumericalCalculus`"]

NLimit[NIntegrate[Cos[Pi * x] x^(1/x), {x, 1, a}, WorkingPrecision → 60,
   PrecisionGoal → 60, MaxRecursion → 60] - NIntegrate[Cos[Pi * x] x^(1/x),
   {x, 1, a - 1}, WorkingPrecision → 60, PrecisionGoal → 60, MaxRecursion → 60],
 a → Infinity, WorkingPrecision → 60, Terms → 15]

-2.542325340 * 10^-28

Needs["NumericalCalculus`"]

NLimit[NIntegrate[Cos[Pi * x] x^(1/x), {x, 1, a}, WorkingPrecision → 64,
   PrecisionGoal → 64, MaxRecursion → 64] - NIntegrate[Cos[Pi * x] x^(1/x),
   {x, 1, a - 1}, WorkingPrecision → 64, PrecisionGoal → 64, MaxRecursion → 64],
 a → Infinity, WorkingPrecision → 64, Terms → 15]

-2.5423253395557 * 10^-28
```

A small increase in the number of NLimit terms give precisely 0:

```
NLimit[NIntegrate[Cos[Pi * x] x^(1/x), {x, 1, a}, WorkingPrecision → 64,
   PrecisionGoal → 64, MaxRecursion → 64] - NIntegrate[Cos[Pi * x] x^(1/x),
   {x, 1, a - 1}, WorkingPrecision → 64, PrecisionGoal → 64, MaxRecursion → 64],
 a → Infinity, WorkingPrecision → 64, Terms → 20]

0. * 10^-32
```

However, a large increase in terms gives an outlier compared to the previous pattern, but still in the neighborhood of 0.

```
Needs["NumericalCalculus`"]

NLimit[NIntegrate[Cos[Pi * x] x^(1/x), {x, 1, a}, WorkingPrecision → 64,
    PrecisionGoal → 64, MaxRecursion → 64] - NIntegrate[Cos[Pi * x] x^(1/x),
    {x, 1, a - 1}, WorkingPrecision → 64, PrecisionGoal → 64, MaxRecursion → 64],
  a → Infinity, WorkingPrecision → 64, Terms → 35]

-2.578371631713526006790566980 45 * 10^-15
```

With the same large number of terms, a more balanced code give a result closer to 0. (In the previous code, Mathematica was complaining that WorkingPrecision should have been higher for the Precision-Goal used.

```
Needs["NumericalCalculus`"]

NLimit[NIntegrate[Cos[Pi * x] x^(1/x), {x, 1, a}, WorkingPrecision → 72,
    PrecisionGoal → 32, MaxRecursion → 32] - NIntegrate[Cos[Pi * x] x^(1/x),
    {x, 1, a - 1}, WorkingPrecision → 72, PrecisionGoal → 32, MaxRecursion → 32],
  a → Infinity, WorkingPrecision → 72, Terms → 35]

-2.195290286055035703361816093663402147246908695 * 10^-26
```

Then we try sine.

```
Needs["NumericalCalculus`"]

-2/Pi - NLimit[
   NIntegrate[Sin[Pi * x] x^(1/x), {x, 1, a}, WorkingPrecision → 50, PrecisionGoal → 50,
     MaxRecursion → 50] - NIntegrate[Sin[Pi * x] x^(1/x), {x, 1, a - 1}, WorkingPrecision → 50,
     PrecisionGoal → 50, MaxRecursion → 50], a → Infinity, WorkingPrecision → 50, Terms → 30]
```

1.6163×10^{-20}

A small increase in the number of NLimit terms show sine in the operation to be closer to showing sine-> -2/Pi:

```
Needs["NumericalCalculus`"]

-2/Pi - NLimit[
   NIntegrate[Sin[Pi * x] x^(1/x), {x, 1, a}, WorkingPrecision → 64, PrecisionGoal → 64,
     MaxRecursion → 64] - NIntegrate[Sin[Pi * x] x^(1/x), {x, 1, a - 1}, WorkingPrecision → 64,
     PrecisionGoal → 64, MaxRecursion → 64], a → Infinity, WorkingPrecision → 64, Terms → 30]

-8.0140508085833126 * 10^-22
```

using a previously used trick, we get precisely 0, showing sine->-2/Pi.

```
-2/Pi - NLimit[NIntegrate[Sin[Pi * x] x^(1/x), {x, 1, a}, WorkingPrecision → 72,
    PrecisionGoal → 72, MaxRecursion → 72] - NIntegrate[Sin[Pi * x] x^(1/x), {x, 1, a - 1},
    WorkingPrecision → 72, PrecisionGoal → 72, MaxRecursion → 72], a → Infinity,
  WorkingPrecision → 70, Terms → 35, Method → SequenceLimit, WynnDegree → 6]
```

```
0. * 10^-27
```

Finally, we try e.

```
-2/Pi I - NLimit[NIntegrate[Exp[Pi * I * x] x^(1/x), {x, 1, a}, WorkingPrecision → 50,
    PrecisionGoal → 50, MaxRecursion → 50] - NIntegrate[Exp[Pi * I * x] x^(1/x),
    {x, 1, a - 1}, WorkingPrecision → 50, PrecisionGoal → 50, MaxRecursion → 50],
  a → Infinity, WorkingPrecision → 50, Terms → 30]
```

$0. \times 10^{-25} + 1.616 \times 10^{-20}\, i$

Here a small increase in the number of NLimit terms give a result closer to 0. Indicating that e -> -2/Pi I.

```
-2/Pi I - NLimit[NIntegrate[Exp[Pi * I * x] x^(1/x), {x, 1, a}, WorkingPrecision → 64,
    PrecisionGoal → 64, MaxRecursion → 64] - NIntegrate[Exp[Pi * I * x] x^(1/x),
    {x, 1, a - 1}, WorkingPrecision → 64, PrecisionGoal → 64, MaxRecursion → 64],
  a → Infinity, WorkingPrecision → 64, Terms → 35]
```

```
0. * 10^-34 + 7.9742539 * 10^-25 I
```

Open Questions

Considering the shape of some of the previous functions are their calculated limits a valid measurement?

Do there exist closed form solutions for the ratios of the later integrals, specifically those with the integrands sine and e? And maybe we might ask about cosine too.

Is $\int_1^\infty \mathrm{Cos}[\mathrm{Pi} * x]\, x^{1/x}\, dx$ indeed convergent as shown to be possible above?

Sumary

The limits of ratios of the hyperbolic integrals and the limits of limits of the differences of the later integrals had exact solutions, or at least exact solutions to the neighborhood of 22 to 32 decimal places, that were easy to recognize. While we have not found exact values for the limits of ratios of the later integrals, except, perhaps cosine which seems to give 1. The limits of differences of the later integrals seem to have known closed forms, but there is room for error there.

A more analytical approach might be needed here. Supplementary papers are welcome to be written.

References

1 S. R. Finch, Mathematical Constants, Cambridge, 2003, pp 450, 452

2 http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.695.5959&rep=rep1&type=pdf page 28.

3 http://www.plouffe.fr/simon/articles/Tableofconstants.pdf

4 www.people.fas.harvard.edu/~sfinch/csolve/erradd.pdf p. 64.

5 http://arxiv.org/abs/0912.3844