Closed forms, (including Gelfond ' s Constant), using MKB constant like integrals.

Marvin Ray Burns
(part – time undergraduate student and number harvester as of 2016)

Abstract

First we will follow the path the author took to find out that for

$$\int_1^\infty \text{Cos}[\text{Pi}*I*x]\, x^{1/x}\, dx \text{ and } \int_1^\infty \text{Sin}[\text{Pi}*I*x]\, x^{1/x}\, dx,$$

the limit of the ratio of *a* to *a* − 1, as *a* goes to infinity, is Gelfond ' *s* Constant, (*e* ^ pi). We will consider that the hypothesis and provide hints for *a* proof using *L*' Hospital' *s* Rule, since we have indeterminate forms as *a* goes to infinity.

We shall struggle with numeric evaluation to compare the limit of the ratios of

$$\int_1^\infty \text{Cos}[\text{Pi}*I*x]\, x^{1/x}\, dx \text{ and } \int_1^\infty \text{Sin}[\text{Pi}*I*x]\, x^{1/x}\, dx \text{ with } \int_1^\infty \text{Cos}[\text{Pi}*x]\, x^{1/x}\, dx \text{ and } \int_1^\infty \text{Sin}[\text{Pi}*x]\, x^{1/x}\, dx.$$

Through some simple plotting and taking into consideration oscillatory nature of the sine and cosine functions, we make judgments on whether we should invest in finding their limit.

Preliminaries

Below, and throughout this paper, we will show the best known (to the author) Mathematica 11 options, found to give the desired results. Many of the computations took several minutes and produced a few warning messages which are not displayed.
In 1999[1], the constant referred to at https://oeis.org/A037077,
Limit[Sum[(-1)^n n^(1/n),{n,1,2x}],x->Infinity] [2], was named the MRB constant, (after its original investigator), by Simon Plouffe[3]. Then on Feb 23, 2009, Marvin Ray Burns named https://oeis.org/A157852 “MKB constant” (MKB) after his wife at the time. Technically, A157852 is the integer sequence of the digits of MKB, (the integral analog of the MRB constant[4]). Hence MKB was named after one with a close relationship to the person the

MRB constant was named after.

MKB= $\left|\lim_{n\to\infty} \int_1^{2n} e^{i\pi x} x^{1/x} \, dx\right|$.

It appears that

$\lim_{n\to a} \int_1^{2n} e^{i\pi x} x^{1/x} \, dx \simeq \lim_{n\to a} \int_1^{2n} \text{Cos}[\text{Pi}*x]\ x^{1/x} \, dx + I*\lim_{n\to a} \int_1^{2n} \text{Sin}[\text{Pi}*x]\ x^{1/x} \, dx$ and $\lim_{n\to a} \int_1^{2n-1} e^{i\pi x} x^{1/x} \, dx \simeq \lim_{n\to a} \int_1^{2n-1} \text{Cos}[\text{Pi}*x]\ x^{1/x} \, dx + I*\lim_{n\to a} \int_1^{2n-1} \text{Sin}[\text{Pi}*x]\ x^{1/x} \, dx,\ a \in \mathbb{N}$.

They are indicated to be true in the result to the following Mathematica code.

```
d := 20;
Table[NIntegrate[x^(1/x) Exp[I*Pi*x], {x, 1, 2 n}, WorkingPrecision → d] -
  (NIntegrate[x^(1/x) Cos[Pi*x], {x, 1, 2 n}, WorkingPrecision → d] +
    I NIntegrate[x^(1/x) Sin[Pi*x], {x, 1, 2 n}, WorkingPrecision → d]), {n, 1, 21}]
```

```
{0. × 10^-21 + 0. × 10^-20 i, 0. × 10^-21 + 0. × 10^-20 i, 0. × 10^-21 + 0. × 10^-20 i,
 0. × 10^-21 + 0. × 10^-20 i, 0. × 10^-21 + 0. × 10^-20 i, 0. × 10^-21 + 0. × 10^-20 i, 0. × 10^-21 + 0. × 10^-20 i,
 -3.105217 × 10^-14 + 0. × 10^-20 i, 0. × 10^-21 + 0. × 10^-20 i, 0. × 10^-21 + 0. × 10^-20 i,
 0. × 10^-21 + 0. × 10^-20 i, 0. × 10^-21 + 0. × 10^-20 i, 0. × 10^-21 + 0. × 10^-20 i, 0. × 10^-21 + 0. × 10^-20 i,
 -3.478646 × 10^-14 + 0. × 10^-20 i, -2.783919 × 10^-14 + 0. × 10^-20 i, -5.530932 × 10^-14 + 0. × 10^-20 i,
 0. × 10^-21 + 0. × 10^-20 i, 0. × 10^-21 + 0. × 10^-20 i, 0. × 10^-21 + 0. × 10^-20 i, 0. × 10^-21 + 0. × 10^-20 i}
```

```
d := 20;
Table[NIntegrate[x^(1/x) Exp[I*Pi*x], {x, 1, 2 n - 1}, WorkingPrecision → d] -
  (NIntegrate[x^(1/x) Cos[Pi*x], {x, 1, 2 n - 1}, WorkingPrecision → d] +
    I NIntegrate[x^(1/x) Sin[Pi*x], {x, 1, 2 n - 1}, WorkingPrecision → d]), {n, 1, 21}]
```

```
{0, 0. × 10^-21 + 0. × 10^-21 i, 0. × 10^-21 + 0. × 10^-21 i, 0. × 10^-21 + 0. × 10^-21 i, 0. × 10^-21 + 0. × 10^-22 i,
 0. × 10^-21 + 0. × 10^-22 i, 0. × 10^-21 + 0. × 10^-22 i, 3.521728 × 10^-14 - 2.7706332 × 10^-14 i,
 0. × 10^-21 + 0. × 10^-22 i, 0. × 10^-21 + 0. × 10^-22 i, 0. × 10^-21 + 0. × 10^-23 i,
 0. × 10^-21 + 1.3585779 × 10^-15 i, 0. × 10^-21 + 0. × 10^-23 i, 0. × 10^-21 - 9.210306 × 10^-15 i,
 3.521728 × 10^-14 - 2.7706332 × 10^-14 i, 0. × 10^-21 + 0. × 10^-22 i, 0. × 10^-21 + 0. × 10^-22 i,
 0. × 10^-21 + 0. × 10^-22 i, 0. × 10^-21 + 0. × 10^-22 i, 0. × 10^-21 + 0. × 10^-22 i, 0. × 10^-21 + 0. × 10^-22 i}
```

Also, it can be proved that

$\lim_{n\to\infty} \int_1^{2n} e^{i\pi x} x^{1/x} \, dx == \lim_{n\to\infty} \int_1^{2n} \text{Cos}[\text{Pi}*x]\ x^{1/x} \, dx + I*\lim_{n\to\infty} \int_1^{2n} \text{Sin}[\text{Pi}*x]\ x^{1/x} \, dx$.

This is shown to be true up to 23 digits of precision in the result to the following Mathematica code. (Some unknown [to this author] form of regularization is used, [at least for the operation with sine in it. See work below as to why the cosine operation might involve convergent integrals.])

We will need the["NumericalCalculus`"] package.

```
Needs["NumericalCalculus`"]
```

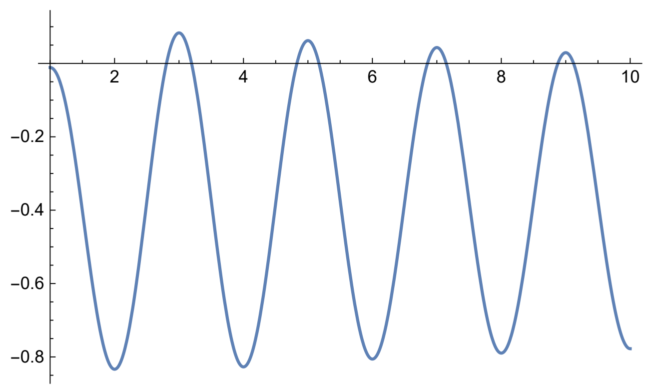
```
digits = 50;
Rationalize[NLimit[NIntegrate[Exp[Pi * I * x] x^(1/x), {x, 1, 2 a},
     WorkingPrecision → 50, PrecisionGoal → 50, MaxRecursion → 50], a → Infinity,
    WorkingPrecision → 50, Terms → 15, Method → SequenceLimit, WynnDegree → 6], 0] -
 (NIntegrate[x^(1/x) * Cos[Pi * x], {x, 1, Infinity}, WorkingPrecision → 2 * digits] +
    I NIntegrate[x^(1/x) Sin[Pi * x], {x, 1, Infinity}, WorkingPrecision → 2 * digits] - I/Pi)
```

$9.7992429134457287475530485684942035472516064643998603655766123150133821383157\times 10^{-23}$ − $3.4229982705687205227429537260755807770601445476110964698935228177554647924 4989\times 10^{-23}\, i$

Here are what $\int_1^a$`Cos[Pi * I * x]` $x^{1/x}$ and $\int_1^a$`Sin[Pi * I * x]` $x^{1/x}$ for a from 1 to 10 look like. At least we notice that the cosine plot seems to lose magnitude quicker than the sine plot, and its sign quickly becomes, and stays negative. The reason we will find the limit of a to a-1 diverges for sine and converges for cosine is the cosine operation always giving the same sign in the calculation of its integrals.

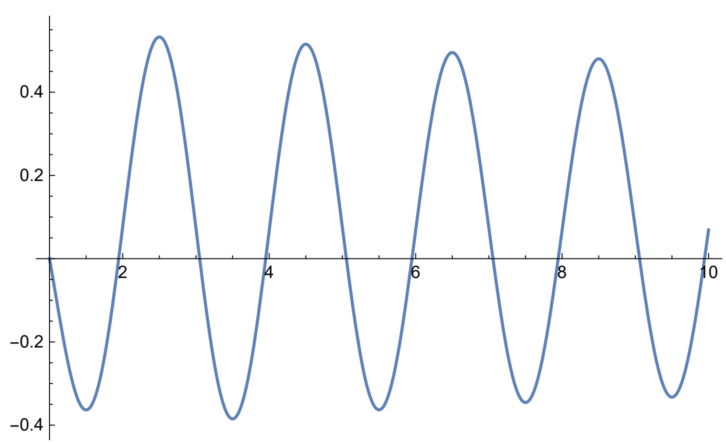
```
In[30] := Plot[Integrate[Sin[Pi * x] x^(1/x), {x, 1, a}], {a, 1, 10}]
```

Out[30] =



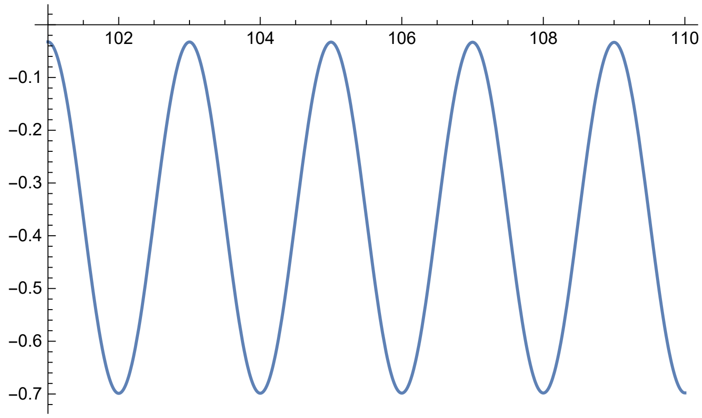

```
In[25] := Plot[Integrate[Cos[Pi * x] x^(1/x), {x, 1, a}], {a, 1, 10}]
```

Out[25] =

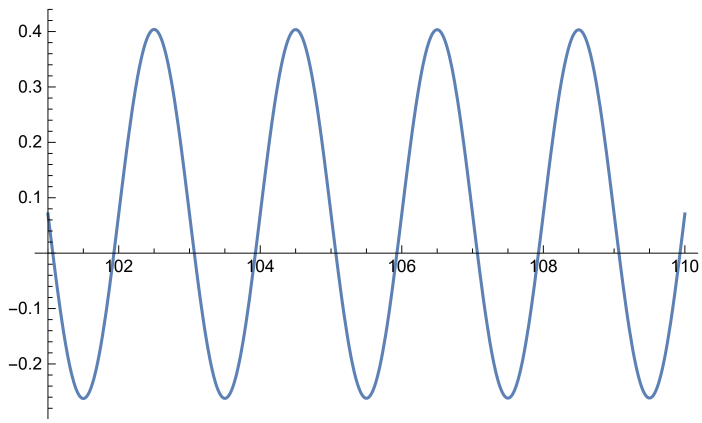


Here are plots of the same operations from 101 to110.

In[26]:=
```
Plot[Integrate[Sin[Pi * x] x^(1/x), {x, 1, a}], {a, 101, 110}]
```

Out[26]=



```
Plot[Integrate[Cos[Pi * x] x^(1/x), {x, 1, a}], {a, 101, 110}]
```



For more precise measurements and a general scheme for calculation of the digits of MKB, by flattening out the oscillatory integral, see Mathar’s work[5], where he also used many integration methods to compute MKB. He called it M1, published many error-bounds of methods, and compared M1 to what he called the MRB constant, M. R. Burns’ constant, and M.

Tools

We will use many integration and limit options available in Mathematica 11.0. We also use a Intel 6 core 3.5 GH extreme edition desktop for the computations.

Operations and Results

Realizing his mistake of confusing integral(Sin[Pi*I*x]x^(1/x),{x,1,a}) with integral(Sin[Pi*x]x^(1/x),{x,1,a}), through the following operations, on Monday, Aug 8, 2016 at 2:00PM, Marvin Ray Burns began to see that the limit of the ratio of a to a-1, as a goes to infinity, in integral(Sin[Pi*I*x]x^(1/x),{x,1,a})/ integral(Sin[Pi*I*x]x^(1/x),{x,1,a-1}) and integral(-Cos[Pi*I*x]x^(1/x),{x,1,a})/integral(Cos[Pi*I*x]x^(1/x),{x,1,a-1}) is Gelfond's Constant, (e^pi).

.

```
i1 = Table[
  NIntegrate[Sin[Pi * I * x] x^(1/x), {x, 1, a}, WorkingPrecision → 20], {a, 9990, 10001}]
```

{2.0989175549269083147 × 10^13629 i, 4.8570402004948497061 × 10^13630 i,
1.1239526514136629501 × 10^13632 i, 2.6009040701590741249 × 10^13633 i,
6.0186716707026764644 × 10^13634 i, 1.3927621974178786927 × 10^13636 i,
3.2229479272454941452 × 10^13637 i, 7.4581241228073160071 × 10^13638 i,
1.7258614376558644163 × 10^13640 i, 3.9937625775425609291 × 10^13641 i,
9.2418424666174747439 × 10^13642 i, 2.1386261832231106579 × 10^13644 i}

```
i2 = Ratios[i1]
```

{23.140690729341148011, 23.140690729698949373,
23.140690730056647960, 23.140690730414243812, 23.140690730771736967,
23.140690731129127467, 23.140690731486415350, 23.140690731843600657,
23.140690732200683426, 23.140690732557663699, 23.140690732914541513}

```
N[E^Pi, 20]
```

23.140692632779269006

```
N[i2 - E^Pi, 20]
```

{-1.903438120995 × 10^-6, -1.903080319633 × 10^-6, -1.902722621046 × 10^-6, -1.902365025194 × 10^-6,
-1.902007532038 × 10^-6, -1.901650141539 × 10^-6, -1.901292853656 × 10^-6, -1.900935668349 × 10^-6,
-1.900578585579 × 10^-6, -1.900221605307 × 10^-6, -1.899864727493 × 10^-6}

Larger a: (Both sine and cosine in the this operation give Gelfond's Constant.)

```
i11 = Table[NIntegrate[Cos[Pi * I * x] x^(1/x), {x, 1, a}, WorkingPrecision → 30],
  {a, 999990, 1000000}]
```

{8.16402704299180404095259449988 × 10^1364361,
1.88921240445149992403409339252 × 10^1364363, 4.37176835688857590343446623243 × 10^1364364,
1.01165747807172488243183607690 × 10^1364366, 2.34104547494103980098507246551 × 10^1364367,
5.41734137742750968168005476577 × 10^1364368, 1.25361031700280490130583526513 × 10^1364370,
2.90094110266711182266711546873 × 10^1364371, 6.71297864017550964458508807654 × 10^1364372,
1.55342975360723204911512762411 × 10^1364374, 3.59474404543783042332075392032 × 10^1364375}

```
i12 = Ratios[i11]
```

{23.140692632482703618667296184 1, 23.140692632482704188661687791 2,
23.140692632482704758654415685 6, 23.140692632482705328645479873 7,
23.140692632482705898634880362 1, 23.140692632482706468622617157 4,
23.140692632482707038608690265 9, 23.140692632482707608593099694 4,
23.140692632482708178575845449 1, 23.140692632482708748556927536 8}

```
N[E^Pi, 20]
```

23.140692632779269006

```
N[i12 - E^Pi, 20]
```

{$-2.9656538706179018 39 \times 10^{-10}$, $-2.965648170673985768 \times 10^{-10}$,
$-2.965642470746706824 \times 10^{-10}$, $-2.965636770836064943 \times 10^{-10}$, $-2.965631070942060058 \times 10^{-10}$,
$-2.965625371064692106 \times 10^{-10}$, $-2.965619671203961020 \times 10^{-10}$, $-2.965613971359866736 \times 10^{-10}$,
$-2.965608271532409188 \times 10^{-10}$, $-2.965602571721588312 \times 10^{-10}$}

Even larger a,(about as big as Mathematica will tolerate)!

```
i21 = Table[NIntegrate[Cos[Pi * I * x] x^(1/x), {x, 1, a}, WorkingPrecision → 40],
  {a, 9 999 990, 10 000 008}]
```

{$1.248750590784479766789088244974362158446 \times 10^{13643749}$,
$2.889695359634080199319377313714578172209 \times 10^{13643750}$,
$6.686955211965069049415827804642155765176 \times 10^{13643751}$,
$1.547407752092216146387935592164153432162 \times 10^{13643753}$,
$3.580808716874046215569412546222015226236 \times 10^{13643754}$,
$8.286239389394650135109617259085898574115 \times 10^{13643755}$,
$1.917493187915811799795191512425441918062 \times 10^{13643757}$,
$4.437212048700115233728772579760159563694 \times 10^{13643758}$,
$1.026801601654186402948091558294664519330 \times 10^{13643760}$,
$2.376090025872139277296579342903098022129 \times 10^{13643761}$,
$5.498436895651140346939155692172340668487 \times 10^{13643762}$,
$1.272376381629768218143854086101177504926 \times 10^{13643764}$,
$2.944367076049786971400355256642342436912 \times 10^{13643765}$,
$6.813469350493284237431892945563446297757 \times 10^{13643766}$,
$1.576684000026034571600891047822655334778 \times 10^{13643768}$,
$3.648555982361789107489894116297179355389 \times 10^{13643769}$,
$8.443011254130941753836957370586199018495 \times 10^{13643770}$,
$1.953771283269108031257718779721784119022 \times 10^{13643772}$,
$4.521162074087431137089658317298126596488 \times 10^{13643773}$}

```
i22 = Ratios[i21]
```

```
{23.140692632775770567040023720954839938 66, 23.140692632775770567716571285781282930 96,
 23.140692632775770568393118652271462097 01, 23.140692632775770569069665820425455382 91,
 23.140692632775770569746212790243340734 74, 23.140692632775770570422759561725196098 52,
 23.140692632775770571099306134871099420 26, 23.140692632775770571775852509681128645 90,
 23.140692632775770572452398686155361721 37, 23.140692632775770573128944664293876592 53,
 23.140692632775770573805490444096751205 24, 23.140692632775770574482036025564063505 28,
 23.140692632775770575158581408695891438 43, 23.140692632775770575835126593492312950 41,
 23.140692632775770576511671579953405986 90, 23.140692632775770577188216368079248493 56,
 23.140692632775770577864760957869918415 98, 23.140692632775770578541305349325493699 75}
```

```
N[E^Pi, 20]
```

```
23.140692632779269006
```

```
N[i22 - E^Pi, 20]
```

```
{-3.4984386890626469937 × 10^-12, -3.4984380125150821673 × 10^-12,
 -3.4984373359677156771 × 10^-12, -3.4984366594205475231 × 10^-12,
 -3.4984359828735777052 × 10^-12, -3.4984353063268062234 × 10^-12,
 -3.4984346297802330774 × 10^-12, -3.4984339532338582674 × 10^-12,
 -3.4984332766876817932 × 10^-12, -3.4984326001417036547 × 10^-12,
 -3.4984319235959238518 × 10^-12, -3.4984312470503423845 × 10^-12,
 -3.4984305705049592527 × 10^-12, -3.4984298939597744562 × 10^-12,
 -3.4984292174147879951 × 10^-12, -3.4984285408699998693 × 10^-12,
 -3.4984278643254100786 × 10^-12, -3.4984271877810186231 × 10^-12}
```

When we switch forms of the operations, Mathematica can give at least 42 digits of accuracy here. Again we will need the [“NumericalCalculus`”] package.

```
Needs["NumericalCalculus`"]
```

```
N[E^Pi -
   Rationalize[NLimit[NIntegrate[Cos[Pi * I * x] x^(1/x), {x, 1, a}, WorkingPrecision → 50,
       PrecisionGoal → 50, MaxRecursion → 50] / NIntegrate[Cos[Pi * I * x] x^(1/x), {x, 1,
        a - 1}, WorkingPrecision → 50, PrecisionGoal → 50, MaxRecursion → 50], a → Infinity,
     WorkingPrecision → 50, Terms → 15, Method → SequenceLimit, WynnDegree → 6], 0], 50]
```

```
-5.9702708245087317429519268920016913700154824404879 × 10^-42
```

```
N[E^Pi -
   Rationalize[NLimit[NIntegrate[Sin[Pi * I * x] x^(1/x), {x, 1, a}, WorkingPrecision → 50,
       PrecisionGoal → 50, MaxRecursion → 50] / NIntegrate[Sin[Pi * I * x] x^(1/x), {x, 1,
        a - 1}, WorkingPrecision → 50, PrecisionGoal → 50, MaxRecursion → 50], a → Infinity,
     WorkingPrecision → 50, Terms → 15, Method → SequenceLimit, WynnDegree → 6], 0], 50]
```

```
-5.9702708245087317429519268920016913700154824404879 × 10^-42
```

The following should help in a proof of the hypothesis: Cos[Pi*I*x] == Cosh[Pi*x], Sin[Pi*I*x] == I Sinh[Pi*x], and Limit[x^(1/x),x->Infinity]==1.
Using L'Hospital's Rule, we have the following:

$$e^{\pi} - \left(\cosh(\pi) - \lim_{a\to\infty}\left(\cos(\pi\, i) - \frac{x^{1/x}\cos(\pi\, i\, x)\;/.\; x \to a}{x^{1/x}\cos(\pi\, i\, x)\;/.\; x \to a-1}\right)\right)$$

0

$$e^{\pi} - \left(i\sin(1) - \lim_{a\to\infty}\left(-\frac{x^{1/x}\sinh(\pi\, x)\;/.\; x \to a}{x^{1/x}\sinh(\pi\, x)\;/.\; x \to a-1} + \sinh(i)\right)\right)$$

0

If we perform the same operations on only the integrals without the "I," (the later integrals), we get the following results, (proof not provided, since there are no **known** closed form solutions for two of the three ratios of the later integrals). Nonetheless, we will make a passing note to a few approximations.

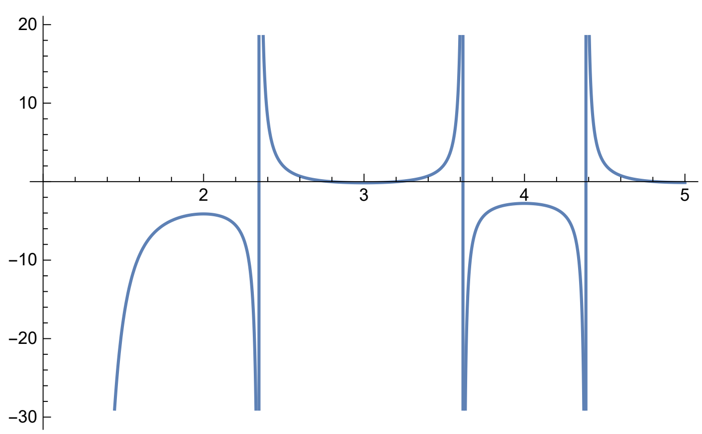
There ' s no point in working on the
Limit of $\int_1^a \text{Sin}[\text{Pi} * x]\, x^{1/x}\, dx - \int_1^{a-1} \text{Sin}[\text{Pi} * x]\, x^{1/x}\, dx$, because it is clearly divergent.

Here is a look at

$$\int_1^a \text{Sin}[\text{Pi} * x]\, x^{1/x}\, dx \Big/ \int_1^{a-1} \text{Sin}[\text{Pi} * x]\, x^{1/x}\, dx.$$

How does one say, take the limit as a->infinity, here?

```
Plot[Integrate[Sin[Pi * x] x^(1/x), {x, 1, a}] /
  Integrate[Sin[Pi * x] x^(1/x), {x, 0, a - 1}], {a, 1, 5}]
```



We will work with

$\int_1^a \mathrm{Cos}[\mathrm{Pi}*x]\, x^{1/x}\, dx \Big/ \int_1^{a-1} \mathrm{Cos}[\mathrm{Pi}*x]\, x^{1/x}\, dx$,
however. Because it probably converges.

```
Needs["NumericalCalculus`"]

NLimit[NIntegrate[Cos[Pi * x] x^(1/x), {x, 1, a}, WorkingPrecision → 70,
   PrecisionGoal → 70, MaxRecursion → 60] / NIntegrate[Cos[Pi * x] x^(1/x),
   {x, 1, a - 1}, WorkingPrecision → 70, PrecisionGoal → 70, MaxRecursion → 60],
 a → Infinity, WorkingPrecision → 70, Terms → 15]
```

```
0.99999999999999999999999999681834364367520582249
```

We arive at about 1. The random digits could be some hard to avoid round of error, but we make no claim of certianty, yet. If those digits are just round off error so that the limit of the ratio is 1, then as far as this author knows, $\int_1^\infty \mathrm{Cos}[\mathrm{Pi}*x]\, x^{1/x}\, dx$ could be convergent.
Increasing the number of NLimit terms here to 20 causes Mathematica to give an unevaluated result. however, if you add the following options to NLimit, you get precisely 1.

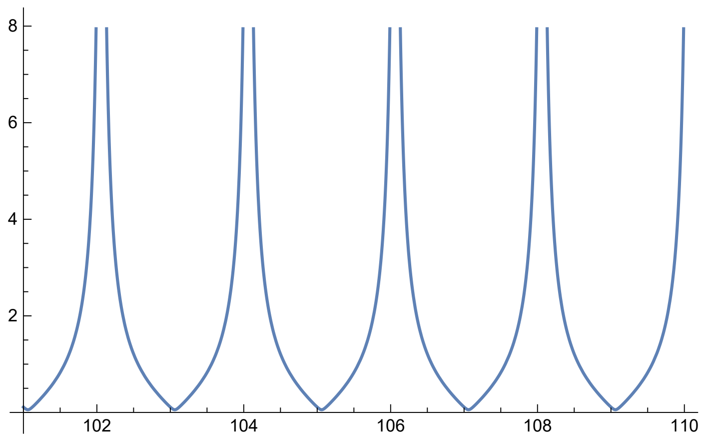
```
NLimit[NIntegrate[Cos[Pi * x] x^(1/x), {x, 1, a}, WorkingPrecision → 70,
   PrecisionGoal → 70, MaxRecursion → 60] / NIntegrate[Cos[Pi * x] x^(1/x), {x, 1, a - 1},
   WorkingPrecision → 70, PrecisionGoal → 70, MaxRecursion → 60], a → Infinity,
 WorkingPrecision → 70, Terms → 30, Method → SequenceLimit, WynnDegree → 6]
```

```
1.000000000000000000000000000000000000000
```

Looking at Limit of Abs $\int_1^a \mathrm{Exp}[\mathrm{Pi}*x]\, x^{1/x}\, dx \Big/ \mathrm{abs} \int_1^{a-1} \mathrm{Exp}[\mathrm{Pi}*x]\, x^{1/x}\, dx$,
it seems to diverge so we won ' *t* try to calculate it :

In[25]:=
```
Plot[Abs[NIntegrate[Exp[Pi * I * x] x^(1/x), {x, 1, a}, WorkingPrecision → 30]] / Abs[
  NIntegrate[Exp[Pi * I * x] x^(1/x), {x, 1, a - 1}, WorkingPrecision → 30]], {a, 101, 110}]
```

Out[25]=



Open Questions

For either of the divergent limits of the ratios of a to a-1 above, does the limit exists for the ratios of a to a-2? If so, what are they?

Sumary

We followed the path the author took to find out that for

$$\int_1^\infty \mathrm{Cos}[\mathrm{Pi} * I * x]\, x^{1/x}\, dx \text{ and } \int_1^\infty \mathrm{Sin}[\mathrm{Pi} * I * x]\, x^{1/x}\, dx,$$

the limit of the ratio of a to $a-1$, as a goes to infinity, is Gelfond' s Constant, (e^pi). We will considered that the hypothesis and provided hints for a proof using L' Hospital' s Rule, since we have indeterminate forms as a goes to infinity.

We struggled with numeric evaluation to compare the limit of the ratios of

$$\int_1^\infty \mathrm{Cos}[\mathrm{Pi} * I * x]\, x^{1/x}\, dx \text{ and } \int_1^\infty \mathrm{Sin}[\mathrm{Pi} * I * x]\, x^{1/x}\, dx \text{ with } \int_1^\infty \mathrm{Cos}[\mathrm{Pi} * x]\, x^{1/x}\, dx \text{ and } \int_1^\infty \mathrm{Sin}[\mathrm{Pi} * x]\, x^{1/x}\, dx.$$

Finally, through some simple plotting and taking into consideration oscillatory nature of the sine and cosine functions, we make judgments on whether we should invest in finding their limit.The reason we will found the limit of a to $a-1$ diverges for sine and converges for cosine is the cosine operation always giving the same sign in the calculation of its integrals.

References

1 S. R. Finch, Mathematical Constants, Cambridge, 2003, pp 450, 452

2 http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.695.5959&rep=rep1&type=pdf page 28.

3 http://www.plouffe.fr/simon/articles/Tableofconstants.pdf

4 www.people.fas.harvard.edu/~sfinch/csolve/erradd.pdf p. 64.

5 http://arxiv.org/abs/0912.3844