SPECTRUM OF (k, r) - REGULAR HYPERGRAPH

K Reji Kumar [1] and Renny P Varghese [2]

Abstract. We present a spectral theory of uniform, regular and linear hypergraph. The main result are the nature of the eigen values of (k, r) - regular linear hypergraph and the relation between its dual and line graph. We also discuss some properties of Laplacian spectrum of a (k, r) - regular hypergraphs.

Keywords : Hypergraph, Spectrum, dual of Hypergraph, line graph, Laplacian Spectrum.

AMS Subject Classification (2010) : 05C65

1. Introduction

The spectral graph theory [2, 3] is the study of the relation between eigen values and eigen vectors of certain associated matrices of a graph and its combinatorial properties. There is some relation between the size of the eigen values and maximum degree of the graph [8, 1]. Connection between spectral characteristics of a graph and other graph theoretic parameters is a well explored area.

A graph structure is obtained when a non empty set (set of vertices) and a subset of all unorderd pairs of elements of the set of vertices (this subset is called the set of edges) are taken together. When the set of unorderd pairs is replaced by order pairs we get directed graphs. For a regular graph G, eigen values of the adjacency matrix are well studied. The second largest eigen value of the adjacency matrix is realted to quantities such as diameter [6], chromatic number [4] etc. The regular graph with small non trivial eigen value can be used as good expanders and superconductors in communication network [12].

From the point of view of applications in social network and allied disciplines a more general structure is very useful. This structure, called *hypergraph* is obtained by taking a subset of the set of all proper subsets of the given set. The elements of the second set are called hyperedges. A hypergraph is denoted by H. For basic ideas and definitions on hypergraph readers may refer the text by [2]. Chung [6] and Wen - Ching et al. [11] proposed the operator attached to a regular and uniform hypergraph and obtained some estimate of the eigen values of the operation. K. Feng and W. C. Li [8] studeid the growth of the second largest eigen value and distribution of the eigen values of the adjacency matrix attached to a regular hypergraph.

In the next section we go through the main definitions and important results needed to understand the concepts included in the subsequent sections.

[1]Department of Mathematics
N. S. S. College, Cherthala, Kerala.
email : rkkmaths@yahoo.co.in

[2]Department of Mathematics
Catholicate College, Pathanamthitta, Kerala

2. Preliminaries

We now discuss some basic terminology of hypergraphs, which we require in the sequel. A hypergraph H is a pair (X, E) where $X = \{v_1, v_2, \cdots, v_n\}$ is a finite set and $E = \{E_1, E_2, \cdots, E_m\}$ of subsets of X, such that $E_j \neq \phi$, $j = 1, 2, 3 \cdots m$ and $\cup_{i=1}^{m} E_i = X$. The elements of X are called vertices or hypervertices and the elements of the sets $\{E_1, E_2, \cdots, E_m\}$ are called hyperedges of H. The cardinality of X is called the *order* of H and cardinality of E is called the *size* of H. In a hypergraph two vertices are adjacent if there is a hyperedge that containing both of these vertices. Two hyperedge are said to be adjacent if $E_i \cap E_j \neq \phi$. A *simple hypergraph* is a hypergraph such that $E_i \subset E_j \Rightarrow i = j$. A hypergraph H is said to be k - *uniform* for an integer $k \geq 2$, if for all E_i in E, $|E_i| = k$ for all i.

A hypergraph H is said to be *linear* if $|E_i \cap E_j| \leq 1$ for all $i \neq j$. A hypergraph in which all vertices have the same degree is said to be regular. An r - *regular hypergraph* is a hypergraph with $d(v_i) = r$ for all $i \leq n$ (r, k) - *regular hypergraph* is a hypergraph which is r - regular and k - uniform. The following results are important.

Theorem 2.1. [4] *Let T be a real $n \times n$ matrix with non negative entries and irreducible then there exists a unique positive real number θ_0 with the following properties.*

- *There is a real number $x_0 > 0$ with $Tx_0 = \theta_0 x_0$*
- *Any non negative right or left eigen vector of T has eigen value θ_0. Suppose $t \in R$ and $x \geq 0$, $x \neq 0$.*
  If $Tx \leq tx$, then $x \geq 0$ and $t \geq \theta_0$. Moreover $t = \theta_0$ iff $Tx = tx$.
  If $Tx \geq tx$, then $t \leq \theta_0$. Mmoreover $t = \theta_0$ iff $Tx = tx$.

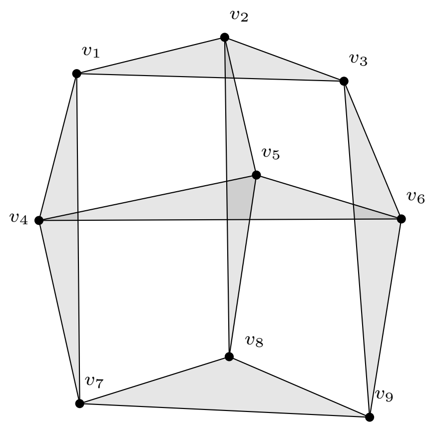
Theorem 2.2. [4] *Consider two sequence of real numbers $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_n$ and $\eta_1 \geq \eta_2 \geq \cdots \geq \eta_m$ with $m < n$. The second is said to interlace the first one whenevr $\theta_i \geq \eta_{n-m+i}$ for $i = 1, 2, ..., m$.*

Theorem 2.3. [4] *Let C be the quotient matrix of a symmetric matrix A whose rows and columns are partitioned according to the partitioning $\{x_1, x_2, \cdots, x_m\}$ then the eigen values of C interlace the eigen values of A.*

Let A be a real symmetric matrix and u be a non zero vector. The *Rayleigh Quotient* of u [4] with respect to A is defined as $\frac{u^T Au}{u^T u}$. The *dual* of the hypergraph $H(X, E)$ [2] is a hypergraph $H^* = H(X^*, E^*)$ where $X^* = \{e_1, e_2, \cdots, e_m\}$ corresponding to the edges of H and $E^* = \{X_1, X_2, \cdots, X_n\}$ where $X_i = \{e_j : x_i \in E_j \text{ in } H\}$. Also $(H^*)^* = H$. Given a hypergraph $H = (X, E)$, where $X = \{E_1, E_2, ..., E_m\}$. Its *representative* graph or *line graph* $\mathcal{L}(H)$, is a graph whose vertices are points $\{e_1, e_2, ..., e_m\}$ representing the edges of H and the vertices e_i and e_j being adjacent if and only if $E_i \cap E_j \neq \phi$.

Lemma 2.4. [2] *The dual of a linear hypergraph is also linear.*

Proof. Given H is linear. Suppose H^* is not linear. Let X_1 and X_2 in H^* intersect at two distinct points e_1 and e_2. Hence $\{e_1, e_2\} \subset E_1$ and E_2. Therefore $|E_1 \cap E_2| \geq 2$, which contradicts $|E_i \cap E_j| \leq 1$. So H* is linear. □



Example of (2,3)- regular and linear hypergraph on 9 vertices

Now we define the *adjacency matrix* [1] of a hypergraph H. Adjacency matrix of H is denoted by $A_H = (a_{ij})$, where

$$a_{ij} = \begin{cases} |\{E_k \in E : \{v_i, v_j\} \subseteq E_k\}| & \text{if } i \neq j \\ 0 & \text{otherwise.} \end{cases}$$

The eigen values of the adjacency matrix A_H is called the eigen values of H. Since A_H is a real symmetric matrix, all the eigen values are real. The spectrum of H is the set of all eigen values of A_H together with their multiplicities. Spectrum of H is denoted by $Spec(H)$ or $Spec(A_H)$. Let $\lambda_1, \lambda_2, \cdots \lambda_s$ be distinct eigen values of H with multiplicities $m_1, m_2, \cdots m_s$ then we write

$$Spec(H) = \begin{pmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_s \\ m_1 & m_2 & \dots & m_s \end{pmatrix}$$

In the next section we derive some properties of the spectrum and Laplacian spectrum of (r, k) - regular hypergraphs. Also discuss the relation between the Line graph and dual of 2 - regular hypergraphs.

3. Spectrum of (r, k) - regular hypergraph

We know that the sum of the entries in each row and column of the adjacency matrix of an (r, k) - regular hypergraph H is $\theta = r(k-1)$. Thus θ is an eigen value of $A(H)$ [9, 11].

Theorem 3.1. *Let H be an (r, k) - regular linear hypergraph then*

(1) $\theta = r\,(k-1)$ is an eigen value of H.
(2) If H is connected, then multiplicity of θ is one.
(3) For any eigen value λ of H we have $|\lambda| \leq \theta$.

Proof. (1) Let $A_H = A$ be the adjacency matrix of H. Also let $u = (1, 1, 1, \cdots, 1)^t$. Since H is k-uniform, each edge has exactly k vertices. ie $|E_j| \leq k$ for $i \leq m$. Since H is r- regular each vertex x_i belongs to exactly r hyperedge. Let $v_i \in E_1, E_2, \cdots, E_k$ Then k hyper edge consist of rk vertices. So different pair of vertices with x_i as one factor is $rk - r = r(k-1)$. Let $\theta = r(k-1)$. Then $Au = \theta u$ ie θ is an eigen value of H.

(2) let $z = (z_1, z_2, \cdots, z_n)^t$, $z \neq 0$ be such that $Az = \theta z$ Let z_i be the entry of z having the largest absolute value $(Az)_i = \theta z_i$

$$\sum_{j=1}^{m} a_{ij} z_j = \theta z_i$$

We have each vertex x_i is associated with $r(k-1)$ other vertices through a hyperedge. By the maximality property of z_i, $z_j = z_i$ for all these vertices. Since H is connected all the vertices of z are equal. ie z is a multiple of $u = (1, 1, \cdots, 1)^t$. Therefore the space of eigen vector associated with the eigen value θ has dimension one. So the multiplicity of θ is one.

(3) Suppose $Ay = \lambda y, y \neq 0$. Let y_i denote the entry of y which is the largest in absolute value.

$$\begin{aligned}
(Ay)_i &= \lambda y_i \\
\sum_{j=1}^{n} a_{ij} y_j &= \lambda y_i \\
|\lambda y_i| &= \left| \sum_{j=1}^{n} a_{ij} y_j \right| \\
&\leq \sum_{j=1}^{n} a_{ij} |y_j| \\
&\leq |y_i| \sum_{j=1}^{n} a_{ij} \\
&= \theta |y_i|
\end{aligned}$$

ie

$$|\lambda| \leq \theta$$

□

Proposition 3.2. *Let H be a k - uniform hypergraph with largest eigen value λ_1. If H is regular of degree r, then $\lambda_1 = \theta$ where $\theta = r(k-1)$. Otherwise $(k-1)\delta_{min} \leq \overline{\delta} \leq \lambda_1 \leq (k-1)\delta_{max}$, where $\delta_{min}, \delta_{max}$ and $\overline{\delta}$ are the minimum, maximum and average degree respectively.*

Proof. Let $\mathbf{1}$ be the vector with all entries equal to 1. Then $A\mathbf{1} \leq (k-1)\delta_{max}\mathbf{1}$. By Theorem 2.1, $\lambda_1 \leq \theta_{max}$ where $\theta_{max} = (k-1)\delta_{max}$ and equality happens if and only if $A\mathbf{1} = \lambda_1 \mathbf{1}$, ie if and only if $\lambda_1 = r(k-1)$ where r is the degree of the vertices. Consider the partition of the vertex set consisting of a single part. By Theorem 2.3 we have $\overline{\delta} \leq \lambda_1$. Equality happens if and only if H is regular. □

Proposition 3.3. *Let H be a linear hypergraph with eigen values $\theta = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, then the following are equivalent.*

(1) *H is an (r,k) - regular hypergraph.*
(2) *$AJ = \theta J$ where $\theta = r(k-1)$ and J is an $n \times n$ matrix with all entries equal to 1.*
(3) *$\sum_{i=1}^{n} \lambda_i^2 = \theta\ n$*

Proof. The statments (1) and (2) are equivalent. Inorder to complete the theorem, we prove that that (1) and (3) are equivalent. First assume H is an (r,k) - regular hypergraph. Then

$$\sum_{i=1}^{n} \lambda_i^2 = tr(A^2) = \theta\ n$$

. Conversily assume (3). Then

$$\frac{1}{n}\sum_{i=1}^{n} \lambda_i^2 = \theta = r(k-1) = \lambda_1.$$

By Proposition 3.2, H is regular. □

Theorem 3.4. *The dual H^* of a (2,k) - regular hypergraph H is k - regular. Hence k is an eigen value of H^*.*

Proof. Let H be a $(2,k)$ - regular hypergraph. We know that $d(v_i) = 2$ for $i \leq n$ and $|E_j| = k$, for $j \leq m$. $X_i = \{e_j / x_i \in E_j\ in\ H\}$. Each x_i belongs to exactly two edges in H. So $|x_j| \leq 2$ for $j \leq n$. Since $|E_j| = k$, each e_i is adjacent to exactly k, $e_j's$ hence $d(e_j) = k$. Therefore H^* is a k - regular simple graph. Hence k is an eigen value of H^*. □

Theorem 3.5. *Let H be a $(2,k)$ - regular linear hypergraph. Its line graph $\mathcal{L} \cong H^*$*

Proof. By Theorem 3.4 , H^* is a k - regular simple graph. Let $\{x_1, x_2, \cdots, x_m\}$ be the vertices of $\mathcal{L}$. In H, each edge E_j has k - vertices. Since the vertex x_j is adjacent to k other vertices, $d(x_j) = k$ for $j \leq m$. $\mathcal{L}$ is a k - regular simple graph. Hence $\mathcal{L}$ and H^* have same number of vertices and edges. Also incidence relation is preserved. Therefore $L(H) \cong H^*$.

□

4. Laplacian Spectrum of (r,k) - regular hypergraph

We define the itLaplacian degree of a vertex $v_i \in X(H)$ as $\delta_l(v_i) = \sum_{j=1}^{n} a_{ij}$. We say that the hypergraph H is Laplacian regular of degree δ_l if any vertex $v \in X(H)$ has Laplacian degree $\delta_l(v) = \delta_l$. If H is a simple graph, then $\delta_l(v_i) = \delta(v_i)$. The Laplacian matrix of a hypergraph H is denoted by $L = L(H)$ and is defined as $L = D - A$ where $D = diag\{\delta_l(v_1), \delta_l(v_2), \cdots, \delta_l(v_n)\}$. The matrix L is symmetric and positive semi definite. All the eigen values are real and non negative. The smallest eigen value is 0 and the corresponding eigen vector is $j = (1,1,...,1)$. Moreover the multiplicity is the number of connected components of H. Second smallest and largest Laplacian eigen values and parameters related are studied by Rodrigues [10]. The eigen values of L are denoted by $0 = \mu_1 < \mu_2 < \cdots < \mu_b$ and their multiplicities $m_1, m_2, m_3, ..., m_b$. Thus the Laplacian spectrum of H and Laplacian degree of its vertices are related by

$$\sum_{l=1}^{b} m_l \mu_l = tr(L(H)) = \sum_{i=1}^{n} \delta_l(v_i).$$

If H is a regular hypergraph of degree $\delta(l)$, then $L = D - A = \delta_l I - A$. Thus the eigen values of A and L are related by $\mu_i = \delta_i - \lambda_i, \quad i = 1, 2, ..., b$.

The second smallest eigen value of the graph gives the most important information contained in the spectrum. This eigen value is called the *algebraic connectivity* and is related to several graph invariants and imposes reasonably good bounds on the value of several parameters of graphs which are very hard to compute. The concept of algeraic connectivity was introduced by Fiedler [7]. Also $\mu_2 \geq 0$ with equality if and only if H is disconnected. Algebraic connectivity is monotone. It doesnot decrease when edges are added to the graph.

Theorem 4.1. *Let H be a k - uniform hypergraph. Also let μ_2 be the algebraic connectivity of H. For any non adjacent vertices s and t in H we have,*

$$\mu_2(H) \leq \frac{k-1}{2}(deg(s) + deg(t)).$$

Proof. We have,

$$\mu_2(H) = \overset{Min}{u} \{\frac{< Lu, u >}{< u, u >} :< u, \mathbf{1} >= 0\}$$

The vector u is defined by,

$$u_x = \begin{cases} 1 & \text{if } x = s \\ -1 & \text{if } x = t \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $< u, \mathbf{1} >= 0$.

$$\mu_2(H) \leq \frac{< Lu, u >}{< u, u >}$$

$$= \frac{\sum_{xy \in E_j} a_{xy}(u_x - u_y)^2}{\sum_{x \in V} u_x^2}$$

$$= \frac{(k-1)(\sum_x a_{xs} + \sum_x a_{tx})}{2}$$

$$= \frac{(k-1)}{2}(deg(s) + deg(t))$$

□

Let G be the weighted graph on the same verex set X. Two vertices x and y are adjacent in X if they are adjacent in H also. The edge weight of xy is equal to (a_{ij}), the number of edges in H containing both x and y. Clearly $L(G) = L(H)$. Rodriguez [10] obtain the result from Fiedler [7] on weighted graph as

$$\mu_2 = 2n \; min\{\frac{\sum_{xy \in E_j} a_{xy}(u_x - u_y)^2}{\sum_{x \in X} \sum_{x \in X} (u_x - u_y)^2} : u \neq \alpha \mathbf{1} \; for \; \alpha \in R\} \tag{4.2}$$

$$\mu_b = 2n \; max\{\frac{\sum_{xy \in E_j} a_{xy}(u_x - u_y)^2}{\sum_{x \in X} \sum_{x \in X} (u_x - u_y)^2} : u \neq \alpha \mathbf{1} \; for \; \alpha \in R\} \tag{4.3}$$

This eigen values μ_2 and μ_b are bounded in terms of maximum and minimum degree of H. For any vertex x, e_x denote the corresponding unit vector of the cannonical basis of R^n by putting $u = e_x$ in equations (4.2) and (4.3) we get,

$$\mu_2 \leq \frac{n}{n-1}\delta_x \leq \mu_b$$

This leads to

$$\mu_2 \leq \frac{n}{n-1}\delta_{l\ min} \leq \frac{n}{n-1}\delta_{l\ max} \leq \mu_b$$

Theorem 4.4. *Let H_1 and H_2 be two edge disjoint hypergraphs on the same vertex set and $H_1 \cup H_2$ be their union. Then*

$$\mu_2(H_1 \cup H_2) \geq \mu_2(H_1) + \mu_2(H_2) \geq \mu_2(H_1)$$

Proof. Let L_1, L_2 and L be the Laplace adjacency matrix of H_1, H_2 and $H = H_1 \cup H_2$ respectively. We have,

$$\mu_2(H) = \overset{Min}{u}\{\frac{< Lu, u >}{< u, u >} :< u, \mathbf{1} >= 0\}$$

$$= \overset{Min}{u}\{\frac{< (L_1 + L_2)u, u >}{< u, u >} :< u, \mathbf{1} >= 0\}$$

$$= \overset{Min}{u}\{\frac{< (L_1 u + L_2 u), u >}{< u, u >} :< u, \mathbf{1} >= 0\}$$

$$\geq \overset{Min}{u}\{\frac{< L_1 u, u >}{< u, u >} :< u, \mathbf{1} >= 0\} + \overset{Min}{u}\{\frac{< L_2 u, u >}{< u, u >} :< u, \mathbf{1} >= 0\}$$

$$\geq \mu_1(H_1) + \mu_2(H_2)$$

$$\geq \mu_1(H_1)$$

□

Acknowledgments. A part of this research was done when the first author visited the Kerala School of Mathematics (KSOM) at Kozhikodu, Kerala in India. He is indebted to KSOM for research facilities given by the institute.

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