Symmetric normal matrices and the Marcus-de Oliveira Determinantal Conjecture

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Abstract

We present notes on the Marcus-de Oliveira conjecture. The conjecture concerns the region in the complex plane covered by the determinants of the sums of two normal matrices with prescribed eigenvalues. Call this region Δ. Let Δ_S be the restriction of Δ to determinants of sums of *symmetric* normal matrices. In this paper, we conjecture that Δ_S has the same boundary as Δ. We prove the conjecture under the restriction that at least one of the two matrices has distinct eigenvalues. If this conjecture is true then proving the Marcus-de Oliveira conjecture for symmetric normal matrices would prove it for the general case. This paper builds on work in [1].

Keywords:
determinantal conjecture; Marcus-de Oliveira; determinants; normal matrices; convex-hull

MSC:
15A15

1 Introduction

Marcus [3] and de Oliveira [2] made the following conjecture. Given two normal matrices A and B with prescribed eigenvalues $a_1, a_2 ... a_n$ and $b_1, b_2 ... b_n$ respectively, $det(A+B)$ lies within the region:

$co\{\prod(a_i + b_{\sigma(i)})\}$

where $\sigma \in S_n$. co denotes the convex hull of the $n!$ points in the complex plane.

As described in [1], the problem can be restated as follows:

Given two diagonal matrices, $A_0 = diag(a_1, a_2 ... a_n)$ and $B_0 = diag(b_1, b_2 ... b_n)$, let:

$\Delta = \{det(A_0 + UB_0U^*) : U \in U(n)\}$, where $U(n)$ is the set of $n \times n$ unitary matrices. Then we can write the conjecture as:

(D.1)
Marcus-de Oliveira conjecture
$\Delta \subseteq co\{\prod(a_i + b_{\sigma(i)})\}$.

(E.1)
let $R(U) = det(A_0 + UB_0U^*)$.

Then the points forming the convex hull are at $R(P_0), R(P_1)...R(P_{n!-1})$, where the P's are the $n \times n$ permutation matrices. We will refer to these as **permutation points** from now on.

2 Restriction to real orthogonal matrices

$\Delta_S = \{det(A_0 + OB_0O^*) : O \in O(n)\}$, where $O(n)$ is the set of $n \times n$ real orthogonal matrices.

As proven in [4], p.207, theorem 4.4.7, a matrix is normal and symmetric if and only if it is diagonalizable by a real orthogonal matrix.

Therefore Δ_S is the set of determinants of sums of normal, symmetric matrices with prescribed eigenvalues. We know Δ_S contains all the permutation points.

(D.2)
Restricted Marcus-de Oliveira conjecture
$\Delta_S \subseteq co\{\prod(a_i + b_{\sigma(i)})\}$.

3 Main Ideas

(C.1)
Symmetric Sufficiency Conjecture
Δ and Δ_S have the same boundary.

This conjecture is supported by computational experiments.

Suppose P is a boundary point of Δ and U is a unitary matrix such that $R(U) = P$, then we call U a **boundary matrix** of Δ. A boundary point may have multiple boundary matrices. A boundary matrix that is not a permutation matrix is called a **regular boundary matrix**.

If every boundary point on Δ has a real orthogonal boundary matrix, this would prove (C.1).

The rest of this paper concerns proving the following restricted form of the conjecture:

(C.2)
Restricted Symmetric Sufficiency Conjecture
Δ and Δ_S have the same boundary when at least A_0 or B_0 has distinct eigenvalues.

4 Properties of unitary matrices given A_0 and B_0

In this section, we will define four properties of unitary matrices that will be very useful when examining boundary matrices of Δ. These properties will be referred to throughout the paper in relation to a given unitary matrix U.

The first three of these properties are matrices related to U. These matrices are defined in [1], p.27. They provide a language to talk about unitary matrices within the context of the determinantal conjecture.

B-matrix
$B = UB_0U^*$

C-matrix

$C = A_0 + UB_0U^*$

Using (E.1), $R(U) = det(C)$

F-matrix
$F = BC^{-1} - C^{-1}B$

We can change the F-matrix into a more useful form:

$F = (C - A_0)C^{-1} - C^{-1}(C - A_0)$
$F = C^{-1}A_0 - A_0C^{-1}$

The F-matrix is only defined when C is invertible or equivalently $R(U) \neq 0$.

Since A_0 is diagonal, we see that F is a zero-diagonal matrix. Also note that if B is symmetric then C is symmetric and F is skew-symmetric.

As demonstrated in [1], p.27, the F-matrix is 0 if and only if U is a permutation matrix.

The fourth property is conditional. Given a unitary matrix U with $R(U) \neq 0$ and with F-matrix $F \neq 0$. let $T = tr(ZF)$, where Z is any skew-hermitian matrix. T is a complex number and can be seen as a vector in the complex plane. If for all possible skew-hermitian matrices Z, all values of T are either parallel or anti-parallel, then we say that U is **trace-argument constant**. We take the zero-vector as being parallel to any vector.

5 Slope of the tangent at the boundary of Δ

Our aim is to examine boundary matrices of Δ. Towards this aim, it is very useful to consider smooth unitary matrix functions going through these boundary matrices and see how they behave under (E.1). For this reason, we introduce the functional form of (E.1):

$R(t) = \det(A_0 + U(t)B_0U^*(t))$, where t is real and $U(t)$ is some smooth function of unitary matrices.

Suppose $U(t)$ goes through a boundary matrix of interest, U_0 at t=0.

Every unitary matrix can be written as an exponential of a skew-hermitian matrix. So we can write:

$U(t) = e^{S(t)}U_0$, where $S(t)$ is a smooth function of skew hermitian matrices with $S(0) = 0$.

Every choice of $S(t)$ with $S(0) = 0$, gives us every possible $U(t)$ that passes through U_0 at $t = 0$.

We wish to examine $U(t)$ and $R(t)$ near $t = 0$.

For small Δt,

$U(\Delta t) = (e^{S(\Delta t)})U_0$

$U(\Delta t) = (e^{S(0)+(\Delta t)S'(0)})U_0$

$U(\Delta t) = (e^{(\Delta t)S'(0)})U_0$

If we take the above function and plug it into $R(t)$ we'll get $R(\Delta t)$, but it won't be in a form useful to us. We use a result from [1], p.27 for this purpose. In order to state this result within the context of this paper, we first need the functional forms of the B-matrix, C-matrix, F-matrix (these were defined in section 4):

$B(t) = U(t)B_0U^*(t)$

$C(t) = A_0 + B(t)$

$F(t) = C^{-1}(t)A_0 - A_0C^{-1}(t)$

Now we can state the result from [1]:

When $F(0) \neq 0$,
$R(\Delta t) = R(0) + (\Delta t)\det(C(0))tr(S'(0)F(0)) + O((\Delta t)^2)$

Therefore,

When $F(0) \neq 0$,
$R'(0) = det(C(0))tr(S'(0)F(0))$

If $F(0) = 0$ then U_0 is a permutation matrix (section 4). The permutation matrix is already real and orthogonal so we needn't be concerned with this possibility.

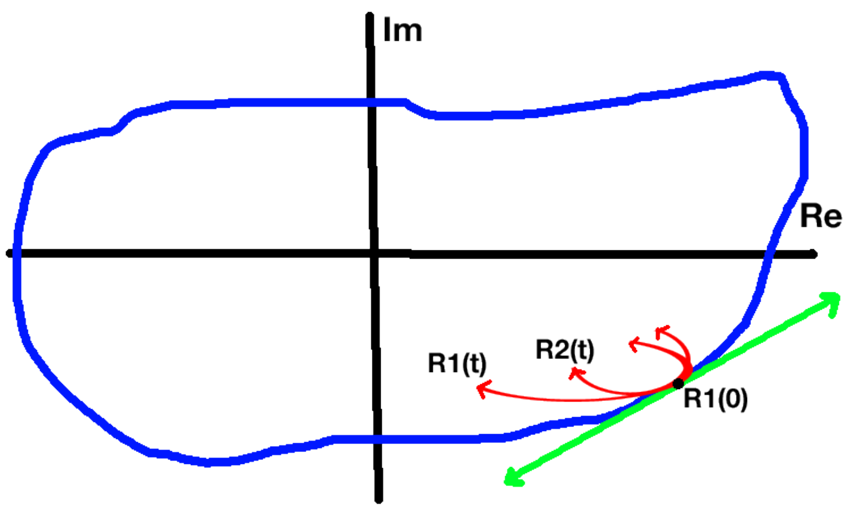
Assume $F(0) \neq 0$.

Note that $C(0)$ is just the C-matrix of U_0 and $F(0)$ is just the F-matrix of U_0. Also, $F(0)$ is only defined as long as $R(0) \neq 0$.

Since U_0 is a regular boundary matrix, the tangent line to the curve $R(t)$ at $t = 0$ must remain the same regardless of our choice of $S(t)$. This is illustrated in (F.1) where the closed curve indicates the boundary of Δ. $R'(0)$ can be seen as a vector in the complex plane. So all possible values of $R'(0)$ are either parallel or anti-parallel.

(F.1)

Figure 1:



$S'(0)$ is a skew hermitian matrix since the difference of skew-hermitian matrices is also skew-hermitian. $S'(0)$ can turn out to be any skew-hermitian matrix. Proof: Suppose we choose an arbitrary skew-hermitian matrix and multiply each element of the matrix by t. Then we get a smooth function of skew-hermitian matrices $S(t)$ with $S(0) = 0$ such that $S'(0)$ is the skew-hermitian matrix we initially chose.

So we can rewrite $R'(0)$ without any reference to the $S(t)$ function:

$R'(0) = det(C(0))tr(ZF(0))$

where Z is a skew-hermitian matrix. All values of $tr(ZF(0))$ are parallel or anti-parallel, regardless of the choice of Z.

So we have the following result:

(T.1)
THEOREM 1
Every regular boundary matrix U of Δ with $R(U) \neq 0$ is trace-argument constant. (as defined in section 4)

The bulk of the rest of the paper concerns proving the following theorem:

(T.2)
THEOREM 2
Given at least A_0 or B_0 has distinct eigenvalues, if a unitary matrix U is trace-argument constant, then there exists a real orthogonal matrix O such that $R(O) = R(U)$.

(T.2) along with (T.1) proves that under the eigenvalue restriction, for every nonzero boundary point P of Δ, there exists a real orthogonal matrix O such that $R(O) = P$. Thereby giving us (C.2) for all cases except when P=0. We will deal with this case after proving Theorem 2.

6 Tools for proving Theorem 2

(T.3)
THEOREM 3
Given a unitary matrix U that is trace-argument constant with F-matrix F, $|F_{ab}| = |F_{ba}|$.

Proof:

We already know that the theorem is true when a = b since F is zero-diagonal.

For n = 3, we define the following 6 skew-hermitian matrices with zero diagonal:

$$Z_{01} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} Z_{02} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} Z_{12} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$Z_{01,i} = \begin{bmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} Z_{02,i} = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix} Z_{12,i} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}$$

Note that the commas do not indicate tensors. They're just used here as a label to distinguish imaginary and real matrices.

We define Z_{ab} and $Z_{ab,i}$ similarly for all n > 3, where $a \neq b$. For a given n we have $\frac{n(n-1)}{2}$ real matrices and $\frac{n(n-1)}{2}$ imaginary matrices.

Suppose $F_{ab} = F_{ab,r} + iF_{ab,i}$
where $F_{ab,r}$ and $F_{ab,i}$ are real numbers.

$tr(Z_{ab}F) = F_{ab} - F_{ba}$

$tr(Z_{ab,i}F) = (F_{ab} + F_{ba})i$

Substitute in for F_{ab} and F_{ba}

$tr(Z_{ab}F) = (F_{ab,r} - F_{ba,r}) + i(F_{ab,i} - F_{ba,i})$

$tr(Z_{ab,i}F) = (-F_{ab,i} - F_{ba,i}) + i(F_{ab,r} + F_{ba,r})$

Since U is trace-argument constant,

$(F_{ab,i} - F_{ba,i})(-F_{ab,i} - F_{ba,i}) = (F_{ab,r} + F_{ba,r})(F_{ab,r} - F_{ba,r})$

We can simplify this to get:

$F_{ab,r}^2 + F_{ab,i}^2 = F_{ba,r}^2 + F_{ba,i}^2$

$|F_{ab}| = |F_{ba}|$

(T.4)

THEOREM 4

Given a unitary matrix U with F-matrix F, and a unitary diagonal matrix D, let $U_2 = D^*U$. then:

1. $R(U_2) = R(U)$

2. the F-matrix of U_2 is D^*FD.

Proof of 1 is trivial.

Proof of 2:

let B_2, C_2 and F_2 be the B,C and F-matrices of U_2.

$B_2 = U_2 B_0 U_2^*$
$B_2 = D^*BD$

$C_2^{-1} = (A_0 + D^*BD)^{-1}$
$C_2^{-1} = D^*C^{-1}D$

$F_2 = C_2^{-1}A_0 - A_0C_2^{-1}$

Substitute in C_2^{-1} and simplify to get:

$F_2 = D^*FD$

(T.5)

THEOREM 5

For every $n \times n$ complex matrix C, and every real $n \times n$ skew-symmetric matrix P, there exists a complex $n \times n$ matrix A and a unitary $n \times n$ diagonal matrix D such that $A = D^*CD$ and $arg(A_{ij}) - arg(A_{ji}) = P_{ij}$

Proof:

The first step is to set the arguments of the elements of A in the first row and column:

$i = 0, 0 \leq j \leq n-1$
$j = 0, 0 \leq i \leq n-1$

Let D_1 be a diagonal matrix of n variable arguments (each element is of the form e^{ϕ_i}). We can expand $A = D_1^* C D_1$ and set $arg(A_{ij}) - arg(A_{ji}) = P_{ij}$ for all elements in the region above. This will give us $n-1$ linear equations with n variables. We have 1 free variable so we can always find a diagonal matrix D_1 to accomplish this.

We repeat the step above for this region (the second row and column of A, excluding elements from the first row and column):

$i = 1, 1 \leq j \leq n-1$
$j = 1, 1 \leq i \leq n-1$

We leave the first row and column unchanged. Let D_2 be a diagonal matrix with 1 in the first spot, and variables in the other spots. Expanding $A = D_2^* D_1^* C D_1 D_2$, we set $arg(A_{ij}) - arg(A_{ji}) = P_{ij}$ for the elements in the above region. This time we have $n-2$ equations with $n-1$ variables. Again we can find a diagonal matrix D_2.

Repeating this process we'll get a diagonal matrix

$D = D_1 D_2 D_3 D_4 ... D_{n-1}$ that sets the arguments of all pairs of elements A_{ij} and A_{ji}.

This is a very useful theorem as it allows us to take any matrix, and construct a unitarily similar one, with the difference of arguments of transpositional elements set to whatever we want.

(T.6)
THEOREM 6
If A_0 has distinct eigenvalues, then given a unitary matrix U, its F-matrix is skew-symmetric if and only if its B-matrix is symmetric.

Proof:

$F_T = -F$

$A_0 C_T^{-1} - C_T^{-1} A_0 = -(C^{-1} A_0 - A_0 C^{-1})$

$A_0 (C_T^{-1} - C^{-1}) = (C_T^{-1} - C^{-1}) A_0$

$Q = C_T^{-1} - C^{-1}$

Q is skew-symmetric and A_0 commutes with Q.

A diagonal matrix with distinct eigenvalues only commutes with another diagonal matrix. So Q must be diagonal. The only diagonal skew-symmetric matrix is the zero-matrix.

This proves that C is symmetric. Therefore the B-matrix is symmetric.

The opposite implication is trivial (mentioned in section 4).

7 Proof of Theorem 2

Given at least A_0 or B_0 has distinct eigenvalues, let U be a trace-argument constant unitary matrix. Without loss of generality, we may choose A_0 to be the matrix with distinct eigenvalues. Let F be U's F-matrix.

F may or may not be skew-symmetric. If it isn't, we can construct a matrix F_2 that is unitarily similar to F, but is skew-symmetric. Using the method in (T.5), we simply set the upper triangular elements of P to π and the lower triangular elements to $-\pi$. Then we can solve for D.

$F_2 = D^*FD$

Since F_2 is zero-diagonal(section 4) with transpositional elements of equal magnitude(T.3), it is skew-symmetric.

We require a unitary matrix U_2 whose F-matrix is F_2. By (T.4), if we choose $U_2 = D^*U$, then U_2's F-matrix is F_2 and $R(U_2) = R(U)$.

By (T.6), U_2's B-matrix, B_2, must be symmetric.

B_2 is a normal, symmetric matrix. So using [4], p.207, theorem 4.4.7, there exists a real orthogonal matrix O, such that:

$B_2 = OB_0O^*$

But we also know that:

$B_2 = U_2B_0U_2^*$

Therefore by (E.1):

$R(O) = R(U_2) = R(U)$.

This proves (T.2), thereby proving (C.2) for nonzero boundary points.

8 Boundary point P=0

Suppose the boundary of Δ includes the point P=0. We can use a topological argument to show that Δ_S contains P.

We know that the unitary matrices are a connected subset of the $n \times n$ complex matrices. So Δ is a connected subset of the complex plane. Therefore we know that P is a limit point of Δ (not an isolated point).

First we'll show that P is a limit point of Δ's boundary using a proof by contradiction. Assume P is an isolated point on the boundary. That means there exists an open ball B centered on P that contains no other boundary points.

Since P is a limit point of Δ, B contains at least one other point in Δ, say Q. And since P is a boundary point, B contains at least one point not in Δ, say R. We can draw a path from Q to R that is completely inside B and goes around P. That means there exists a boundary point on this path. But this contradicts the fact that B has no boundary points other than P. So our initial assumption that P is an isolated point of Δ's boundary is false, and P is a limit point of Δ's boundary.

We already know that the boundary of Δ_S contains all of Δ's boundary points other than P. Since the boundary of a set is closed, it must contain all of its limit points. Hence the boundary of Δ_S contains P.

So this proves (C.2) in full.

9 Conclusions - Implication of the Symmetric Sufficiency Conjecture

(T.7)
THEOREM 7
If the Symmetric Sufficiency Conjecture is true, then the restricted Marcus-de Oliveira conjecture implies the full Marcus-de Oliveira conjecture.

Proof:

The unitary group and the real orthogonal group are closed subsets of the $n \times n$ complex matrices. Hence Δ and Δ_S are closed subsets of the complex plane.

Suppose we know that (D.2) is true. Then Δ_S along with its boundary is within the convex-hull. Suppose we also know that the symmetric sufficiency conjecture is true. Then we know that the boundary of Δ is inside the convex-hull. Can we have a unitary matrix U such that $R(U)$ is outside the convex-hull? No, because that would mean we have points of Δ on both the inside and outside of the boundary of Δ. This is impossible since Δ is a closed set. So Δ is within the convex hull proving (D.1).

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