Symmetric normal matrices and the Marcus-de Oliveira Determinantal Conjecture

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December 23, 2017

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Abstract

We present notes on the Marcus-de Oliveira conjecture. The conjecture concerns the region in the complex plane covered by the determinants of the sums of two normal matrices with prescribed eigenvalues. Call this region Δ. Let Δ_S be the restriction of Δ to determinants of sums of *symmetric* normal matrices. This paper focuses on boundary matrices of Δ. We derive some properties of boundary matrices and boundary points. We conjecture that $\partial\Delta \subseteq \partial\Delta_S$. Speculations on how to prove this conjecture are given. We also present a second conjecture with regards to the form of normal matrices with magnitude symmetry. This paper builds on work in [1].

Keywords:
determinantal conjecture; Marcus-de Oliveira; determinants; normal matrices; convex-hull

MSC:
15A15

1 Introduction

Marcus [3] and de Oliveira [2] made the following conjecture. Given two normal matrices A and B with prescribed eigenvalues $a_1, a_2 ... a_n$

and $b_1, b_2 ... b_n$ respectively, $det(A + B)$ lies within the region:

$co\{\prod(a_i + b_{\sigma(i)})\}$

where $\sigma \in S_n$. co denotes the convex hull of the $n!$ points in the complex plane.

As described in [1], the problem can be restated as follows:

Given two diagonal matrices, $A_0 = diag(a_1, a_2 ... a_n)$ and $B_0 = diag(b_1, b_2 ... b_n)$, let:

$\Delta = \{det(A_0 + UB_0U^*) : U \in U(n)\}$, where $U(n)$ is the set of $n \times n$ unitary matrices. Then we can write the conjecture as:

(D.1)
Marcus-de Oliveira conjecture
$\Delta \subseteq co\{\prod(a_i + b_{\sigma(i)})\}$.

(E.1)
let $R(U) = det(A_0 + UB_0U^*)$.

Then the points forming the convex hull are at $R(P_0), R(P_1) ... R(P_{n!-1})$, where the P's are the $n \times n$ permutation matrices. We will refer to these as **permutation points** from now on.

2 Restriction to real orthogonal matrices

$\Delta_S = \{det(A_0 + OB_0O^*) : O \in O(n)\}$, where $O(n)$ is the set of $n \times n$ real orthogonal matrices.

As proven in [4], p.207, theorem 4.4.7, a matrix is normal and symmetric if and only if it is diagonalizable by a real orthogonal matrix.

Therefore Δ_S is the set of determinants of sums of normal, symmetric matrices with prescribed eigenvalues. We know Δ_S contains all the permutation points.

(D.2)

Restricted Marcus-de Oliveira conjecture
$\Delta_S \subseteq co\{\prod(a_i + b_{\sigma(i)})\}$.

3 Two conjectures

(C.1)
Boundary Conjecture
$\boldsymbol{\partial\Delta \subseteq \partial\Delta_S}$**.**

This conjecture is supported by computational experiments.

Suppose P is a boundary point of Δ and U is a unitary matrix such that $R(U) = P$, then we call U a **boundary matrix** of Δ. A boundary point may have multiple boundary matrices. A **regular boundary point** is a point where the boundary is smooth. A non-permutation boundary matrix for a regular boundary point is called a **regular boundary matrix**.

If every boundary point on Δ has a real orthogonal boundary matrix, this would prove (C.1).

In order to state the next conjecture we need to define some terms:

A **quasi-symmetric** matrix is a matrix that can be written as DSD^* where S is a complex, symmetric matrix and D is a unitary diagonal matrix.

A **quasi-hermitian** matrix is a matrix that can be written as $e^{i\theta}H$ where θ is real and H is hermitian.

A matrix M is **magnitude-symmetric** if $|M_{ij}| = |M_{ji}|$ for all i,j.

(C.2)
Magnitude-Symmetry Conjecture
Given a complex, normal matrix N that is magnitude-symmetric, N is quasi-symmetric or quasi-hermitian.

The conjecture is trivial for n = 2. If n = 2, we can always find a unitary diagonal matrix D such that $S = D^*ND$ is symmetric. So N is quasi-symmetric.

We came upon (C.2) while attempting to prove (C.1).

4 Properties of unitary matrices given A_0 and B_0

In this section, we define four properties of unitary matrices that will be very useful when examining boundary matrices of Δ. These properties will be referred to throughout the paper in relation to a given unitary matrix.

The first three of these properties are matrices related to U. These matrices are defined in [1], p.27. They provide a language to talk about unitary matrices within the context of the determinantal conjecture.

B-matrix
$B = UB_0U^*$

C-matrix
$C = A_0 + UB_0U^*$

Using (E.1), $R(U) = det(C)$

F-matrix
$F = BC^{-1} - C^{-1}B$

We can change the F-matrix into a more useful form:

$F = (C - A_0)C^{-1} - C^{-1}(C - A_0)$
$F = C^{-1}A_0 - A_0C^{-1}$

The F-matrix is only defined when C is invertible or equivalently $R(U) \neq 0$.

Since A_0 is diagonal, we see that F is a zero-diagonal matrix.

As demonstrated in [1], p.27, the F-matrix is 0 if and only if U is a permutation matrix.

The fourth property is conditional. Given a unitary matrix U with $R(U) \neq 0$ and with F-matrix $F \neq 0$. let $T = tr(ZF)$, where Z is any skew-hermitian matrix. T is a complex number and can be seen as a vector in the complex plane. If for all possible skew-hermitian matrices Z, all values of T are either parallel or anti-parallel, then we say that U is **trace-argument constant**. We take the zero-vector as being parallel to any vector.

5 Tangent at $\partial\Delta$

Our aim is to examine boundary matrices of Δ. Towards this aim, it is useful to consider smooth unitary matrix functions going through these boundary matrices and see how they behave under (E.1). For this reason, we introduce the functional form of (E.1):

$R(t) = \det(A_0 + U(t)B_0U^*(t))$, where t is real and $U(t)$ is some smooth function of unitary matrices.

Suppose $U(t)$ goes through a boundary matrix of interest, U_0 at t = 0.

Every unitary matrix can be written as an exponential of a skew-hermitian matrix. So we can write:

$U(t) = e^{S(t)}U_0$, where $S(t)$ is a smooth function of skew hermitian matrices with $S(0) = 0$.

Every choice of $S(t)$ with $S(0) = 0$, gives us every possible $U(t)$ that passes through U_0 at $t = 0$.

We wish to examine $U(t)$ and $R(t)$ near $t = 0$.

For small Δt,

$U(\Delta t) = (e^{S(\Delta t)})U_0$

$U(\Delta t) = (e^{S(0)+(\Delta t)S'(0)})U_0$

$U(\Delta t) = (e^{(\Delta t)S'(0)})U_0$

If we take the above function and plug it into $R(t)$ we'll get $R(\Delta t)$, but it won't be in a form useful to us. We use a result from [1], p.27 for this purpose. In order to state this result within the context of this paper, we first need the functional forms of the B-matrix, C-matrix, F-matrix (these were defined in section 4):

$B(t) = U(t)B_0U^*(t)$

$C(t) = A_0 + B(t)$

$F(t) = C^{-1}(t)A_0 - A_0C^{-1}(t)$

Now we can state the result from [1]:

When $F(0) \neq 0$,
$R(\Delta t) = R(0) + (\Delta t)\det(C(0))tr(S'(0)F(0)) + O((\Delta t)^2)$

Therefore,

When $F(0) \neq 0$,
$R'(0) = det(C(0))tr(S'(0)F(0))$

If $F(0) = 0$ then U_0 is a permutation matrix (section 4). The permutation matrix is already real and orthogonal so we needn't be concerned with this possibility.

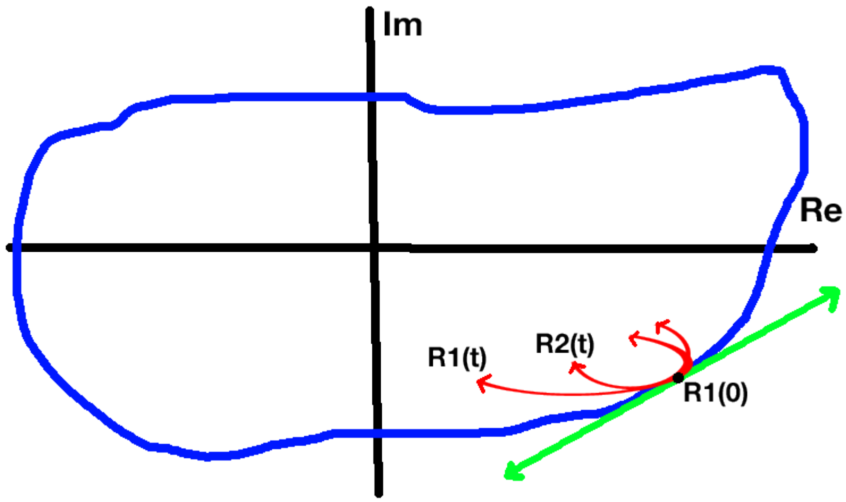
Assume $F(0) \neq 0$.

Note that $C(0)$ is just the C-matrix of U_0 and $F(0)$ is just the F-matrix of U_0. Also, $F(0)$ is only defined as long as $R(0) \neq 0$.

Suppose U_0 is a regular boundary matrix, the tangent line to the curve $R(t)$ at $t = 0$ must remain the same regardless of our choice of $S(t)$. This is illustrated in (F.1) where the closed curve indicates $\partial\Delta$. $R'(0)$ can be seen as a vector in the complex plane. So all possible values of $R'(0)$ are either parallel or anti-parallel.

(F.1)

Figure 1:



$S'(0)$ is a skew hermitian matrix since the difference of skew-hermitian matrices is also skew-hermitian. $S'(0)$ can turn out to be any skew-hermitian matrix. Proof: Suppose we choose an arbitrary skew-hermitian matrix and multiply each element of the matrix by t. Then we get a smooth function of skew-hermitian matrices $S(t)$ with $S(0) = 0$ such that $S'(0)$ is the skew-hermitian matrix we initially chose.

So we can rewrite $R'(0)$ without any reference to the $S(t)$ function:

$R'(0) = det(C(0))tr(ZF(0))$

where Z is a skew-hermitian matrix. All values of $tr(ZF(0))$ are parallel or anti-parallel, regardless of the choice of Z.

So we have the following result:

(T.1)
THEOREM 1
Every regular boundary matrix U of Δ with $R(U) \neq 0$ is trace-argument constant. (as defined in section 4)

6 $\partial\Delta$ is smooth at all non-zero, non-permutation points.

In [1], p.26, Theorem 4, Bebiano and Queiró prove that if within the neighborhood of a non-zero point $z \in \partial\Delta$, Δ is contained within an angle less than π, then z must be a permutation point.

We extend this result here.

(T.2)
THEOREM 2
Given a non-zero point $z \in \partial\Delta$. If within the neighborhood of z, Δ is contained within an angle greater than π, then z must be a permutation point.

Proof:

Assume we have a non-zero point $z \in \partial\Delta$, such that within the neighborhood of z, Δ is contained within an angle greater than π . Since the angle is greater than π, we can find two smooth functions $R_1(t) \subseteq \Delta$ and $R_2(t) \subseteq \Delta$ such that $R_1(0) = R_2(0) = z$ and $R_1'(0)$ is not parallel or anti-parallel to $R_2'(0)$.

Assume z is not a permutation point. Let U be a boundary matrix for z and let F be the F-matrix of U. Then using the tools developed in section 5,

$R_1'(0) = det(C)tr(Z_1F)$
$R_2'(0) = det(C)tr(Z_2F)$

where Z_1 and Z_2 are two skew-hermitian matrices. But since $R_1'(0)$ and $R_2'(0)$ are not parallel or anti-parallel, they form a basis for all the complex numbers as a vector space over the real numbers.

So $V = a \times det(C)tr(Z_1F) + b \times det(C)tr(Z_2F)$ goes in any direction depending on the choice of real numbers a and b.

$V = det(C)(a \times tr(Z_1F) + b \times tr(Z_2F))$
$V = det(C)tr((a \times Z_1 + b \times Z_2)F)$

$Z_n = a \times Z_1 + b \times Z_2$ is also a skew-hermitian matrix.

So given any direction, there exists a skew-hermitian matrix Z_n such that $det(C)tr(Z_n F)$ goes in that direction. Hence there exists a smooth function $R_n(t) \subseteq \Delta$ such that $R_n(0) = z$, and $R'_n(0)$ is parallel or anti-parallel to that direction.

So there are functions going through z in all directions, contained within Δ. So z is not a boundary point. We arrive at a contradiction, and so z must be a permutation point.

So within the neighborhood of a non-zero, non-permutation point $z \in \partial\Delta$, Δ is contained within an angle of π. This means the boundary is smooth at all non-zero, non-permutation points.

7 Properties of trace-argument constant matrices

(T.3)
THEOREM 3
Given a unitary matrix U that is trace-argument constant, its F-matrix is quasi-hermitian.

Proof:

For n = 3, we define the following 12 skew-hermitian matrices with zero diagonal:

$$Z_{12} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad Z_{13} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad Z_{23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$Z_{21} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad Z_{31} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \qquad Z_{32} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$Z_{12,i} = Z_{21,i} = \begin{bmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad Z_{13,i} = Z_{31,i} = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix} \quad Z_{23,i} = Z_{32,i} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}$$

Note that the commas do not indicate tensors. They're just used here as a label to distinguish imaginary and real matrices.

We define Z_{ab} and $Z_{ab,i}$ similarly for all n > 3, where $a \neq b$. For a given n we have $n(n-1)$ real matrices and $n(n-1)$ imaginary matrices.

Suppose $F_{ab} = F_{ab,r} + iF_{ab,i}$
where $F_{ab,r}$ and $F_{ab,i}$ are real numbers.

$tr(Z_{ab}F) = F_{ab} - F_{ba}$

$tr(Z_{ab,i}F) = (F_{ab} + F_{ba})i$

Substitute in for F_{ab} and F_{ba}

$tr(Z_{ab}F) = (F_{ab,r} - F_{ba,r}) + i(F_{ab,i} - F_{ba,i})$

$tr(Z_{ab,i}F) = (-F_{ab,i} - F_{ba,i}) + i(F_{ab,r} + F_{ba,r})$

Since U is trace-argument constant,

$(F_{ab,i} - F_{ba,i})(-F_{ab,i} - F_{ba,i}) = (F_{ab,r} + F_{ba,r})(F_{ab,r} - F_{ba,r})$

We can simplify this to get:

$F_{ab,r}^2 + F_{ab,i}^2 = F_{ba,r}^2 + F_{ba,i}^2$

$|F_{ab}| = |F_{ba}|$

We can write:
$F_{ab} = |F_{ab}|\angle\theta_{ab}$
$F_{ba} = |F_{ab}|\angle\theta_{ba}$

slope of $tr(Z_{ab}F)$:

$\frac{\sin(\theta_{ab})-\sin(\theta_{ba})}{\cos(\theta_{ab})-\cos(\theta_{ba})} = -\cot(\frac{\theta_{ab}+\theta_{ba}}{2})$

similarly,
slope of $tr(Z_{cd}F) = -\cot(\frac{\theta_{cd}+\theta_{dc}}{2})$, where $c \neq d$

$\cot(\frac{\theta_{cd}+\theta_{dc}}{2}) = \cot(\frac{\theta_{ab}+\theta_{ba}}{2})$

therefore either:

$\frac{\theta_{cd}+\theta_{dc}}{2} = \frac{\theta_{ab}+\theta_{ba}}{2}$

or,

$\frac{\theta_{cd}+\theta_{dc}}{2} = \frac{\theta_{ab}+\theta_{ba}}{2} + \pi$

For some specific x, y where $x \neq y$

let $\beta = \frac{\theta_{xy}+\theta_{yx}}{2}$

let $H = e^{-i\beta}F$

For any $a \neq b$,

$H_{ab} = |H_{ab}|\angle\alpha_{ab}$

$\frac{\alpha_{ab}+\alpha_{ba}}{2} = 0$ or π

H is zero-diagonal, with transpositional elements of equal magnitude and opposite arguments. Therefore H is hermitian and F is quasi-hermitian.

(CR.1)
COROLLARY
Given a unitary matrix that is trace-argument constant with C-matrix C, C^{-1} is magnitude-symmetric.

Proof:

$F = C^{-1}A_0 - A_0C^{-1}$

$F_{ij} = C^{-1}_{ij}(a_j - a_i)$ where a_i and a_j are the ith and jth eigenvalues of A_0

$F_{ji} = C^{-1}_{ji}(a_i - a_j)$

Since F is magnitude-symmetric by (T.3), C^{-1} must be magnitude-symmetric.

(T.4)
THEOREM 4
Suppose we are given that for every unitary matrix that is trace-argument constant, its C-matrix is quasi-symmetric. Then the boundary conjecture is true.

Given C is the C-matrix of an arbitrary trace-argument constant unitary matrix.

Assume C is quasi-symmetric. Then using [4], p.207, theorem 4.4.7,

$C = A_0 + DOB_0O^TD^*$, for some diagonal unitary matrix D and real orthogonal matrix O.

Using (E.1), $R(O) = det(C)$. Then using (T.1) every non-zero boundary point of Δ has an orthogonal boundary matrix.

We're left to deal with a possible zero boundary point. We can use a topological argument to include zero.

We'll prove the following theorem first before finishing Theorem 4:

(T.5)
THEOREM 5
Let S be of a connected subset of a euclidean space E. If S has more than one point, then ∂S has no isolated points.

Given S is a connected subset of some euclidean space E. Given S has more than one point.

We will use a proof by contradiction. Assume ∂S has an isolated point P. Then there exists an open ball B centered on P that contains no other boundary points. Since S is a connected subset with more than one point, B contains at least one other point in S, say Q. And since P ∈ ∂S, B contains at least one point not in S, say R.

We can draw a path from Q to R that is completely inside B and goes around P. That means there exists a boundary point on this path. But this contradicts the fact that B has no boundary points other than P. So our initial assumption that P is an isolated point of ∂S is false.

Now we can complete our proof of (T.4).

Suppose $\partial\Delta$ includes the point P = 0. We already know that $\partial\Delta_S$ contains all of Δ's boundary points other than P. Since the unitary matrices are a connected subset of the $n \times n$ complex matrices, we know that Δ is a connected subset of the complex plane. By (T.5) we know that P is a limit point of $\partial\Delta$. Therefore it is a limit point of $\partial\Delta_S$ as well. Since the boundary of a set is closed, it must contain all of its limit points. Hence $\partial\Delta_S$ contains P.

We've shown that $\partial\Delta \subseteq \partial\Delta_S$ given the quasi-symmetry assumption.

This completes our proof (T.4).

8 Speculations on how to prove C is quasi-symmetric.

$C = A_0 + B$

We know by (CR.1) that C^{-1} is magnitude-symmetric.

Suppose we can prove that if C^{-1} is magnitude-symmetric then C is magnitude-symmetric. And suppose we can prove the magnitude-symmetry conjecture, (C.2). That would mean that C would have to be quasi-symmetric or quasi-hermitian. Now we'd have to eliminate the possibility that C is quasi-hermitian but not quasi-symmetric.

9 Conclusion - Implication of the boundary conjecture

(T.6)
THEOREM 6
If the boundary conjecture is true, then the restricted Marcus-de Oliveira conjecture implies the full Marcus-de Oliveira conjecture.

Proof:

The unitary group and the real orthogonal group are compact subsets of the $n \times n$ complex matrices. Since a continuous image of a compact set is compact, Δ and Δ_S are compact subsets of the complex plane. Hence they are both closed.

Suppose we know that (D.2) is true. Then Δ_S along with its boundary is within the convex-hull. Suppose we also know that the boundary conjecture is true. Then we know that $\partial\Delta$ is inside the convex-hull. Can we have a unitary matrix U such that $R(U)$ is outside the convex-hull? No, because that would mean we have points of Δ on both the inside and outside of $\partial\Delta$. This is impossible since Δ is a closed set. So Δ is within the convex hull proving (D.1).

Many thanks to the referee whose suggestions vastly improved the paper.

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