THE VALUE OF THE COSMOLOGICAL CONSTANT

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Abstract

This paper presents a derivation of the value of the cosmological constant. The approach was based on the Einstein's gravitational field equations and the Hubble's law. The value of the cosmological constant Λ was found to be: $\Lambda = \frac{H_0^2}{3}$, here H_0 is the Hubble constant.

The Einstein's gravitational field equation with the cosmological constant Λ is

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \quad [1] \tag{1}$$

Where
$R_{\mu\nu}$ is the Ricci tensor.
R is the curvature scalar.
Λ is the cosmological constant.
$g_{\mu\nu}$ is the metric tensor.
G is the gravitational constant.
$T_{\mu\nu}$ is the energy–momentum tensor.

The Hubble's Law [2] is

$$\dot{D} = H_0 D \tag{2}$$

Where
$\dot{D}$ is the recessional velocity, typically expressed in km/s.
H_0 is the Hubble constant. $H_0 = 67.6^{+0.7}_{-0.6} km\ s^{-1} Mpc^{-1}$ [3]
D is the proper distance, measured in mega parsecs (Mpc).

We will calculate the expansion shell layer ΔS of an arbitrarily selected sphere with the radius D in the space by using the Einstein's gravitational field equation (1), and the Hubble's law (2). The radius D shall be a large cosmological distance measured in mega parsecs (Mpc).

Both approaches should give the same result of the volume of the expansion shell layer ΔS as shown in yellow in Figure 1.
Figure 1 is not to scale, the ΔS actually is very thin, such as D = 10Mpc = 3.0857 × 10^{20}km, thus $\dot{D} \approx 675 km/s$.

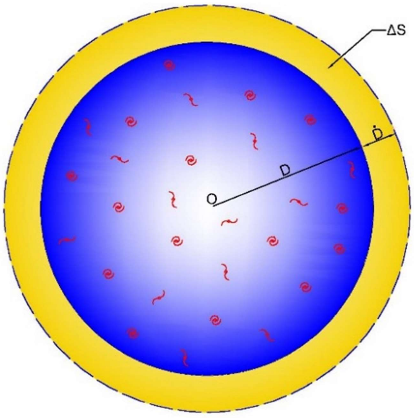

Figure 1: (not to scale) An expanding sphere with the radius D; ΔS is the increased shell layer with the expansion speed $\dot{D}$ on the surface of the sphere.

What we are concerned with is the cosmological constant, therefore we only investigate the third item $\Lambda g_{\mu\nu}$ on the left side of the Einstein's gravitational field equation (1). By using the third item $\Lambda g_{\mu\nu}$ as the integrand function in the spherical coordinates, we shall get the expansion shell layer ΔS of the sphere as

$$\Delta S = \iiint \Lambda g_{\mu\nu}\, dV \tag{3}$$

Next, we need to find the appropriate metric $g_{\mu\nu}$.

The Schwarzschild solution [4] of the Einstein's field equations (1) is

$$ds^2 = -\left(1-\frac{r_G}{r}\right)c^2dt^2+\left(1-\frac{r_G}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + sin^2\theta d\varphi^2) \tag{4}$$

Here, $r_G = \frac{2GM}{c^2}$ is the Schwarzschild radius.

$g_{\mu\nu}$ are the metric tensors of the Schwarzschild geometry

$$g_{\mu\nu} = \begin{bmatrix} g_{00} & 0 & 0 & 0 \\ 0 & g_{11} & 0 & 0 \\ 0 & 0 & g_{22} & 0 \\ 0 & 0 & 0 & g_{33} \end{bmatrix} \tag{5}$$

Here $g_{00} = -\left(1-\frac{r_G}{r}\right)$; $g_{11} = \left(1-\frac{r_G}{r}\right)^{-1}$; $g_{22} = r^2$; $g_{33} = r^2 sin^2\theta$.

The absolute value of the determinant of Schwarzschild metric is

$$\begin{aligned} |g| &= -g_{00} \cdot g_{11} \cdot g_{22} \cdot g_{33} \\ &= \left(1-\frac{r_G}{r}\right)\cdot\left(1-\frac{r_G}{r}\right)^{-1} \cdot r^2 \cdot r^2 sin^2\theta \\ &= r^4 sin^2\theta \end{aligned}$$

Then $\sqrt{|g|} = r^2 \sin\theta$

For integrating the shell layer ΔS, the volume element shall be

$$\sqrt{|g|}dV = r^2 \sin\theta \; cdtdrd\theta d\varphi \tag{6}$$

The Schwarzschild metric is static and spherically symmetric, it is time independent, therefore

$$g_{\mu\nu} = g_{\mu\nu}(x^i),\; g_{0i} = 0$$

We calculate the expansion shell layer ΔS of the sphere at an instantaneous time t_o with the expansion speed $\dot{D}$, therefore the Schwarzschild metric does not include the the time metric component cdt.

Hence, for integrating shell layer ΔS, the volume element becomes

$$\sqrt{|g|}dV = r^2 \sin\theta \; drd\theta \tag{7}$$

The calculation of the integrating shell layer ΔS of the sphere is shown as followings.

$$\begin{aligned}
\Delta S &= \iiint \Lambda g_{\mu\nu}\, dV \\
&= \Lambda \iiint \sqrt{|g|}\, dV \\
&= \Lambda \iiint r^2 \sin\theta \; drd\theta d\varphi \\
&= \Lambda \int_D^{D+\dot{D}} r^2 dr \int_0^{\pi} \sin\theta \; d\theta \int_0^{2\pi} d\varphi \\
&= \Lambda 4\pi D^3 H_0
\end{aligned} \tag{8}$$

Next, we use the Hubble's law to calculate the volume of the expansion shell layer ΔS of the sphere with the radius D.

The volume of the sphere with the radius D is

$$S = \frac{4}{3}\pi D^3 \tag{9}$$

Multiplying the Hubble constant H_0 to the radius D in equation (9), we get the expansion shell layer ΔS shown in yellow in Figure 1.

$$\Delta S = \frac{4}{3}\pi D^3 H_0^3 \tag{10}$$

Combining both equations (8) and (10), we obtained the following equation

$$\Lambda 4\pi D^3 H_0 = \frac{4}{3}\pi D^3 H_0^3 \tag{11}$$

Finally, solving the equation (11) for Λ, then we get the value of the cosmological constant

$$\Lambda = \frac{H_0^2}{3} \tag{12}$$

$$\Lambda = 1.5998 \times 10^{-36} S^{-2}$$

References

[1] Sean M. Carroll (arXiv:astro-ph/0004075v2 8 Apr. 2000)

[2] E.P. Hubble, Proc. Natl. Acad. Sci. 15, 168 (1929)

[3] Grieb, Jan N.; Sanchez, Ariel G.; Salazar-Albornoz (arXiv: 1607.03143[astro-ph.CO] 11 Jul 2016)

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