Infinite Product Representations for Gamma Function and Binomial Coefficient

By Edigles Guedes

November 16, 2016

"In the beginning was the Word, and the Word was with God, and the Word was God." - John 1:1.

Abstract. In this paper, I demonstrate one new infinite product for binomial coefficient and news Euler's and Weierstrass's infinite product for Gamma function among other things.

1. Introduction

In 1729, Leonhard Euler (1707-1783) gave the infinite product expansion for gamma function [1, p. 33]

$$\Gamma(z) = \frac{1}{z}\prod_{j=1}^{\infty}\left(1+\frac{1}{j}\right)^{z}\left(1+\frac{z}{j}\right)^{-1}, \tag{1}$$

which is valid in $\mathbb{C}$, except for $z \in \{0, -1, -2, \ldots\}$.

In 1854, Karl Weierstrass (1815-1897) gave the infinte product expansion for gamma function [1, p. 34-35]

$$\Gamma(z) = z e^{\gamma z}\prod_{j=1}^{\infty}\left(1+\frac{z}{j}\right)e^{-z/j}, \tag{2}$$

which is valid for all $\mathbb{C}$.

The binomial coefficient may be defined by the finite product [2]

$$\binom{\ell}{n} = \frac{(\ell)_n}{n!} = \prod_{r=1}^{n}\left(\frac{\ell+r-1}{r}\right), \tag{3}$$

which is valid for $\ell \in \mathbb{C} - \{0, -1, -2, \ldots\}$ and $n \in \mathbb{N}$.

In [3, p. 3, formula (1.1.1.2)], the Pochhammer's symbol is defined by

$$(\ell)_n = \frac{\Gamma(\ell+n)}{\Gamma(\ell)}. \tag{4}$$

In [4], I gave the infinite product representation for the Pochhammer's symbol

$$(\ell)_n = \prod_{j=1}^{\infty}\left(1+\frac{1}{j}\right)^{n}\left(1+\frac{n}{j+\ell-1}\right)^{-1}, \tag{5}$$

which is valid for $\ell \in \mathbb{C} - \{0, -1, -2, \ldots\}$ and $n \in \mathbb{N}$.

The beta function may be defined by [5]

$$B(a,b) \equiv \frac{(a-1)!(b-1)!}{(a+b-1)!}, \tag{6}$$

when a, b are positive integers.

In this paper, I prove the infinite product representation for binomial coefficient given by

$$\binom{\ell}{n} = \frac{(\ell)_n}{n!} = \prod_{j=1}^{\infty}\left(1+\frac{n}{j}\right)\left(1+\frac{n}{j+\ell-1}\right)^{-1},$$

and the gamma function is given by news infinite product representations

$$z\Gamma(z+n) = \prod_{j=1}^{\infty}\left(1+\frac{1}{j}\right)^{z+n}\left[\left(1+\frac{z}{j}\right)\left(1+\frac{n}{j+z-1}\right)\right]^{-1}$$

and

$$\frac{1}{z\Gamma(z+n)} = e^{\gamma(z+n)}\prod_{j=1}^{\infty}\left(1+\frac{z}{j}\right)\left(1+\frac{n}{j+z-1}\right)e^{-(z+n)/j};$$

$$(z-n)\Gamma(z) = \prod_{j=1}^{\infty}\left(1+\frac{1}{j}\right)^{z}\left[\left(1+\frac{z-n}{j}\right)\left(1+\frac{n}{j+z-n-1}\right)\right]^{-1}$$

and

$$\frac{1}{(z-n)\Gamma(z)} = e^{\gamma z}\prod_{j=1}^{\infty}\left(1+\frac{z-n}{j}\right)\left(1+\frac{n}{j+z-n-1}\right)e^{-z/j};$$

among other things.

2. Preliminary

Lemma 1. *If* $a, b \in \mathbb{R}$ *and* $b \neq 0$, *then*

$$\frac{a}{b} = \prod_{j=1}^{\infty}\frac{(a+j-1)(b+j)}{(a+j)(b+j-1)}.$$

Proof. I well-know the identity

$$\frac{a}{b}\cdot\frac{b!(a-1)!}{[b+(a-1)+1]!} = \frac{a!(b-1)!}{[a+(b-1)+1]!}, \tag{7}$$

using the definition for beta function (6), I obtain

$$\frac{a}{b} = \frac{B(a+1,b)}{B(a,b+1)}. \tag{8}$$

On the other hand, the beta function have the following infinite product representation [6, p. 899]

$$(a+b+1)B(a+1,b+1) = \prod_{j=1}^{\infty}\frac{j(a+b+j)}{(a+j)(b+j)}, \tag{9}$$

valid for $a, b \neq -1, -2, \ldots$ Setting $a \to a-1$ and $b \to b-1$, respectively, in both members of (9), I find

$$B(a,b+1) = \frac{1}{a+b}\prod_{j=1}^{\infty}\frac{j(a+b+j-1)}{(a+j-1)(b+j)}, \tag{10}$$

and

$$B(a+1,b) = \frac{1}{a+b}\prod_{j=1}^{\infty}\frac{j(a+b+j-1)}{(a+j)(b+j-1)}. \tag{11}$$

Substituting (10) and (11) into the right hand side of (9), I get

$$\frac{a}{b} = \frac{a+b}{a+b}\prod_{j=1}^{\infty}\frac{j(a+b+j-1)(a+j-1)(b+j)}{(a+j)(b+j-1)j(a+b+j-1)}.$$

Eliminate the same terms in the numerator and denominator of the above equation, and encounter

$$\frac{a}{b} = \prod_{j=1}^{\infty}\frac{(a+j-1)(b+j)}{(a+j)(b+j-1)},$$

which is the desired result. □

3. Binomial Coefficient: the Infinite Product

3.1. Infinite Product Representation for Binomial Coefficient.

Theorem 2. *If* $\ell \in \mathbb{C} - \{-1, -2, \ldots\}$ *and* $n \in \mathbb{N}$, *then*

$$\binom{\ell}{n} = \frac{(\ell)_n}{n!} = \prod_{j=1}^{\infty}\left(1+\frac{n}{j}\right)\left(1+\frac{n}{j+\ell-1}\right)^{-1},$$

where $\left(\begin{array}{c}\ell\\ n\end{array}\right)$ *denotes the binomial coefficient,* $(\ell)_n$ *denotes the Pochhammer's symbol and* $n!$ *denotes the factorial.*

Proof. Setting $a=\ell+r-1$ and $b=r$ in both members of the Lemma 1, I obtain

$$\frac{\ell+r-1}{r}=\prod_{j=1}^{\infty}\frac{(\ell+r+j-2)(r+j)}{(\ell+r+j-1)(r+j-1)}. \tag{12}$$

Substituting (12) into the right hand side of (3), I find

$$\begin{aligned}\left(\begin{array}{c}\ell\\ n\end{array}\right)=\frac{(\ell)_n}{n!}&=\prod_{r=1}^{n}\prod_{j=1}^{\infty}\frac{(\ell+r+j-2)(r+j)}{(\ell+r+j-1)(r+j-1)}\\ &=\prod_{j=1}^{\infty}\prod_{r=1}^{n}\frac{(\ell+r+j-2)(r+j)}{(\ell+r+j-1)(r+j-1)}\\ &=\prod_{j=1}^{\infty}\frac{(j+n)(j+\ell-1)}{j(j+\ell+n-1)}=\prod_{j=1}^{\infty}\left(1+\frac{n}{j}\right)\left(1+\frac{n}{j+\ell-1}\right)^{-1},\end{aligned}$$

which is the desired result. □

4. News Euler's and Weierstrass's Infinite Product Representation for Gamma Function

4.1. New Euler's Infinite Product Representation for Gamma Function.

Theorem 3. *If* $z\in\mathbb{C}-\{0,-1,-2,...\}$ *and* $n\in\mathbb{N}$*, then*

$$z\Gamma(z+n)=\prod_{j=1}^{\infty}\left(1+\frac{1}{j}\right)^{z+n}\left[\left(1+\frac{z}{j}\right)\left(1+\frac{n}{j+z-1}\right)\right]^{-1},$$

where $\Gamma(z)$ *denotes the gamma function.*

Proof. From definition (4), I give

$$(\ell)_n=\frac{\Gamma(\ell+n)}{\Gamma(\ell)}\Rightarrow\Gamma(\ell+n)=(\ell)_n\cdot\Gamma(\ell). \tag{13}$$

Setting the right hand side of (1) and (5) into the right hand side of (13), I get

$$\begin{aligned}\Gamma(\ell+n)&=\prod_{j=1}^{\infty}\left(1+\frac{1}{j}\right)^{n}\left(1+\frac{n}{j+\ell-1}\right)^{-1}\cdot\frac{1}{\ell}\prod_{j=1}^{\infty}\left(1+\frac{1}{j}\right)^{\ell}\left(1+\frac{\ell}{j}\right)^{-1}\\ \Rightarrow\ell\Gamma(\ell+n)&=\prod_{j=1}^{\infty}\left(1+\frac{1}{j}\right)^{n}\left(1+\frac{n}{j+\ell-1}\right)^{-1}\prod_{j=1}^{\infty}\left(1+\frac{1}{j}\right)^{\ell}\left(1+\frac{\ell}{j}\right)^{-1}\\ &=\prod_{j=1}^{\infty}\left(1+\frac{1}{j}\right)^{n}\left(1+\frac{n}{j+\ell-1}\right)^{-1}\left(1+\frac{1}{j}\right)^{\ell}\left(1+\frac{\ell}{j}\right)^{-1}\\ &=\prod_{j=1}^{\infty}\left(1+\frac{1}{j}\right)^{\ell+n}\left[\left(1+\frac{\ell}{j}\right)\left(1+\frac{n}{j+\ell-1}\right)\right]^{-1}.\end{aligned}$$

Changing ℓ by z in previous equation, I obtain the desired result. □

4.2. New Weierstrass's Infinite Product Representation for Gamma Function.

Theorem 4. *If* $z\in\mathbb{C}$ *and* $n\in\mathbb{N}$*, then*

$$\frac{1}{z\Gamma(z+n)}=e^{\gamma(z+n)}\prod_{j=1}^{\infty}\left(1+\frac{z}{j}\right)\left(1+\frac{n}{j+z-1}\right)e^{-(z+n)/j},$$

where $\Gamma(z)$ *denotes the gamma function,* e^x *denotes the exponential function and* γ *denotes the Euler-Mascheroni constant.*

Proof. The inverse of the Theorem 3, give me

$$\frac{1}{z\Gamma(z+n)} = \prod_{j=1}^{\infty}\left(1+\frac{1}{j}\right)^{-(z+n)}\left(1+\frac{z}{j}\right)\left(1+\frac{n}{j+z-1}\right)$$
$$= \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{z+n}\lim_{m\to\infty}\prod_{j=1}^{m}\left(1+\frac{1}{j}\right)^{-(z+n)}\left(1+\frac{z}{j}\right)\left(1+\frac{n}{j+z-1}\right)$$
$$= \lim_{m\to\infty}\left[\left(1+\frac{1}{m}\right)^{z+n}\prod_{j=1}^{m}\left(1+\frac{1}{j}\right)^{-(z+n)}\left(1+\frac{z}{j}\right)\left(1+\frac{n}{j+z-1}\right)\right]$$
$$= \lim_{m\to\infty}\left[\prod_{j=1}^{m-1}\left(1+\frac{1}{j}\right)^{-(z+n)}\prod_{j=1}^{m}\left(1+\frac{z}{j}\right)\left(1+\frac{n}{j+z-1}\right)\right]$$
$$= \lim_{m\to\infty}\left[m^{-(z+n)}\prod_{j=1}^{m}\left(1+\frac{z}{j}\right)\left(1+\frac{n}{j+z-1}\right)\right]$$
$$= \lim_{m\to\infty}\left[\exp\left(\left(1-1+\frac{1}{2}-\frac{1}{2}+\ldots+\frac{1}{m}-\frac{1}{m}-\ln m\right)(z+n)\right)\prod_{j=1}^{m}\left(1+\frac{z}{j}\right)\left(1+\frac{n}{j+z-1}\right)\right]$$
$$= \lim_{m\to\infty}\left[\exp\left(\left(1+\frac{1}{2}+\ldots+\frac{1}{m}-\ln m\right)(z+n)\right)\exp\left(\frac{-(z+n)}{1}+\frac{-(z+n)}{2}+\ldots+\frac{-(z+n)}{m}\right)\right.$$
$$\left.\prod_{j=1}^{m}\left(1+\frac{z}{j}\right)\left(1+\frac{n}{j+z-1}\right)\right]$$
$$= \lim_{m\to\infty}\left[\exp\left(\left(1+\frac{1}{2}+\ldots+\frac{1}{m}-\ln m\right)(z+n)\right)\prod_{j=1}^{m}\left(1+\frac{z}{j}\right)\left(1+\frac{n}{j+z-1}\right)e^{-(z+n)/j}\right]$$
$$= \lim_{m\to\infty}\left(\exp\left(\left(1+\frac{1}{2}+\ldots+\frac{1}{m}-\ln m\right)(z+n)\right)\right)\lim_{m\to\infty}\left(\prod_{j=1}^{m}\left(1+\frac{z}{j}\right)\left(1+\frac{n}{j+z-1}\right)e^{-(z+n)/j}\right)$$
$$= e^{\gamma(z+n)}\prod_{j=1}^{\infty}\left(1+\frac{z}{j}\right)\left(1+\frac{n}{j+z-1}\right)e^{-(z+n)/j},$$

which is the desired result. □

Example 5. Set $n=1$ in Theorem 3 and Theorem 4, and encounter

$$z^2\,\Gamma(z) = \prod_{j=1}^{\infty}\left(1+\frac{1}{j}\right)^{z+1}\left[\left(1+\frac{z}{j}\right)\left(1+\frac{1}{j+z-1}\right)\right]^{-1}$$

and

$$\frac{1}{z^2\,\Gamma(z)} = e^{\gamma(z+1)}\prod_{j=1}^{\infty}\left(1+\frac{z}{j}\right)\left(1+\frac{1}{j+z-1}\right)e^{-(z+1)/j}.$$

4.3. New Euler's Infinite Product Representation for Gamma Function.

Theorem 6. *If* $z\in\mathbb{C}-\{0,-1,-2,\ldots\}$, $n\in\mathbb{N}$ *and* $\mathrm{Re}(z)>n$, *then*

$$(z-n)\Gamma(z) = \prod_{j=1}^{\infty}\left(1+\frac{1}{j}\right)^{z}\left[\left(1+\frac{z-n}{j}\right)\left(1+\frac{n}{j+z-n-1}\right)\right]^{-1},$$

where $\Gamma(z)$ *denotes the gamma function.*

Proof. In [3, p. 240, I.29], I found the formula

$$\Gamma(z-n) = \frac{\Gamma(z)}{(z-n)_n} \Rightarrow \Gamma(z) = \Gamma(z-n)\cdot(z-n)_n. \tag{14}$$

Setting the right hand side of (1) and (5) into the right hand side of (14), I get

$$\begin{aligned}
\Gamma(z) &= \frac{1}{z-n}\prod_{j=1}^{\infty}\left(1+\frac{1}{j}\right)^{z-n}\left(1+\frac{z-n}{j}\right)^{-1}\cdot\prod_{j=1}^{\infty}\left(1+\frac{1}{j}\right)^{n}\left(1+\frac{n}{j+z-n-1}\right)^{-1}\\
&= \frac{1}{z-n}\prod_{j=1}^{\infty}\left(1+\frac{1}{j}\right)^{z-n}\left(1+\frac{z-n}{j}\right)^{-1}\left(1+\frac{1}{j}\right)^{n}\left(1+\frac{n}{j+z-n-1}\right)^{-1}\\
&= \frac{1}{z-n}\prod_{j=1}^{\infty}\left(1+\frac{1}{j}\right)^{z}\left[\left(1+\frac{z-n}{j}\right)\left(1+\frac{n}{j+z-n-1}\right)\right]^{-1}\\
\Rightarrow (z-n)\Gamma(z) &= \prod_{j=1}^{\infty}\left(1+\frac{1}{j}\right)^{z}\left[\left(1+\frac{z-n}{j}\right)\left(1+\frac{n}{j+z-n-1}\right)\right]^{-1},
\end{aligned}$$

which is the desired result. □

4.4. New Weierstrass's Infinite Product Representation for Gamma Function.

Theorem 7. *If $z\in\mathbb{C}$, $n\in\mathbb{N}$and $\operatorname{Re}(z)>n$, then*

$$\frac{1}{(z-n)\Gamma(z)} = e^{\gamma z}\prod_{j=1}^{\infty}\left(1+\frac{z-n}{j}\right)\left(1+\frac{n}{j+z-n-1}\right)e^{-z/j},$$

where $\Gamma(z)$ denotes the gamma function, e^x denotes the exponential function and γ denotes the Euler-Mascheroni constant.

Proof. The inverse of the Theorem 3, give me

$$\begin{aligned}
\frac{1}{(z-n)\Gamma(z)} &= \prod_{j=1}^{\infty}\left(1+\frac{1}{j}\right)^{-z}\left(1+\frac{z-n}{j}\right)\left(1+\frac{n}{j+z-n-1}\right)\\
&= \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{z}\lim_{m\to\infty}\prod_{j=1}^{m}\left(1+\frac{1}{j}\right)^{-z}\left(1+\frac{z-n}{j}\right)\left(1+\frac{n}{j+z-n-1}\right)\\
&= \lim_{m\to\infty}\left[\left(1+\frac{1}{m}\right)^{z}\prod_{j=1}^{m}\left(1+\frac{1}{j}\right)^{-z}\left(1+\frac{z-n}{j}\right)\left(1+\frac{n}{j+z-n-1}\right)\right]\\
&= \lim_{m\to\infty}\left[\prod_{j=1}^{m-1}\left(1+\frac{1}{j}\right)^{-z}\prod_{j=1}^{m}\left(1+\frac{z-n}{j}\right)\left(1+\frac{n}{j+z-n-1}\right)\right]\\
&= \lim_{m\to\infty}\left[m^{-z}\prod_{j=1}^{m}\left(1+\frac{z-n}{j}\right)\left(1+\frac{n}{j+z-n-1}\right)\right]\\
&= \lim_{m\to\infty}\left[\exp\left(\left(1-1+\frac{1}{2}-\frac{1}{2}+\ldots+\frac{1}{m}-\frac{1}{m}-\ln m\right)z\right)\prod_{j=1}^{m}\left(1+\frac{z-n}{j}\right)\left(1+\frac{n}{j+z-n-1}\right)\right]\\
&= \lim_{m\to\infty}\left[\exp\left(1+\frac{1}{2}+\ldots+\frac{1}{m}-\ln m\right)\exp\left(\frac{-z}{1}+\frac{-z}{2}+\ldots+\frac{-z}{m}\right)\right.\\
&\qquad\left.\prod_{j=1}^{m}\left(1+\frac{z-n}{j}\right)\left(1+\frac{n}{j+z-n-1}\right)\right]\\
&= \lim_{m\to\infty}\left[\exp\left(\left(1+\frac{1}{2}+\ldots+\frac{1}{m}-\ln m\right)z\right)\prod_{j=1}^{m}\left(1+\frac{z-n}{j}\right)\left(1+\frac{n}{j+z-n-1}\right)e^{-z/j}\right]\\
&= \lim_{m\to\infty}\left(\exp\left(\left(1+\frac{1}{2}+\ldots+\frac{1}{m}-\ln m\right)z\right)\right)\lim_{m\to\infty}\left(\prod_{j=1}^{m}\left(1+\frac{z-n}{j}\right)\left(1+\frac{n}{j+z-n-1}\right)e^{-z/j}\right)\\
&= e^{\gamma z}\prod_{j=1}^{\infty}\left(1+\frac{z-n}{j}\right)\left(1+\frac{n}{j+z-n-1}\right)e^{-z/j},
\end{aligned}$$

which is the desired result. □

Example 8. Set $n=1$ in Theorem 6 and Theorem 7, and encounter

$$(z-1)\Gamma(z)=\prod_{j=1}^{\infty}\left(1+\frac{1}{j}\right)^{z}\left[\left(1+\frac{z-1}{j}\right)\left(1+\frac{1}{j+z-2}\right)\right]^{-1}$$

and

$$\frac{1}{(z-1)\Gamma(z)}=e^{\gamma z}\prod_{j=1}^{\infty}\left(1+\frac{z-1}{j}\right)\left(1+\frac{1}{j+z-2}\right)e^{-z/j}.$$

5. The Gamma Function at Rational Argument as Finite Product of Gamma Functions

5.1. Gamma Function at Rational Argument.

Theorem 9. *If p and q are positive integers and $p\leqslant q$, $n\in\mathbb{N}$, and $p/q<n$, then*

$$\frac{p}{q}\Gamma\left(\frac{p}{q}+n\right)=\prod_{s=1}^{q}\left(\frac{\Gamma\left(\frac{s}{q}\right)}{\Gamma\left(\frac{s+1}{q}\right)}\right)^{\frac{p}{q}+n}\left(\frac{\Gamma\left(\frac{p}{q^2}+\frac{n+s-1}{q}\right)}{\Gamma\left(\frac{p}{q^2}+\frac{s-1}{q}\right)}\cdot\frac{\Gamma\left(\frac{p}{q^2}+\frac{s}{q}\right)}{\Gamma\left(\frac{s}{q}\right)}\right),$$

where $\Gamma(z)$ denotes the gamma function.

Proof. Consider the infinite product representation for gamma function in Theorem 3, let $z=p/q$, with $p\in\mathbb{Z}$ and $q\in\mathbb{N}$, and encounter

$$\begin{aligned}\frac{p}{q}\Gamma\left(\frac{p}{q}+n\right)&=\prod_{j=1}^{\infty}\left(1+\frac{1}{j}\right)^{\frac{p}{q}+n}\left[\left(1+\frac{p}{jq}\right)\left(1+\frac{nq}{jq+p-q}\right)\right]^{-1}\\&=\prod_{k=0}^{\infty}\left(1+\frac{1}{k+1}\right)^{\frac{p}{q}+n}\left[\left(1+\frac{p}{(k+1)\,q}\right)\left(1+\frac{nq}{(k+1)\,q+p-q}\right)\right]^{-1}\\&=\prod_{k=0}^{\infty}\left(1+\frac{1}{k+1}\right)^{\frac{p}{q}+n}\left[\left(1+\frac{p}{(k+1)\,q}\right)\left(1+\frac{nq}{kq+p}\right)\right]^{-1}\end{aligned}$$

Now, notice that for any $a\in\mathbb{Z}$ and $b\in\mathbb{N}$, there exists unique $c,d\in\mathbb{Z}$, such that $a=bc+d$ and $0\leqslant d<b$ (division law in $\mathbb{Z}$, see [7, Lemma 7, p. 4]). Hither, this means that any ($k\in\mathbb{N}_0$, $q\in\mathbb{N}$) uniquely determine the integer r and s, such that $k=qr+s$, where $r=0,1,2,\ldots$ and $s=1,2,\ldots,q-1$. Thereupon, it follows (by uniform convergence) that

$$\begin{aligned}\frac{p}{q}\Gamma\left(\frac{p}{q}+n\right)&=\prod_{r=0}^{\infty}\prod_{s=0}^{q-1}\left(1+\frac{1}{qr+s+1}\right)^{\frac{p}{q}+n}\left[\left(1+\frac{p}{(qr+s+1)\,q}\right)\left(1+\frac{nq}{(qr+s)\,q+p}\right)\right]^{-1}\\&=\prod_{s=0}^{q-1}\prod_{r=0}^{\infty}\left(1+\frac{1}{qr+s+1}\right)^{\frac{p}{q}+n}\left[\left(1+\frac{p}{(qr+s+1)\,q}\right)\left(1+\frac{nq}{(qr+s)\,q+p}\right)\right]^{-1}\\&=\prod_{s=0}^{q-1}\left(1+\frac{1}{s+1}\right)^{\frac{p}{q}+n}\left(\frac{\Gamma\left(1+\frac{s+1}{q}\right)}{\Gamma\left(1+\frac{s+2}{q}\right)}\right)^{\frac{p}{q}+n}\left(\frac{\Gamma\left(\frac{p+q(n+s)}{q^2}\right)}{\Gamma\left(\frac{p+qs}{q^2}\right)}\cdot\frac{\Gamma\left(\frac{p+q(s+1)}{q^2}\right)}{\Gamma\left(\frac{s+1}{q}\right)}\right)\\&=\prod_{s=1}^{q}\left(1+\frac{1}{s}\right)^{\frac{p}{q}+n}\left(\frac{\Gamma\left(1+\frac{s}{q}\right)}{\Gamma\left(1+\frac{s+1}{q}\right)}\right)^{\frac{p}{q}+n}\left(\frac{\Gamma\left(\frac{p+q(n+s-1)}{q^2}\right)}{\Gamma\left(\frac{p+q\,(s-1)}{q^2}\right)}\cdot\frac{\Gamma\left(\frac{p+qs}{q^2}\right)}{\Gamma\left(\frac{s}{q}\right)}\right),\end{aligned}$$

using the identity $\Gamma(1+z)=z\Gamma(z)$ in previous equation, I have

$$\frac{p}{q}\,\Gamma\left(\frac{p}{q}+n\right)=\prod_{s=1}^{q}\left(\frac{s+1}{s}\right)^{\frac{p}{q}+n}\left(\frac{\frac{s}{q}\Gamma\left(\frac{s}{q}\right)}{\frac{s+1}{q}\Gamma\left(\frac{s+1}{q}\right)}\right)^{\frac{p}{q}+n}\left(\frac{\Gamma\left(\frac{p+q(n+s-1)}{q^2}\right)}{\Gamma\left(\frac{p+q\,(s-1)}{q^2}\right)}\cdot\frac{\Gamma\left(\frac{p+qs}{q^2}\right)}{\Gamma\left(\frac{s}{q}\right)}\right)$$

$$=\prod_{s=1}^{q}\left(\frac{\Gamma\left(\frac{s}{q}\right)}{\Gamma\left(\frac{s+1}{q}\right)}\right)^{\frac{p}{q}+n}\left(\frac{\Gamma\left(\frac{p}{q^2}+\frac{n+s-1}{q}\right)}{\Gamma\left(\frac{p}{q^2}+\frac{s-1}{q}\right)}\cdot\frac{\Gamma\left(\frac{p}{q^2}+\frac{s}{q}\right)}{\Gamma\left(\frac{s}{q}\right)}\right),$$

which is the desired result. □

Example 10. Put $p=1$, $q=2$ in previous Theorem, and let n to be a non-negative integer, thus

$$\frac{1}{2}\,\Gamma\left(\frac{1}{2}+n\right)=\prod_{s=1}^{2}\left(\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)}\right)^{\frac{1}{2}+n}\left(\frac{\Gamma\left(\frac{1}{4}+\frac{n+s-1}{2}\right)}{\Gamma\left(\frac{1}{4}+\frac{s-1}{2}\right)}\cdot\frac{\Gamma\left(\frac{1}{4}+\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}\right)$$

or

$$\Gamma\left(\frac{1}{2}+n\right)=2^{n-1}\sqrt{\frac{2}{\pi}}\,\Gamma\left(\frac{1}{4}+\frac{n}{2}\right)\Gamma\left(\frac{3}{4}+\frac{n}{2}\right)=\frac{2^n}{\sqrt{2\pi}}\,\Gamma\left(\frac{1}{4}+\frac{n}{2}\right)\Gamma\left(\frac{3}{4}+\frac{n}{2}\right).$$

References

[1] Remmert, Reinhold, *Classical Topics in Complex Function*, Graduate Texts in Mathematics, 172, Springer-Verlag, New York, 1998.

[2] en.wikipedia.org/wiki/Binomial_coefficient, available in November 16, 2016.

[3] Slater, Lucy Joan, *Generalized Hypergeometric Functions*, Cambridge University Press, 1966.

[4] Guedes, Edigles, *Infinite Product Representations for Binomial Coefficient, Pochhammer's Symbol, Newton's Binomial and Exponential Function*, viXra.org/pdf/1611.0049v1.pdf.

[5] en.wikipedia.org/wiki/Beta_function, available in November 16, 2016.

[6] Gradshteyn, I. S. and Ryzhik, I. M., *Table of Integrals, Series and Products*, Academic Press, San Diego, 2000.

[7] Remmert, Reinhold, *A Classical Introduction to Modern Number Theory*, Graduate Texts in Mathematics, 84, Second Edition, Springer-Verlag, New York, 1990.