Critical loss factor in 2-DOF in-series system with hysteretic friction and its use for vibration control

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Although the classical theory of lumped mechanical systems employs the viscous friction mechanisms (dashpots), the loss factors of most solid structures are largely controlled by hysteresis. This paper presents new relationships for the dynamics of 2-DOF in-series systems with hysteresis damping. The most important among them is a close-form equation for the critical loss factor that was derived as the marginal condition for the degenerate case where the higher-frequency resonance peak fully vanishes in the vibration spectrum of the second mass. The critical loss factor can take values between 0 and $2^{-3/2} \approx 0.354$ and depends on the ratio of the natural frequencies of 2-DOF system: the closer the undamped natural frequencies, the lower the critical loss factor. The equation may help to interpret the vibration spectra for the second mass in the real 2-DOF systems, in particular on sweep-sine shaker tests. The single resonance peak in the degenerate case for a 2-DOF grows up notably as the natural frequencies get close to each other. By a formal analogy with 1-DOF systems, the peak magnitude can be reduced by increasing the loss factor. But in 2-DOF systems, the vibration can be effectively attenuated for the same loss factor by making the natural frequencies more different from each other (in particular, via increasing the stiffness of the second spring).

Keywords: 2-DOF mechanical system, hysteretic damping, critical loss factor, resonance, NVH (noise, vibration, and harshness), sweep-sine test.

1. INTRODUCTION

The resonances of 1-DOF and 2-DOF mechanical systems are well studied in the classical case of viscous friction [1-4]. However, most solid structures exhibit non-viscous damping mechanisms resulted from hysteresis, structural losses (caused by looseness of joints, internal strain, energy leaks to the adjacent structures, etc.), and coulomb friction [5-13]. All the damping mechanisms can be simulated using the loss factor η calculated from energy considerations. This parameter is linearly proportional to the vibration frequency in case of viscous friction, does not depend on the frequency for the hysteresis damping mechanism, and tends to reduce with the frequency because of the energy leaks to the adjacent structures. The assumption of hysteresis damping is reasonably valid if the loss factor does not vary notably in the important frequency range.

The close-form relationship between the resonance (damped natural) frequency and undamped natural frequency was derived for 1-DOF system with an arbitrary loss factor $\eta << 1$ [12]. Under forced vibration, the frequency response of every 1-DOF system with viscous friction has one resonance peak that can vanish in the degenerate case (when the damping ratio exceeds its critical value). The magnitude of the resonance peak for the 1-DOF system with hysteretic friction reduces with the loss factor but does not vanish. Every 2-DOF in-series system has two undamped natural frequencies but the higher-frequency resonance peak may vanish entirely, in particular for the second mass. This effect is of practical interest because the vibration of multiple solid structures can be often simulated using the simplified 2-DOF in-series models with hysteresis damping: in particular for the cars or trucks where the auxiliary cooling module (the second mass) is attached to the main radiator (the first mass); the appropriate sets of vibration isolators play the role of the first and second springs, and the car frame serves as the vibrating base. The goal of this paper is to obtain clear analytical relationships that can helpful for the engineers working in the areas of noise and vibration control.

2. GENERAL MATHEMATICAL MODEL

2.1. Basic equations for the 2-DOF in-series system made of two 1-DOF subsystems

Consider the 2-DOF in-series system incorporating two rigid bodies and two springs with hysteresis dampers (Fig. 1). The masses of the bodies are m_1 and m_2, the spring constants are k_1 and k_2. Hence, the partial natural angular frequencies are given by the equations

$\omega_{p1} = \sqrt{k_1/m_1}$ and $\omega_{p2} = \sqrt{k_2/m_2}$.

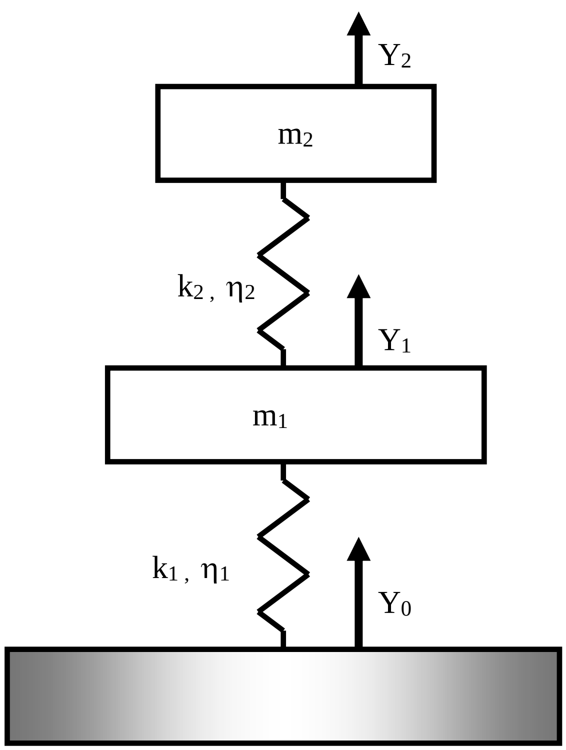

Fig. 1. 2-DOF in-series system with the hysteresis damping (the lowest body simulates a shaker, the dampers are not shown).

The system is attached to a rigid base vibrating harmonically with the angular frequency ω and displacement amplitude y_0 , so the vibration displacement of the base can be defined as

$Y_0 = y_0 \exp(-i\,\omega\,t)$ where $i = \sqrt{-1}$ is the imaginary unit. The vibration displacements of the first and second bodies can be expressed in the form $Y_1 = y_1 \exp(-i\,\omega\,t)$ and $Y_2 = y_2 \exp(-i\,\omega\,t)$, where y_1 and y_2 are the displacement amplitudes. The complex spring constants are defined as $K_1 = k_1\left(1 - i\,\eta_{p1}\right)$ and $K_2 = k_2\left(1 - i\,\eta_{p2}\right)$ where η_{p1} and η_{p2} are the partial loss factors associated with the first and second springs [8, 11]. Therefore, the differential equations of motion can be written in the form

$$\begin{cases} m_1 \ddot{Y}_1 + K_1 Y_1 + K_2\left(Y_1 - Y_2\right) = K_1 Y_0, \\ m_2 \ddot{Y}_2 + K_2\left(Y_2 - Y_1\right) = 0. \end{cases} \qquad (1)$$

The characteristic equation of this linear dynamic system

$$m_1 m_2 \Big\{ \omega^4 - \omega^2 \left[\omega_{p1}^2 (1 - i\,\eta_{p2}) + \omega_{p2}^2 (1 - i\,\eta_{p2}) (1 + \mu) \right] + \\ + \omega_{p1}^2\, \omega_{p2}^2 (1 - i\,\eta_{p1}) (1 - i\,\eta_{p2}) \Big\} = 0 \qquad (2)$$

is quadratic relative to the unknown quantity ω^2 and has two roots

$$\tilde{\omega}_{min,\,max}^2 = \omega_{min,\,max}^2 \left(1 - i\,\eta_{min,max}\right) \qquad (3)$$

where the undamped natural angular frequencies of the 2-DOF system

$$\omega_{min,\,max} = \omega_{p1} \sqrt{\mathrm{Re}\,\{D_{min,\,max}\}} \qquad (4)$$

and the loss factors

$$\eta_{min,\,max} = -\frac{\mathrm{Im}\left\{D_{min,\,max}\right\}}{\mathrm{Re}\left\{D_{min,\,max}\right\}}. \qquad (5)$$

Here, the dimensionless parameters

$$D_{min,max} = \frac{a+b}{2} \left(1 \mp \sqrt{1 - \frac{1}{1+\mu} \frac{4ab}{(a+b)^2}} \right), \tag{6}$$

$$\begin{cases} a = 1 - i\,\eta_{p1}, \\ b = p^2 (1+\mu)(1 - i\,\eta_{p2}), \end{cases} \tag{7}$$

$$p = \frac{\omega_{p2}}{\omega_{p1}}, \tag{8}$$

$$\mu = \frac{m_2}{m_1}. \tag{9}$$

2.2. Calculation of the relative displacements in the 2-DOF system

Using Eqs (1) - (3), calculate the ratios of the displacement amplitude y_2 to the displacement amplitude y_0:

$$\frac{y_2}{y_0} = \frac{(1 - i\,\eta_{min})(1 - i\,\eta_{max})}{\Psi(\omega)} \tag{10}$$

where the polynomial

$$\Psi(\omega) = [1 - (\omega/\omega_{min})^2 - i\,\eta_{min}]\,[1 - (\omega/\omega_{max})^2 - i\,\eta_{max}]. \tag{11}$$

It is noteworthy that Eqs (10) are also valid if the base does not move, the vibrating force $F = F_0 \exp(-i\,\omega\,t)$ is applied to the first body, and $y_0 = F_0 / K_1$.

3. DEDUCTION OF IMPORTANT CLOSE-FORM RELATIONSHIPS

3.1. The relationship between the 2-DOF system loss factors and partial loss factors

Applying the Vieta's formula [14] for quadratic equations to Eq. (2) and using Eq. (3), obtain the equality $\omega_{p1}^2\,\omega_{p2}^2\,(1 - i\,\eta_{p1})(1 - i\,\eta_{p2}) = \omega_{min}^2\,\omega_{max}^2\,(1 - i\,\eta_{min})(1 - i\,\eta_{max})$

that can be approximated (after neglecting the small values $\eta_{p1}\eta_{p2}$ and $\eta_{min}\eta_{max}$) to the form

$$\omega_{p1}^2\,\omega_{p2}^2\;[1-i\,(\eta_{p1}+\eta_{p2})]=\omega_{min}^2\;\omega_{max}^2\;[1-i\,(\eta_{min}+\eta_{max})]. \qquad (12)$$

Comparing the imaginary parts on both sides of Eq. (12), obtain:

$$\eta_{p1}+\eta_{p2}=\eta_{min}+\eta_{max}. \qquad (13)$$

As Eq. (13) indicates, provided that all the loss factors are small, the sum of the partial loss factors equals the sum of the loss factors of the 2-DOF system.

3.2. The affinity state for the undamped natural frequencies of 2-DOF in-series system

Using Eqs (4) and (6)-(9), express the ratio of the undamped natural frequencies ω_{min} and ω_{max} as

$$q=\frac{\omega_{min}}{\omega_{max}}=\sqrt{\frac{1-\sqrt{1-r}}{1+\sqrt{1-r}}}=\sqrt{\frac{1}{r}}-\sqrt{\frac{1}{r}-1} \qquad (14)$$

where the parameter

$$r=\frac{1}{1+\mu}\;\frac{4\,p^2\,(1+\mu)}{[1+p^2\,(1+\mu)]^2}. \qquad (15)$$

In line with the well-known inequality $(u+v)^2\geq 2uv$ for the positive values u and v, the the parameter r attains its maximum

$$r_{max}=\frac{1}{1+\mu}<1 \qquad (16)$$

on the condition that

$$p=p_{aff}=\frac{1}{\sqrt{1+\mu}}\;, \qquad (17)$$

or in terms of mass and stiffness,

$$k_2 = \frac{m_2 k_1}{m_1 + m_2} \; . \tag{18}$$

Calculate the first derivative of function $q(r)$ given by Eq. (14):

$$\frac{dq}{dr} = \frac{1}{2\,r^{3/2}} \left(\frac{1}{\sqrt{1-r}} - 1 \right) > 0 \, .$$

Hence, the parameter q monotonically grows with the value r and attains the maximum value

$$q_{max} = \sqrt{1+\mu} - \sqrt{\mu} \tag{19}$$

at the maximum value of the parameter r defined by Eq. (16).

The relationships between the parameters $q = \omega_{min}/\omega_{max}$ and $p = \omega_{p2}/\omega_{p1}$ for various values of the parameter $\mu = m_2/m_1$ are plotted in Fig. 2.

Therefore, for the parameters m_1, m_2 and k_1 given, the undamped natural frequencies of the 2-DOF system get most close to each other if the second spring constant k_2 fits Eq. (18).

Let's define the case described by Eqs (17) or (18) as the "affinity state" for the undamped natural frequencies of a 2-DOF in-series system.

In the affinity state, after neglecting the small value $\eta_{min}\eta_{max}$, Eq. (6) can be reduced to the form

$$D_{min,\,max} \approx (1 - i\,\eta) \left(1 \mp \sqrt{\frac{\mu}{1+\mu}} \right) \tag{20}$$

where

$$\eta = \eta_{min} = \eta_{max} = \frac{\eta_{p1} + \eta_{p2}}{2} . \tag{21}$$

As follows from Eq. (21), in the affinity state the 2-DOF system loss factors are similar, even if the partial loss factors are not identical. It should be noted that Eq (21) is in agreement with the more general Eq. (13).

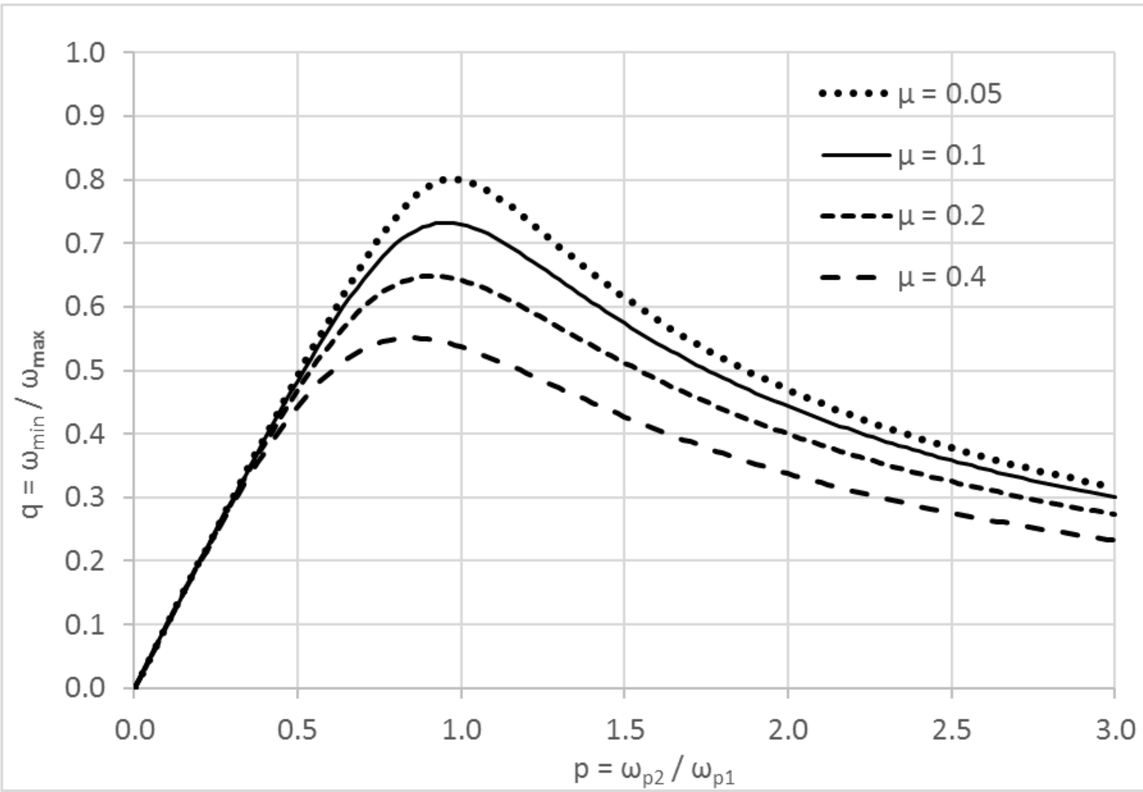

Fig. 2. Relationships between the parameters $q = \omega_{min}/\omega_{max}$ and $p = \omega_{p2}/\omega_{p1}$ for various values of the parameter $\mu = m_2/m_1$.

3.3. The degenerate case: no higher-frequency resonance for the second mass

Let's analyze the vibration of the second mass.

Using Eqs (10) and (11), calculate the transmissibility

$$T_2(\omega) = \left|\frac{y_2}{y_0}\right| = \sqrt{\frac{\left(1+\eta_{min}^2\right)\left(1+\eta_{max}^2\right)}{\xi(\omega)}} \quad . \tag{22}$$

where

$$\xi(\omega) = |\Psi(\omega)|^2 = \{[1-(\omega/\omega_{min})^2]^2 + \eta_{min}^2\}\{[1-(\omega/\omega_{max})^2]^2 + \eta_{max}^2\}. \qquad (23)$$

The damped natural frequencies fit the minimums of the function $\xi(\omega)$.

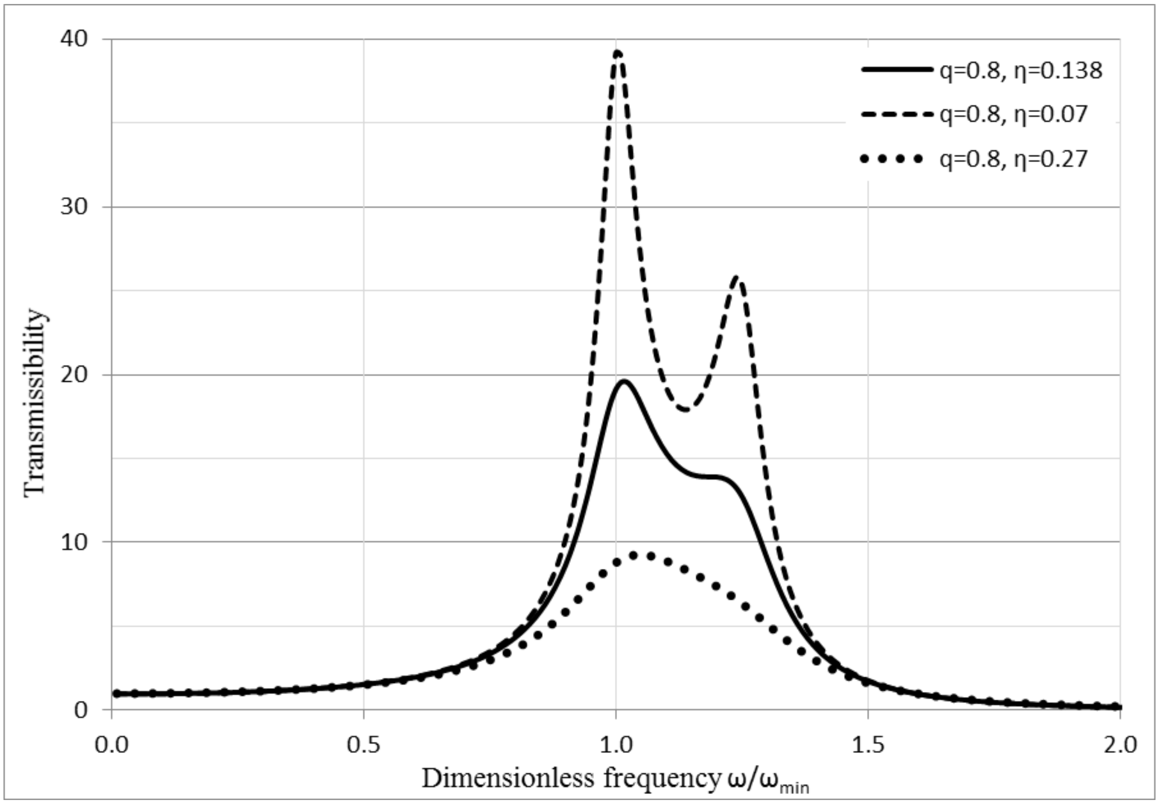

Fig. 3. Transmissibility $T_2(\omega)$ as a function of the dimensionless frequency ω/ω_{min} at various loss factors η (0.07, 0.138, and 0.27) and q = 0.8.

Consider that both loss factors are similar:

$$\eta_{min} = \eta_{max} = \eta. \qquad (24)$$

This assumption is strictly valid only for the affinity state or if $\eta_{p1} = \eta_{p2} = \eta$ but can also be applied if such conditions are approximately valid.

Using Eq. (24), transform Eq. (23) to a simpler form

$$\xi(\omega) = [(1-x)^2 + \eta^2\,][\,(1-\beta x)^2 + \eta^2\,] \tag{25}$$

with the dimensionless parameter

$$\beta = q^2 = \left(\frac{\omega_{min}}{\omega_{max}}\right)^2 \tag{26}$$

and dimensionless independent variable

$$x = \left(\frac{\omega}{\omega_{min}}\right)^2 \tag{27}$$

From the necessary condition of minimum $\dfrac{d\,\xi}{d\,x} = 4\beta^2\,\Phi(x) = 0,$ derive the general equation

for the resonance (undamped natural) frequencies

$$\Phi(x) = x^3 - \frac{3\,(\beta+1)}{2\beta}\,x^2 + \\ + \left[\frac{(\beta+1)^2 + 2\beta + (\beta^2+1)\,\eta^2}{2\beta^2}\right]x - \frac{(\beta+1)\,(1+\eta^2)}{2\beta^2} = 0\,. \tag{28}$$

In the “no-friction” case $\eta = 0,$ Eq. (28) has three real roots:

$$x_1 = 1,\;\; x_2 = \frac{x_1 + x_3}{2} = \frac{1}{2}\left(1 + \frac{1}{\beta}\right),\;\; x_3 = \frac{1}{\beta}\,. \tag{29}$$

Here, the first and third roots represent two undamped natural frequencies, and the second root defines the anti-resonance frequency (located in the middle between the two resonance peaks) at which the transmissibility $T_2(\omega)$ attains its minimum.

3.4. Critical loss factor of 2-DOF in-series system with hysteretic friction

As seen in Fig. 3, if the loss factor exceeds some critical value, the degenerate case occurs: the higher-frequency resonance peak interacts with the anti-resonance dip and vanishes.

Let’s derive a general equation for the critical loss factor.

Substituting Tartaglia’s substitution [14]

$$x = z + \frac{\beta + 1}{2\beta} \tag{30}$$

into Eq. (28), obtain a more simple cubic equation

$$z^3 + 3Pz + 2Q = 0 \tag{31}$$

with the coefficients

$$\begin{cases} P = -\dfrac{(1-\beta)^2 - 2(1+\beta^2)\eta^2}{12\beta^2}, \\ Q = \dfrac{(1-\beta)^2(1+\beta)\eta^2}{8\beta^3}. \end{cases} \tag{32}$$

Of great importance is the parameter

$$W = P^3 + Q^2 \tag{33}$$

because in line with the classical theory [14]:

(1) if $W < 0$, then all roots are real and unequal (there should be resonance peaks and one anti-resonance dip in the vibration spectrum),

(2) if $W = 0$, then all roots are real, at least two of them being are equal,

(3) if $W > 0$, then one root is real and two other roots are complex conjugates (just one resonance peak is available).

Hence, the transition from two resonance peaks to one resonance peak occurs if $W = 0$. Using Eqs (32) and (33), express this condition in the form

$$\frac{(1-\beta)^2 - 2(1+\beta^2)\eta_{cr}^2}{12\beta^2} = \left[\frac{(1-\beta)^2(1+\beta)\eta_{cr}^2}{8\beta^3}\right]^{2/3} \tag{34}$$

where the value η_{cr} is the critical loss factor. Introducing the auxiliary variable

$$y = \left(\frac{1-\beta}{\eta_{cr}}\right)^{2/3}, \tag{35}$$

convert Eq. (34) to the form

$$y^3 + 3\,P_1\,y + 2\,Q_1 = 0 \tag{36}$$

with the coefficients

$$\begin{cases} P_1 = -(1\text{-}\beta^2)^{2/3}, \\ Q_1 = -(1+\beta^2). \end{cases} \tag{37}$$

Here, the important parameter of the cubic equation

$$W_1 = P_1^3 + Q_1^2 = -(1\text{-}\beta^2)^2 + (1+\beta^2)^2 = 4\beta^2 > 0. \tag{38}$$

Applying the theory [14] to Eqs (37) and (38), calculate the only real root of Eq. (36)

$$y = \sqrt[3]{-Q_1 + \sqrt{W_1}} + \sqrt[3]{\text{-}\,Q_1 - \sqrt{W_1}} = (1+\beta)^{2/3} + (1-\beta)^{2/3}. \tag{39}$$

Equating the right parts of Eqs. (35) and (39) and using Eq. (26), calculate the critical loss factor

$$\eta_{cr} = \frac{1-q^2}{[(1-q^2)^{2/3} + (1+q^2)^{2/3}]^{3/2}} \tag{40}$$

where the parameter $q = \dfrac{\omega_{min}}{\omega_{max}}$.

As follows from Eq. (40), the critical loss factor approaches its maximum $2^{-3/2} \approx 0.354$ if $q \to 0$ **(**that is, for $\omega_{min} << \omega_{max}$ **)** and approaches 0 if $q \to 1$. Applying the Maclaurin series expansions, reduce Eq. (40) to the approximate form

$$\eta_{cr} \approx 0.354\,\left(1-q^2\right). \tag{41}$$

The accurate (40) and approximate (41) relationships for the critical loss factor are graphically compared in Fig. 4. As seen, the difference between the two plots is little.

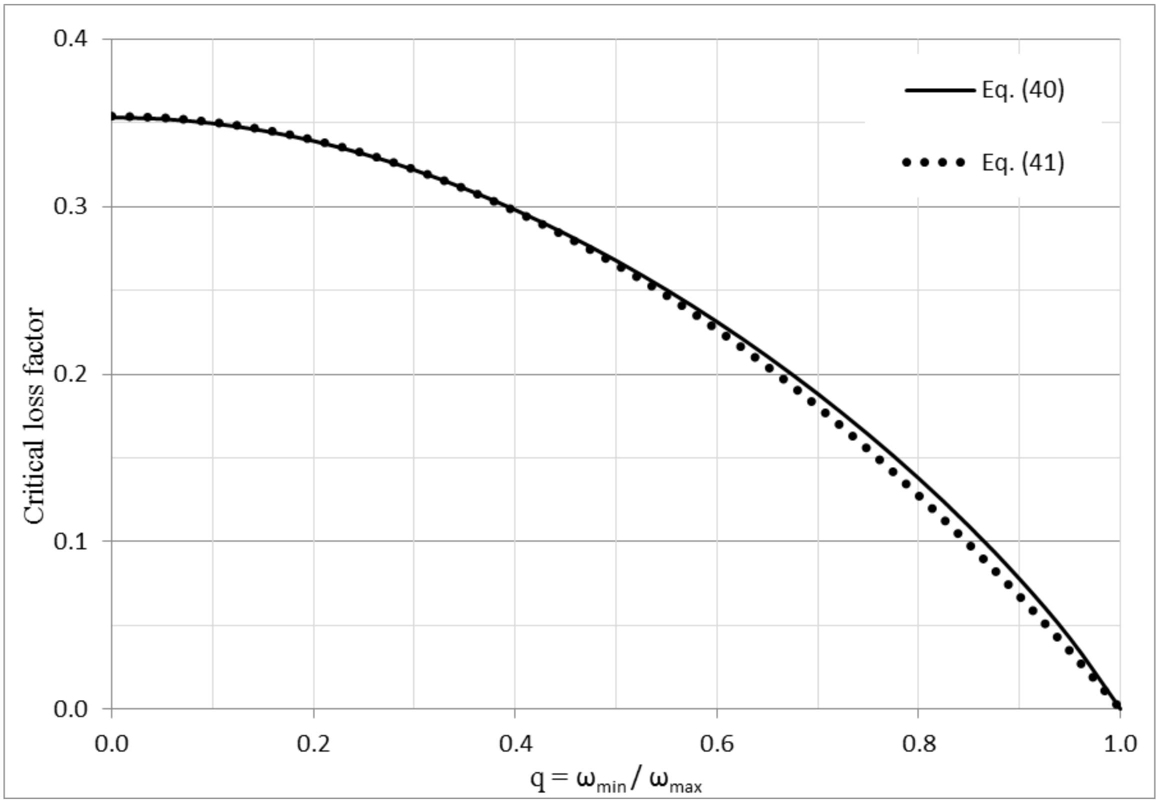

Fig. 4. Critical loss factor η_{cr} as a function of the ratio of the undamped natural frequencies, plotted using the accurate Eq. (40) and approximate Eq. (41).

3.5. Resonance frequencies of 2-DOF system in the critical degenerative case

Using the method [14] and Eqs (32) and (40), calculate the roots of Eq. (31) in the critical case:

$$\begin{cases} z_1 = -2\sqrt[3]{Q} = \dfrac{(\beta - 1)}{\beta} \dfrac{(1-\beta^2)^{1/3}}{(1-\beta)^{2/3} + (1+\beta)^{2/3}}, \\ z_{2,3} = \dfrac{z_1}{2}. \end{cases} \quad (43)$$

Substituting the first of Eqs (43) into Eq. (30), calculate the first root of Eq. (31) and estimate its first approximation using Maclaurin series expansions:

$$x_{1cr} = \frac{\beta+1}{2\beta} + \frac{(\beta - 1)}{\beta} \frac{(1-\beta^2)^{1/3}}{(1-\beta)^{2/3} + (1+\beta)^{2/3}} \approx 1 + \frac{\beta\,(1-\beta)}{9}. \quad (44)$$

Using Eqs (26), (27) and (44), obtain

$$\frac{\omega_{1cr}}{\omega_{min}} \approx \sqrt{1+\frac{\beta(1-\beta)}{9}} = \sqrt{1+\frac{q^2(1-q^2)}{9}} \approx 1+\frac{q^2(1-q^2)}{18} \leq 1\frac{1}{72} \approx 1.014. \qquad (45)$$

As follows from Eq. (45), in the critical case the fundamental damped natural frequency is very close to the fundamental undamped natural frequency (the relative deviation does not exceed 2%). The numeric calculations by Eqs (22) and (23) also indicate (see the plots in Fig. 3) that the difference between the fundamental damped and undamped frequencies is commonly minor. On the contrary, if the loss factor grows, the higher-frequency resonance for the second mass may notably shift to the lower frequencies and disappear in the degenerate case.

4. POTENTIAL APPLICATIONS FOR VIBRATION CONTROL

The vibration of the second mass of 2-DOF in-series systems may become severe if its natural frequencies are close together. In the practically important case $\eta_{min} = \eta_{max} = \eta$, using Eqs (22) and (23), calculate the transmissibility at the fundamental resonance frequency

$$T_2(\omega_{min}) \approx \frac{1+\eta^2}{\eta\sqrt{\left(1-q^2\right)^2+\eta^2}} \qquad (46)$$

Here, we employ the fact that the fundamental damped resonance frequency all but coincide with the fundamental undamped natural frequency ω_{min}.

From Eq. (46), the transmissibility for the second mass significantly grows if the parameter $q \to 1$, that is, the natural frequencies ω_{min} and ω_{max} get close to each other.

In this case, the resonance vibration of a 2-DOF in-series system can be notably reduced just by making the natural frequencies more different (in particular, via replacing the second spring with a stiffer one). In the degenerate case, the engineer could suggest that the second mass (in particular, an auxiliary cooling module attached to a car radiator) vibrates as a 1-DOF mechanical system. As a result, he might look for the vibration isolators with a much higher loss factor. In practical cases, this decision may not be practical. As shown in Fig. 5, the resonance peak in the degenerate case ($q = 0.8$ and $\eta = 0.138$) can be notably attenuated (1) via halving the ratio q of the natural frequencies from 0.8 to to 0.4 for the same loss factor, or (2) by doubling the lost factor value from 0.138 to 0.27 for the same parameter $q = 0.8$. The first method looks more effective.

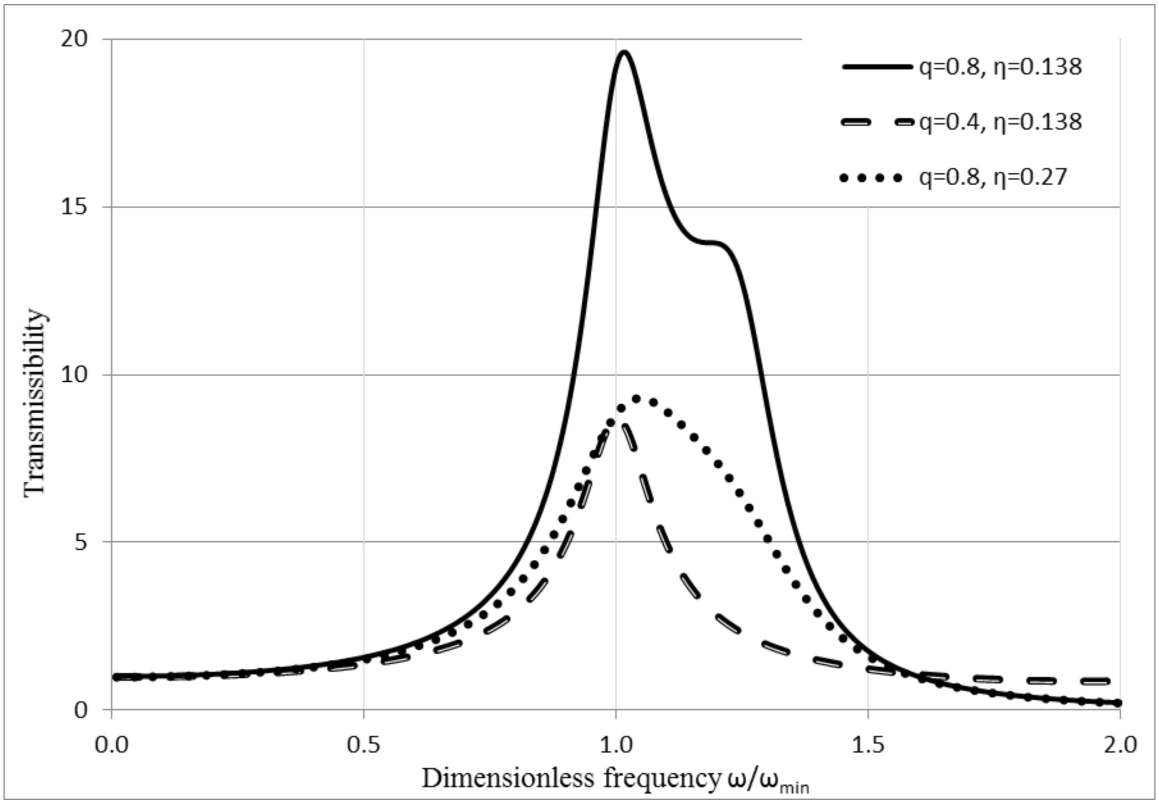

Fig. 5. The transmissibility calculated using Eq. (46) for three different cases.

5. CONCLUSIONS

The new relationships describing the dynamics of 2-DOF in-series systems with hysteresis damping and vibrating base were deduced.

(1) Eq. (13): The sum of the loss factors of the 2-DOF system equals the sum of the partial loss factors.

(2) Eq. (21): In the so-called affinity state (where the undamped natural frequencies of the 2-DOF system are most close to each other) defined by Eq. (17), the loss factors of the 2-DOF system get similar and equal to the arithmetic average of the partial loss factors.

(3) Eq. (40): A close-form equation for the critical loss factor was derived as the marginal condition of the degenerate case (when the higher-frequency resonance peak fully vanishes in the frequency response for the second mass). The critical loss factor can take values between 0 and $2^{-3/2} \approx 0.354$ and depends on the ratio of the natural frequencies of 2-DOF system: the closer the natural frequencies, the lower the critical loss factor.

(4) Eq. (41): The accurate equation for the critical loss factor was reduced to the approximate form. The difference between the values computed by Eqs (40) and (41) is little but Eq. (41) has a much simpler form.

The presented theory can help the practical engineers to understand if the real single-resonance system vibrates like the 1-DOF model or 2-DOF model in the degenerate case. For the first option, the main way to reduce the peak magnitude is to increase the loss factor. For the second option, the vibration can be effectively reduced for the same loss factor by making the natural frequencies more different from each other (in particular, via increasing the stiffness of the second spring).

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