

Convergence of the Newton-Kantorovich Method for the Gram Points

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Abstract

Some conjectures and corollaries regarding the convergence of the iterates of Newton's method $\lim_{m \rightarrow \infty} \left\{ N_{\vartheta_n(\tilde{y}_n^{(+)})}^{om} \right\}$ with starting points $\tilde{y}_n^{(+)} = \frac{(8n-7)\pi}{4W\left(\frac{8n-7}{8e}\right)}$ to the Gram points $g_n = y_n^{(+)}$ where $\vartheta_n(t) = \vartheta(t) - (n-1)\pi$ and $\vartheta(t)$ is the Riemann-Siegel vartheta function are given.

The Hardy Z function

$$Z(t) = e^{i\vartheta(t)} \zeta\left(\frac{1}{2} + it\right) \quad (1)$$

is isomorphic to the Riemann ζ function

$$\zeta(t) = e^{-i\vartheta\left(\frac{i}{2} - it\right)} Z\left(\frac{i}{2} - it\right) \quad (2)$$

where

$$\vartheta(t) = -\frac{i}{2} \left(\ln \Gamma\left(\frac{1}{4} + \frac{it}{2}\right) - \ln \Gamma\left(\frac{1}{4} - \frac{it}{2}\right) \right) - \frac{\ln(\pi)t}{2} \quad (3)$$

is the Riemann-Siegel theta(aka vartheta) function and $\ln \Gamma$ is the principal branch of the logarithm of the Γ function which is the analytic continuation of the factorial, $\Gamma(s+1) = s! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot s$. [6] The n -th Gram point's [2, 6.5] location $y_n^{(+)}$ is approximated very well by

$$\tilde{y}_n^{(+)} = \tilde{g}_n = \frac{(8n-7)\pi}{4W\left(\frac{8n-7}{8e}\right)} \quad (4)$$

where W is the Lambert W function which is derived by replacing $\tilde{\vartheta} \rightarrow \vartheta(t)$ with its asymptotic expansion

$$\tilde{\vartheta}(t) = \frac{1}{2} t \ln\left(\frac{t}{2\pi e}\right) - \frac{\pi}{8} = \vartheta(t) + O(0.05) \forall t \geq 1 \quad (5)$$

and using the fact that equation

$$\tilde{\vartheta}(t) = y \quad (6)$$

is invertible having a unique solution given by $t = \frac{\pi + 8y}{4W\left(\frac{\pi + 8y}{8\pi e}\right)}$ so that

$$\tilde{\vartheta}\left(\frac{\pi + 8y}{4W\left(\frac{\pi + 8y}{8\pi e}\right)}\right) = \frac{1}{2} \frac{\pi + 8y}{4W\left(\frac{\pi + 8y}{8\pi e}\right)} \ln\left(\frac{\frac{\pi + 8y}{4W\left(\frac{\pi + 8y}{8\pi e}\right)}}{2\pi e}\right) - \frac{\pi}{8} = y \quad (7)$$

and the derivative of $\tilde{\vartheta}(t)$ is given by

$$\tilde{\vartheta}'(t) = \frac{1}{2} \ln\left(\frac{t}{2\pi e}\right) + \frac{1}{2} + O(0.1) \forall t \geq 1 \quad (8)$$

[3, Eq. 163]^{[1][8]} Newton's method for solving $\vartheta(t) = (n-1)\pi$ is guaranteed to converge if started from a point $t \in B(\tilde{y}_n^{(+)}, r_n^*)$ within the closure (inside or on the boundary) of ball of radius r_n^* , to be derived below, centered at $\tilde{y}_n^{(+)}$. The Gram points have the property that the imaginary part of ζ on the critical line vanishes at their locations, that is

$$\text{Im}\left(\zeta\left(\frac{1}{2} + iy_n^{(+)}\right)\right) = 0 \quad (9)$$

1. The authors notation leads to the impression that \tilde{g}_n is actually the n -th Gram point, rather than a very good approximation to it

To determine the sequence r_n^* in (4) let

$$a_n = \left| \frac{\vartheta_n(\tilde{y}_n^{(+)})}{\dot{\vartheta}(\tilde{y}_n^{(+)})} \right| = \left| \frac{\vartheta(\tilde{y}_n^{(+)}) - (n-1)\pi}{\dot{\vartheta}(\tilde{y}_n^{(+)})} \right| \quad (10)$$

$$b_n = \left| \frac{\frac{\dot{\vartheta}(\tilde{y}_{(n+1)}^{(+)}) - \dot{\vartheta}(\tilde{y}_{(n-1)}^{(+)})}{\dot{\vartheta}(\tilde{y}_n^{(+)})}}{\tilde{y}_{(n+1)}^{(+)} - \tilde{y}_{(n-1)}^{(+)}} \right| \quad (11)$$

where

$$r_n^* = \frac{1 - \sqrt{2a_nb_n}}{b_n} \quad (12)$$

then, by applying [4, Theorem 3] or the more general [1, Theorem 2.2.4], it can be shown that the Newton-Kantorovich method[7] defined by the iteration function

$$N_{\vartheta_n}(t) = t - \frac{-\frac{i}{2} \left(\ln \Gamma\left(\frac{1}{4} + \frac{it}{2}\right) - \ln \Gamma\left(\frac{1}{4} - \frac{it}{2}\right) \right) - \frac{\ln(\pi)}{2}t - n\pi + \pi}{\frac{1}{4} \left(\Psi\left(\frac{1}{4} - \frac{it}{2}\right) + \Psi\left(\frac{1}{4} + \frac{it}{2}\right) \right) - \frac{\ln(\pi)}{2}} \quad (13)$$

which, when applied applied to itself repeatedly, converges to a solution $\vartheta_n(y_n^{(+)}) = 0$ if the initial point of the iteration is within $B(\tilde{y}_n^{(+)}, r_n^*)$, the closure of a ball of radius r_n^* centered at $\tilde{y}_n^{(+)}$, that is

$$y_n^{(+)} = \lim_{m \rightarrow \infty} N_{\vartheta_n}^{\circ m}(\tilde{y}_n^{(+)}) \subset \overline{B(\tilde{y}_n^{(+)}, r_n^*)} \quad (14)$$

if

$$h_n = a_n b_n = \left| \frac{\vartheta_n(\tilde{y}_n^{(+)})}{\dot{\vartheta}(\tilde{y}_n^{(+)})} \right| \left| \frac{\frac{\dot{\vartheta}(\tilde{y}_{(n+1)}^{(+)}) - \dot{\vartheta}(\tilde{y}_{(n-1)}^{(+)})}{\dot{\vartheta}(\tilde{y}_n^{(+)})}}{\tilde{y}_{(n+1)}^{(+)} - \tilde{y}_{(n-1)}^{(+)}} \right| = \left| \left(\frac{\vartheta_n(\tilde{y}_n^{(+)})}{\dot{\vartheta}(\tilde{y}_n^{(+)})} \right) \left(\frac{\dot{\vartheta}(\tilde{y}_{(n+1)}^{(+)}) - \dot{\vartheta}(\tilde{y}_{(n-1)}^{(+)})}{\tilde{y}_{(n+1)}^{(+)} - \tilde{y}_{(n-1)}^{(+)}} \right) \right| \leq \frac{1}{2} \forall n \geq 1 \quad (15)$$

and furthermore the solution $y_n^{(+)}$ is unique within the interior of a ball centered at the same location $\tilde{y}_n^{(+)}$ of radius

$$r_n^{**} = \frac{1 + \sqrt{2a_nb_n}}{b_n} > r_n^* \quad (16)$$

that is

$$\left\{ \operatorname{Im} \left(\zeta \left(\frac{1}{2} + it \right) \right) \neq 0 : t \in B(\tilde{y}_n^{(+)}, r_n^{**}) \setminus B(\tilde{y}_n^{(+)}, r_n^*) \right\} \quad (17)$$

If $h_n = a_n b_n = \frac{1}{2}$ then $r_n^* = r_n^{**}$ and the solution both exists and is unique within the same ball, a situation which would be denoted by

$$\left\{ \operatorname{Im} \left(\zeta \left(\frac{1}{2} + it \right) \right) = 0 : t \in B(\tilde{y}_n^{(+)}, r_n^* = r_n^{**}) \right\} \quad (18)$$

Letting $n = 1$, we see that

$$a_1 = \left| \frac{\vartheta_1(\tilde{y}_1^{(+)})}{\dot{\vartheta}(\tilde{y}_1^{(+)})} \right| = 0.002236840082... \quad (19)$$

$$b_1 = \left| \frac{\frac{\dot{\vartheta}(\tilde{y}_{(2)}^{(+)}) - \dot{\vartheta}(\tilde{y}_{(0)}^{(+)})}{\dot{\vartheta}(\tilde{y}_1^{(+)})}}{\tilde{y}_{(2)}^{(+)} - \tilde{y}_{(0)}^{(+)}} \right| = 0.06201224977... \quad (20)$$

so that

$$h_1 = a_1 b_1 = 0.0001387114859 \leq 0.5 \quad (21)$$

and thus convergence is assured so that

$$y_1^{(+)} \subset \overline{B(\tilde{y}_1^{(+)}, r_1^*)} \quad (22)$$

where

$$r_1^* = 15.85725376... \quad (23)$$

and furthermore the solution is unique within the larger ball centered at the same location of radius r_1^{**} , that is

$$\left\{ \operatorname{Im} \left(\zeta \left(\frac{1}{2} + it \right) \right) \neq 0 : t \in B(\tilde{y}_1^{(+)}, r_1^{**}) \setminus B(\tilde{y}_1^{(+)}, r_1^*) \right\} \quad (24)$$

where

$$r_1^{**} = 16.39443856... \quad (25)$$

Conjecture 1. *The sequence a_n is monotonically decreasing, $a_n < a_{n-1} \forall n \geq 1$*

Conjecture 2. *The sequence b_n is monotonically decreasing, $b_n < b_{n-1} \forall n \geq 1$*

Let \tilde{a}_n and \tilde{b}_n be the sequences obtained by replacing $\vartheta(t)$ and its derivative $\dot{\vartheta}(t)$ with their asymptotic expansions $\tilde{\vartheta}(t)$

$$\tilde{a}_n = \left| \frac{\tilde{\vartheta}_n(\tilde{y}_n^{(+)})}{\tilde{\dot{\vartheta}}_n(\tilde{y}_n^{(+)})} \right| = \left| \frac{\tilde{\vartheta}_n \left(\frac{(8n-7)\pi}{4W\left(\frac{8n-7}{8e}\right)} \right)}{\tilde{\dot{\vartheta}} \left(\frac{(8n-7)\pi}{4W\left(\frac{8n-7}{8e}\right)} \right)} \right| = \left| \frac{\left(\frac{1}{2} \frac{(8n-7)\pi}{4W\left(\frac{8n-7}{8e}\right)} \ln \left(\frac{\frac{(8n-7)\pi}{4W\left(\frac{8n-7}{8e}\right)}}{2\pi e} \right) - \frac{\pi}{8} \right) - (n-1)\pi + O(0.05)}{\left(\frac{1}{2} \ln \left(\frac{\frac{(8n-7)\pi}{4W\left(\frac{8n-7}{8e}\right)}}{2\pi e} \right) + \frac{1}{2} \right) + O(0.1)} \right| \quad (26)$$

$$\tilde{b}_n = \left| \frac{\tilde{\vartheta} \left(\frac{(8(n+1)-7)\pi}{4W\left(\frac{8(n+1)-7}{8e}\right)} \right) - \tilde{\vartheta} \left(\frac{(8(n-1)-7)\pi}{4W\left(\frac{8(n-1)-7}{8e}\right)} \right)}{\tilde{\dot{\vartheta}} \left(\frac{(8n-7)\pi}{4W\left(\frac{8n-7}{8e}\right)} \right)} \right| = \left| \frac{\left(\frac{1}{2} \ln \left(\frac{\frac{(8(n+1)-7)\pi}{4W\left(\frac{8(n+1)-7}{8e}\right)}}{2\pi e} \right) + \frac{1}{2} + O(0.05) \right) - \frac{1}{2} \ln \left(\frac{\frac{(8(n-1)-7)\pi}{4W\left(\frac{8(n-1)-7}{8e}\right)}}{2\pi e} \right) + \frac{1}{2} + O(0.05)}{\frac{1}{2} \ln \left(\frac{\frac{(8n-7)\pi}{4W\left(\frac{8n-7}{8e}\right)}}{2\pi e} \right) + \frac{1}{2} + O(0.1)} \right| \quad (27)$$

$$= \left| \frac{\frac{(8(n+1)-7)\pi}{4W\left(\frac{8(n+1)-7}{8e}\right)} - \frac{(8(n-1)-7)\pi}{4W\left(\frac{8(n-1)-7}{8e}\right)}}{\left(\frac{(8(n+1)-7)\pi}{4W\left(\frac{8(n+1)-7}{8e}\right)} \right) - \frac{(8(n-1)-7)\pi}{4W\left(\frac{8(n-1)-7}{8e}\right)}} \right|$$

then it can be shown that

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \tilde{a}_n = 0 \quad (28)$$

$$\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \tilde{b}_n = 0 \quad (29)$$

TODO: use the asymptotic expansion

$$W\left(\frac{xt \pm n}{xe}\right) = W\left(\frac{t}{e}\right) + O\left(\frac{1}{t}\right) \quad (30)$$

and the fact that

$$\ln(t) - \ln(\ln(t)) < W(t) < \ln(t) \quad (31)$$

[5]

Conjecture 3. *The sequence h_n is monotonically decreasing, $h_n < h_{n-1} \forall n \geq 1$*

Let

$$\tilde{h}_n = \tilde{a}_n \tilde{b}_n = \left(\frac{\left(\frac{1}{2} \frac{(8n-7)\pi}{4W\left(\frac{8n-7}{8e}\right)} \ln \left(\frac{\frac{(8n-7)\pi}{4W\left(\frac{8n-7}{8e}\right)}}{2\pi e} \right) - \frac{\pi}{8} \right) - (n-1)\pi}{\frac{1}{2} \ln \left(\frac{\frac{(8n-7)\pi}{4W\left(\frac{8n-7}{8e}\right)}}{2\pi e} \right) + \frac{1}{2}} \right) \left(\frac{\left(\frac{1}{2} \ln \left(\frac{\frac{(8(n+1)-7)\pi}{4W\left(\frac{8(n+1)-7}{8e}\right)}}{2\pi e} \right) + \frac{1}{2} \right) - \frac{1}{2} \ln \left(\frac{\frac{(8(n-1)-7)\pi}{4W\left(\frac{8(n-1)-7}{8e}\right)}}{2\pi e} \right) + \frac{1}{2} \right)}{\frac{1}{2} \ln \left(\frac{\frac{(8n-7)\pi}{4W\left(\frac{8n-7}{8e}\right)}}{2\pi e} \right) + \frac{1}{2}} \right) \left(\frac{\frac{(8(n+1)-7)\pi}{4W\left(\frac{8(n+1)-7}{8e}\right)} - \frac{(8(n-1)-7)\pi}{4W\left(\frac{8(n-1)-7}{8e}\right)}}{\left(\frac{(8(n+1)-7)\pi}{4W\left(\frac{8(n+1)-7}{8e}\right)} \right) - \frac{(8(n-1)-7)\pi}{4W\left(\frac{8(n-1)-7}{8e}\right)}} \right) \quad (32)$$

then it can be shown that

$$\lim_{n \rightarrow \infty} \tilde{h}_n = \lim_{n \rightarrow \infty} \tilde{a}_n \tilde{b}_n = 0 \quad (33)$$

Corollary 4. *If Conjecture 3 is true and h_n is monotonically decreasing then $h_n \leq \frac{1}{2} \forall n \geq 1$ since $h_1 = 0.001387114859... > h_n \forall n \geq 1$ which means that the n th Gram point $y_n^{(+)}$ exists and is a unique and well-defined number $\forall n \geq 1$ since Theorem 3 in [4] proves that the accumulation point $y_n^{(+)}$ of the Cauchy sequence generated by repeated application of N_{ϑ_n} [9, 3.10] converges quadratically to a well-defined and unique point*

$$\{\tilde{y}_n^{(+)}, N_{\vartheta_n}(\tilde{y}_n^{(+)}), N_{\vartheta_n}(N_{\vartheta_n}(\tilde{y}_n^{(+)})), N_{\vartheta_n}(N_{\vartheta_n}(N_{\vartheta_n}(\tilde{y}_n^{(+)}))), \dots\} \rightarrow y_n^{(+)} \in \overline{B(\tilde{y}_n^{(+)}, r_n^*)} \quad (34)$$

since $\tilde{y}_n^{(+)}$ is a good starting point converging to $y_n^{(+)} \forall n \geq 1$.

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