Convergence of the Newton-Kantorovich Method for the Gram Points

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Abstract

Some conjectures and corollaries regarding the convergence of the iterates of Newton's method $\lim_{m \to \infty} \left\{ N_{\vartheta_n\left(\tilde{y}_n^{(+)}\right)}^{om} \right\}$ with starting points $\tilde{y}_n^{(+)} = \frac{(8n-7)\pi}{4W\left(\frac{8n-7}{8e}\right)}$ to the Gram points $g_n = y_n^{(+)}$ where $\vartheta_n(t) = \vartheta(t) - (n-1)\pi$ and $\vartheta(t)$ is the Riemann-Siegel vartheta function are given.

The Hardy Z function

$$Z(t) \ = e^{i\vartheta(t)} \zeta\left(\frac{1}{2} + it\right) \tag{1}$$

is isomorphic to the Riemann ζ function

$$\zeta(t) = e^{-i\vartheta\left(\frac{i}{2} - it\right)} Z\left(\frac{i}{2} - it\right) \tag{2}$$

where

$$\vartheta(t) = -\frac{i}{2}\left(\ln\Gamma\left(\frac{1}{4} + \frac{it}{2}\right) - \ln\Gamma\left(\frac{1}{4} - \frac{it}{2}\right)\right) - \frac{\ln(\pi)\,t}{2} \tag{3}$$

is the Riemann-Siegel theta(aka vartheta) function and $\ln\Gamma$ is the principal branch of the logarithm of the Γ function which is the analytic continuation of the factorial, $\Gamma(s+1) = s! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot s$. [6] The n-th Gram point's[2, 6.5] location $y_n^{(+)}$ is approximated very well by

$$\tilde{y}_n^{(+)} = \tilde{g}_n = \frac{(8n-7)\pi}{4W\left(\frac{8n-7}{8e}\right)} \tag{4}$$

where W is the Lambert W function which is derived by replacing $\tilde{\vartheta} \to \vartheta(t)$ with its asymptotic expansion

$$\tilde{\vartheta}(t) = \frac{1}{2} t \ln\left(\frac{t}{2\pi e}\right) - \frac{\pi}{8} = \vartheta(t) + O(0.05) \forall t \geqslant 1 \tag{5}$$

and using the fact that equation

$$\tilde{\vartheta}(t) = y \tag{6}$$

is invertible having a unique solution given by $t = \frac{\pi + 8y}{4W\left(\frac{\pi + 8y}{8\pi e}\right)}$ so that

$$\tilde{\vartheta}\left(\frac{\pi + 8y}{4W\left(\frac{\pi + 8y}{8\pi e}\right)}\right) = \frac{1}{2}\frac{\pi + 8y}{4W\left(\frac{\pi + 8y}{8\pi e}\right)} \ln\left(\frac{\frac{\pi + 8y}{4W\left(\frac{\pi + 8y}{8\pi e}\right)}}{2\pi e}\right) - \frac{\pi}{8} = y \tag{7}$$

and the derivative of $\tilde{\vartheta}(t)$ is given by

$$\dot{\tilde{\vartheta}}(t) = \frac{1}{2}\ln\left(\frac{t}{2\pi e}\right) + \frac{1}{2} + O(0.1) \forall t \geqslant 1 \tag{8}$$

[3, Eq. 163][1][8] Newton's method for solving $\vartheta(t) = (n-1)\pi$ is guaranteed to converge if started from a point $t \in B\left(\tilde{y}_n^{(+)}, r_n^*\right)$ within the closure(inside or on the boundary) of ball of radius r_n^*, to be derived below, centered at $\tilde{y}_n^{(+)}$. The Gram points have the property that the imaginary part of ζ on the critical line vanishes at their locations, that is

$$\operatorname{Im}\left(\zeta\left(\frac{1}{2} + iy_n^{(+)}\right)\right) = 0 \tag{9}$$

1. The authors notation leads to the impression that $\tilde{g}_n$ is actually the n-th Gram point, rather than a very good approximation to it

To determine the sequence r_n^* in (4) let

$$a_n = \left|\frac{\vartheta_n(\tilde{y}_n^{(+)})}{\dot{\vartheta}(\tilde{y}_n^{(+)})}\right| = \left|\frac{\vartheta(\tilde{y}_n^{(+)}) - (n-1)\pi}{\dot{\vartheta}(\tilde{y}_n^{(+)})}\right| \tag{10}$$

$$b_n = \left|\frac{\frac{\dot{\vartheta}(\tilde{y}_{(n+1)}^{(+)}) - \dot{\vartheta}(\tilde{y}_{(n-1)}^{(+)})}{\dot{\vartheta}(\tilde{y}_n^{(+)})}}{\tilde{y}_{(n+1)}^{(+)} - \tilde{y}_{(n-1)}^{(+)}}\right| \tag{11}$$

where

$$r_n^* = \frac{1 - \sqrt{2 a_n b_n}}{b_n} \tag{12}$$

then, by applying [4, Theorem 3] or the more general [1, Theorem 2.2.4], it can be shown that the Newton-Kantorovich method[7] defined by the iteration function

$$N_{\vartheta_n}(t) \;= t - \frac{-\frac{i}{2}\left(\ln\Gamma\left(\frac{1}{4} + \frac{i\,t}{2}\right) - \ln\Gamma\left(\frac{1}{4} - \frac{i\,t}{2}\right)\right) - \frac{\ln(\pi)}{2}t - n\pi + \pi}{\frac{1}{4}\left(\Psi\left(\frac{1}{4} - \frac{i\,t}{2}\right) + \Psi\left(\frac{1}{4} + \frac{i\,t}{2}\right)\right) - \frac{\ln(\pi)}{2}} \tag{13}$$

which, when applied applied to itself repeatedly, converges to a solution $\vartheta_n(y_n^{(+)}) = 0$ if the initial point of the iteration is within $B(\tilde{y}_n^{(+)}, r_n^*)$, the closure of a ball of radius r_n^* centered at $\tilde{y}_n^{(+)}$, that is

$$y_n^{(+)} \;= \lim_{m\to\infty} N_{\vartheta_n}^{\circ m}(\tilde{y}_n^{(+)}) \;\subset \overline{B(\tilde{y}_n^{(+)}, r_n^*)} \tag{14}$$

if

$$h_n = a_n\, b_n = \left|\frac{\vartheta_n(\tilde{y}_n^{(+)})}{\dot{\vartheta}(\tilde{y}_n^{(+)})}\right| \left|\frac{\frac{\dot{\vartheta}(\tilde{y}_{(n+1)}^{(+)}) - \dot{\vartheta}(\tilde{y}_{(n-1)}^{(+)})}{\dot{\vartheta}(\tilde{y}_n^{(+)})}}{\tilde{y}_{(n+1)}^{(+)} - \tilde{y}_{(n-1)}^{(+)}}\right| = \left|\left(\frac{\vartheta_n(\tilde{y}_n^{(+)})}{\dot{\vartheta}(\tilde{y}_n^{(+)})}\right)\left(\frac{\frac{\dot{\vartheta}(\tilde{y}_{(n+1)}^{(+)}) - \dot{\vartheta}(\tilde{y}_{(n-1)}^{(+)})}{\dot{\vartheta}(\tilde{y}_n^{(+)})}}{\tilde{y}_{(n+1)}^{(+)} - \tilde{y}_{(n-1)}^{(+)}}\right)\right| \leqslant \frac{1}{2} \forall n \geqslant 1 \tag{15}$$

and furthermore the solution $y_n^{(+)}$ is unique within the interior of a ball centered at the same location $\tilde{y}_n^{(+)}$ of radius

$$r_n^{**} = \frac{1 + \sqrt{2 a_n b_n}}{b_n} > r_n^* \tag{16}$$

that is

$$\left\{\mathrm{Im}\left(\zeta\left(\frac{1}{2} + i t\right)\right) \neq 0 \colon t \in B(\tilde{y}_n^{(+)}, r_n^{**}) \backslash B(\tilde{y}_n^{(+)}, r_n^*)\right\} \tag{17}$$

If $h_n = a_n b_n = \frac{1}{2}$ then $r_n^* = r_n^{**}$ and the solution both exists and is unique within the same ball, a situation which would be denoted by

$$\left\{\mathrm{Im}\left(\zeta\left(\frac{1}{2} + i t\right)\right) = 0 \colon t \in B(\tilde{y}_n^{(+)}, r_n^* = r_n^{**})\right\} \tag{18}$$

Letting $n = 1$, we see that

$$a_1 = \left|\frac{\vartheta_1(\tilde{y}_1^{(+)})}{\dot{\vartheta}(\tilde{y}_1^{(+)})}\right| = 0.002236840082... \tag{19}$$

$$b_1 = \left|\frac{\frac{\dot{\vartheta}(\tilde{y}_{(2)}^{(+)}) - \dot{\vartheta}(\tilde{y}_{(0)}^{(+)})}{\dot{\vartheta}(\tilde{y}_1^{(+)})}}{\tilde{y}_{(2)}^{(+)} - \tilde{y}_{(0)}^{(+)}}\right| = 0.06201224977... \tag{20}$$

so that

$$h_1 = a_1\, b_1 = 0.0001387114859 \leqslant 0.5 \tag{21}$$

and thus convergence is assured so that

$$y_1^{(+)} \subset \overline{B(\tilde{y}_1^{(+)}, r_1^*)} \tag{22}$$

where

$$r_1^{*}=15.85725376... \tag{23}$$

and furthermore the solution is unique within the larger ball centered at the same location of radius r_1^{**}, that is

$$\left\{\mathrm{Im}\left(\zeta\left(\frac{1}{2}+it\right)\right)\neq 0 : t\in B\left(\tilde{y}_1^{(+)},r_1^{**}\right)\backslash B\left(\tilde{y}_1^{(+)},r_1^{*}\right)\right\} \tag{24}$$

where

$$r_1^{**}=16.39443856... \tag{25}$$

Conjecture 1. *The sequence a_n is monotonically decreasing, $a_n<a_{n-1}\forall n\geqslant 1$*

Conjecture 2. *The sequence b_n is monotonically decreasing, $b_n<b_{n-1}\forall n\geqslant 1$*

Conjecture 3. *Let $\tilde{a}_n$ and $\tilde{b}_n$ be the sequences obtained by replacing $\vartheta(t)$ and its derivative $\dot{\vartheta}(t)$ with their asymptotic expansions $\tilde{\vartheta}(t)$*

$$\tilde{a}_n=\left|\frac{\tilde{\vartheta}_n\left(\tilde{y}_n^{(+)}\right)}{\dot{\tilde{\vartheta}}\left(\tilde{y}_n^{(+)}\right)}\right|=\left|\frac{\tilde{\vartheta}_n\left(\frac{(8n-7)\pi}{4W\left(\frac{8n-7}{8e}\right)}\right)}{\dot{\tilde{\vartheta}}\left(\frac{(8n-7)\pi}{4W\left(\frac{8n-7}{8e}\right)}\right)}\right|=\left|\frac{\left(\frac{1}{2}\frac{(8n-7)\pi}{4W\left(\frac{8n-7}{8e}\right)}\ln\left(\frac{\frac{(8n-7)\pi}{4W\left(\frac{8n-7}{8e}\right)}}{2\pi e}\right)-\frac{\pi}{8}\right)-(n-1)\pi+O(0.05)}{\left(\frac{1}{2}\ln\left(\frac{\frac{(8n-7)\pi}{4W\left(\frac{8n-7}{8e}\right)}}{2\pi e}\right)+\frac{1}{2}\right)+O(0.1)}\right| \tag{26}$$

$$\tilde{b}_n=\left|\frac{\frac{\dot{\tilde{\vartheta}}\left(\frac{(8(n+1)-7)\pi}{4W\left(\frac{8(n+1)-7}{8e}\right)}\right)-\dot{\tilde{\vartheta}}\left(\frac{(8(n-1)-7)\pi}{4W\left(\frac{8(n-1)-7}{8e}\right)}\right)}{\dot{\tilde{\vartheta}}\left(\frac{(8n-7)\pi}{4W\left(\frac{8n-7}{8e}\right)}\right)}}{\frac{(8(n+1)-7)\pi}{4W\left(\frac{8(n+1)-7}{8e}\right)}-\frac{(8(n-1)-7)\pi}{4W\left(\frac{8(n-1)-7}{8e}\right)}}\right|=\left|\frac{\frac{\left(\frac{1}{2}\ln\left(\frac{\frac{(8(n+1)-7)\pi}{4W\left(\frac{8(n+1)-7}{8e}\right)}}{2\pi e}\right)+\frac{1}{2}+O(0.05)\right)-\frac{1}{2}\ln\left(\frac{\frac{(8(n-1)-7)\pi}{4W\left(\frac{8(n-1)-7}{8e}\right)}}{2\pi e}+\frac{1}{2}+O(0.05)\right)}{\frac{1}{2}\ln\left(\frac{\frac{(8n-7)\pi}{4W\left(\frac{8n-7}{8e}\right)}}{2\pi e}\right)+\frac{1}{2}+O(0.1)}}{\left(\frac{(8(n+1)-7)\pi}{4W\left(\frac{8(n+1)-7}{8e}\right)}\right)-\frac{(8(n-1)-7)\pi}{4W\left(\frac{8(n-1)-7}{8e}\right)}}\right| \tag{27}$$

then it can be shown that

$$\lim_{n\to\infty}a_n=\lim_{n\to\infty}\tilde{a}_n=0 \tag{28}$$

$$\lim_{n\to\infty}b_n=\lim_{n\to\infty}\tilde{b}_n=0 \tag{29}$$

using the fact that

$$W\left(\frac{xt\pm n}{xe}\right)=W\left(\frac{t}{e}\right)+O\left(\frac{1}{t}\right) \tag{30}$$

and

$$\ln(t)-\ln(\ln(t))<W(t)<\ln(t) \tag{31}$$

[5]

Conjecture 4. *The sequence h_n is monotonically decreasing, $h_n<h_{n-1}\forall n\geqslant 1$*

Let

$$\tilde{h}_n=\tilde{a}_n\tilde{b}_n=\left|\left(\frac{\left(\frac{1}{2}\frac{(8n-7)\pi}{4W\left(\frac{8n-7}{8e}\right)}\ln\left(\frac{\frac{(8n-7)\pi}{4W\left(\frac{8n-7}{8e}\right)}}{2\pi e}\right)-\frac{\pi}{8}\right)-(n-1)\pi}{\frac{1}{2}\ln\left(\frac{\frac{(8n-7)\pi}{4W\left(\frac{8n-7}{8e}\right)}}{2\pi e}\right)+\frac{1}{2}}\right)\left(\frac{\frac{\left(\frac{1}{2}\ln\left(\frac{\frac{(8(n+1)-7)\pi}{4W\left(\frac{8(n+1)-7}{8e}\right)}}{2\pi e}\right)+\frac{1}{2}\right)-\frac{1}{2}\ln\left(\frac{\frac{(8(n-1)-7)\pi}{4W\left(\frac{8(n-1)-7}{8e}\right)}}{2\pi e}+\frac{1}{2}\right)}{\frac{1}{2}\ln\left(\frac{\frac{(8n-7)\pi}{4W\left(\frac{8n-7}{8e}\right)}}{2\pi e}\right)+\frac{1}{2}}}{\frac{(8(n+1)-7)\pi}{4W\left(\frac{8(n+1)-7}{8e}\right)}-\frac{(8(n-1)-7)\pi}{4W\left(\frac{8(n-1)-7}{8e}\right)}}\right)\right| \tag{32}$$

then it can be shown that

$$\lim_{n\to\infty} \tilde{h}_n = \lim_{n\to\infty} \tilde{a}_n \tilde{b}_n = 0 \tag{33}$$

Corollary 5. *If Conjecture 4 is true and h_n is monotonically decreasing then $h_n \leqslant \frac{1}{2} \forall n \geqslant 1$ since $h_1 = 0.001387114859... > h_n \forall n \geqslant 1$ which means that the nth Gram point $y_n^{(+)}$ exists and is a unique and well-defined number $\forall n \geqslant 1$ since Theorem 3 in [4] proves that the accumulation point $y_n^{(+)}$ of the Cauchy sequence generated by repeated application of N_{ϑ_n} [9, 3.10] converges quadratically to a well-defined and unique point*

$$\{\tilde{y}_n^{(+)}, N_{\vartheta_n}(\tilde{y}_n^{(+)}), N_{\vartheta_n}(N_{\vartheta_n}(\tilde{y}_n^{(+)})), N_{\vartheta_n}(N_{\vartheta_n}(N_{\vartheta_n}(\tilde{y}_n^{(+)})), ...\} \to y_n^{(+)} \in \overline{B(\tilde{y}_n^{(+)}, r_n^*)} \tag{34}$$

since $\tilde{y}_n^{(+)}$ is a good starting point converging to $y_n^{(+)} \forall n \geqslant 1$.

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