The use of the electrodynamics canonical spin tensor

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It is demonstrated that a dielectric, which absorbs a circularly polarized plane electromagnetic wave, absorbs the angular momentum, which is contained in the wave, according to the canonical spin tensor. The given calculations show that spin occurs to be the same natural property of a plane electromagnetic wave, as energy and momentum.
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1. Introduction. The Poynting vector

Let a plane monochromatic circularly polarized electromagnetic wave

$$\breve{\mathbf{E}}_1 = E_1(\mathbf{x}+i\mathbf{y})\exp(ikz - i\omega t)\ \text{[V/m]},\quad \breve{\mathbf{H}}_1 = -i\varepsilon_0 c\breve{\mathbf{E}}_1\ \text{[A/m]},\quad ck=\omega \tag{1.1}$$

impinges normally on a flat surface of lossy dielectric, which is characterized by a complex permittivity $\breve{\varepsilon}$ (we mark complex numbers and vectors by *breve*). As is known, in this case, the reflected and the passed waves have the forms

$$\breve{\mathbf{E}}_2 = \breve{E}_2(\mathbf{x}+i\mathbf{y})\exp(-ikz - i\omega t),\quad \breve{\mathbf{H}}_2 = i\varepsilon_0 c\breve{\mathbf{E}}_2\quad \breve{E}_2 = \frac{1-\breve{k}}{1+\breve{k}}E_1\quad \breve{k}=\sqrt{\breve{\varepsilon}} = k' + ik'' \tag{1.2}$$

$$\breve{\mathbf{E}}_3 = \breve{E}_3(\mathbf{x}+i\mathbf{y})\exp(ik\breve{k}z - i\omega t),\quad \breve{\mathbf{H}}_3 = -i\varepsilon_0 c\breve{k}\breve{\mathbf{E}}_3,\quad \breve{E}_3 = \frac{2}{1+\breve{k}}E_1 \tag{1.3}$$

($\breve{k}$ designates the complex refraction coefficient).

The mass-energy flux density, which impinges normally on the dielectric surface and then is absorbed by the dielectric, i.e. the total Poynting vector, can be calculated taking into account that the incident and reflected beams do not interfered with each other, and, so, one may simply subtract energy flux density of the reflected beam from energy flux density of the incident beam.

$$\Pi = \Pi_1 - \Pi_2 = \varepsilon_0 c E_1^2 - \varepsilon_0 c|\breve{E}_2|^2 = \varepsilon_0 c E_1^2\left(1 - \left|\frac{1-\breve{k}}{1+\breve{k}}\right|^2\right) = \varepsilon_0 c E_1^2 \frac{4k'}{(1+k')^2 + k''^2}\ \left[\frac{\text{J}}{\text{m}^2\text{s}}\right]. \tag{1.4}$$

2. The absorption of energy an angular momentum by the dielectric

The energy flux density (1.4) enters into the dielectric and can be calculated by the use of expression (1.3) for the wave inside the dielectric. A mechanism of the absorption such a wave was explained by Feynman [1] very good: the rotating electric field $\breve{\mathbf{E}}_3 = \breve{E}_3(\mathbf{x}+i\mathbf{y})\exp(-i\omega t)$ exerts a moment of force $\tau = \breve{\mathbf{p}}\times\breve{\mathbf{E}}_3$ on rotating dipole moments of molecules of the polarized dielectric and makes a work. The power volume density of this work has the form

$$w = |\breve{\mathbf{P}}\times\breve{\mathbf{E}}_3|\omega\ \ [\text{W/m}^3],\quad \breve{\mathbf{P}} = (\breve{\varepsilon}-1)\varepsilon_0\breve{\mathbf{E}}_3,\quad \breve{\varepsilon} = \breve{k}^2 = k'^2 - k''^2 + 2ik'k'', \tag{2.1}$$

$\breve{\mathbf{P}}$ is the polarization vector. The calculation gives

$$w = \frac{\omega}{2}\Re\{\breve{P}_x\overline{E}_{3y} - \breve{P}_y\overline{E}_{3x}\} = \frac{\omega\varepsilon_0}{2}\Re\{(\breve{\varepsilon}-1)(\breve{E}_{3x}\overline{E}_{3y} - \breve{E}_{3y}\overline{E}_{3x})\} = \frac{\omega\varepsilon_0}{2}\exp(-2kk''z)\Re\{(\breve{\varepsilon}-1)(-i-i)\}|\breve{E}_3|^2$$

$$= \omega\varepsilon_0\exp(-2kk''z)\Im(\breve{\varepsilon}-1)|\breve{E}_3|^2 = \omega\varepsilon_0\exp(-2kk''z)2k'k''|\breve{E}_3|^2. \tag{2.2}$$

The energy flux density, which comes on the surface of dielectric from the waves, can be obtained by an integration of the power volume density (2.2) over z

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$$\Pi = \int_0^\infty w dz = \omega\varepsilon_0 \int_0^\infty \exp(-2kk''z)2k'k''\left|\breve{E}_3\right|^2 dz = \omega\varepsilon_0 \frac{k'}{k}\left|\breve{E}_3\right|^2 = \varepsilon_0 c E_1^2 \frac{4k'}{(1+k')^2 + k''^2}. \tag{2.3}$$

This expression is coincident with (1.4).

But we must recognize that the volume density of moment of force[2] $\tau_{\sim} = \breve{\mathbf{P}} \times \breve{\mathbf{E}}_3$, which supply with energy inside the dielectric, in the same time, is a volume flux density of angular momentum, which comes inside the dielectric. This volume density of moment of force $\breve{\mathbf{P}} \times \breve{\mathbf{E}}_3$ produces specific mechanical stresses in the dielectric [2]. And, as a volume flux density of angular momentum, it gives angular momentum flux density, which comes on the dielectric surface from the waves, when it is integrated over z.

$$\Upsilon = \int_0^\infty \left|\breve{\mathbf{P}} \times \breve{\mathbf{E}}_3\right| dz = \frac{1}{\omega}\int_0^\infty w dz = \frac{\Pi}{\omega} = \frac{\varepsilon_0 c E_1^2}{\omega}\frac{4k'}{(1+k')^2 + k''^2} \quad \left[\frac{\text{J s}}{\text{m}^2\text{s}}\right]. \tag{2.4}$$

The results of this Section were first published in paper [3].

Now our task is to make sure that electromagnetic waves (1.1), (1.2) contain angular momentum flux density (2.4).

3. Calculation of an angular momentum flux density, which is contained in electromagnetic waves

By the account, angular momentum (2.4) is absorbed under every square meter of the dielectric surface per second, one can concludes that the angular momentum is brought to the surface by waves (1.1), (1.2). To calculate this bringing angular momentum flux, it is natural to use the canonical spin tensor [4,5]

$$\Upsilon^{\lambda\mu\nu} = -2A^{[\lambda}F^{\mu]\nu}, \tag{3.1}$$

here $F_{\mu\lambda}$ is the electromagnetic field tensor, and A^λ is the magnetic vector potential.

Angular momentum flux density, which is directed along z-axis at xy surface, is given by the component

$$\Upsilon^{xyz} = -2A^{[x}F^{y]z} = A_x H_x + A_y H_y \ \text{[J/m}^2\text{]}. \tag{3.2}$$

Note that the lowering of the spatial index of the vector potential is related to the change of the sign in the view of the metric signature $(+---)$. Since for a monochromatic field $A_k = -\int E_k dt = -iE_k/\omega$, densitiy (3.2) can be expressed through the electromagnetic field:

$$\Upsilon^{xyz} = (-iE_x H_x - iE_y H_y)/\omega. \tag{3.3}$$

In our case, we have for the incident and reflected waves, similarly to (1.4)

$$\begin{aligned}\Upsilon^{xyz} &= \Re\{-iE_{1x}\overline{H}_{1x} - iE_{1y}\overline{H}_{1y} - i\breve{E}_{2x}\overline{H}_{2x} - i\breve{E}_{2y}\overline{H}_{2y}\}/2\omega \\ &= \frac{\varepsilon_0 c}{\omega}\left(E_1^2 - \left|\breve{E}_2\right|^2\right) = \frac{\varepsilon_0 c E_1^2}{\omega}\frac{4k'}{(1+k')^2 + k''^2}.\end{aligned} \tag{3.4}$$

Expressions (2.4) and (3.4) are coincident. This result was also presented in paper [3]. The observable equality $\Upsilon = \Pi/\omega$ is a consequence of the relation between photon spin $\hbar$ and photon energy $\hbar\omega$.

6. Conclusion

The given calculations show that spin occurs to be the same natural property of a plane electromagnetic wave, as energy and momentum. Recognizing the existence of photons with momentum, energy and spin in a plane electromagnetic wave, it is strange to deny the existence of spin in such a wave, as is done in modern electrodynamics.

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[2] We mark pseudo densities by index *tilda*. The volume density of moment of force $\tau_{\sim}$ .is a pseudo density, as opposed to the moment of force τ..

I am eternally grateful to Professor Robert Romer, having courageously published my question: "Does a plane wave really not carry spin?” [6]

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