

On the Navier–Stokes equations

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The problem on the existence and smoothness of the Navier–Stokes equations is considered.

1. Problem description

The Navier–Stokes equations are thought to govern the motion of a viscous incompressible fluid in \mathbb{R}^3 , see Batchelor 1967. Let $\mathbf{u} = \mathbf{u}(\mathbf{x}, t) \in \mathbb{R}^3$, $p = p(\mathbf{x}, t) \in \mathbb{R}$, and $\mathbf{f} = \mathbf{f}(\mathbf{x}, t) \in \mathbb{R}^3$ be the velocity, pressure, and given externally applied force respectively, each dependent on position $\mathbf{x} \in \mathbb{R}^3$ and time $t \geq 0$. The fluid is assumed to be incompressible with constant viscosity $\nu > 0$ and to fill all of \mathbb{R}^3 . The Navier–Stokes equations can then be written as

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = \nu \nabla^2 \mathbf{u} - \nabla p + \mathbf{f}, \quad (1)$$

$$\nabla \cdot \mathbf{u} = 0 \quad (2)$$

with initial condition

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0 \quad (3)$$

where $\mathbf{u}_0 = \mathbf{u}_0(\mathbf{x}) \in \mathbb{R}^3$. In these equations ∇ is the gradient operator and ∇^2 is the Laplacian operator. When $\nu = 0$, equations (1), (2), (3) are called the Euler equations. Solutions of (1), (2), (3) are to be found with

$$\mathbf{u}_0(\mathbf{x} + e_j) = \mathbf{u}_0(\mathbf{x}), \quad \mathbf{f}(\mathbf{x} + e_j, t) = \mathbf{f}(\mathbf{x}, t) \quad \text{for } 1 \leq j \leq 3 \quad (4)$$

where $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$. The initial condition \mathbf{u}_0 is a given C^∞ divergence-free vector field on \mathbb{R}^3 and

$$|\partial_{\mathbf{x}}^\alpha \partial_t^\beta \mathbf{f}| \leq C_{\alpha\beta\gamma} (1 + |t|)^{-\gamma} \quad \text{on } \mathbb{R}^3 \times [0, \infty) \quad \text{for any } \alpha, \beta, \gamma. \quad (5)$$

A solution of (1), (2), (3) would then be accepted to be physically reasonable if

$$\mathbf{u}(\mathbf{x} + e_j, t) = \mathbf{u}(\mathbf{x}, t), \quad p(\mathbf{x} + e_j, t) = p(\mathbf{x}, t) \quad \text{on } \mathbb{R}^3 \times [0, \infty) \quad \text{for } 1 \leq j \leq 3 \quad (6)$$

and

$$\mathbf{u}, p \in C^\infty(\mathbb{R}^3 \times [0, \infty)). \quad (7)$$

I consider a proof of the following statement (D), see Fefferman 2000.

(D) Breakdown of Navier–Stokes Solutions on $\mathbb{R}^3/\mathbb{Z}^3$.

Take $\nu > 0$. Then there exist a smooth, divergence-free vector field \mathbf{u}_0 on \mathbb{R}^3 and a smooth \mathbf{f} on $\mathbb{R}^3 \times [0, \infty)$, satisfying (4), (5), for which there exist no solutions (\mathbf{u}, p) of (1), (2), (3), (6), (7) on $\mathbb{R}^3 \times [0, \infty)$.

2. Proof of statement (D)

Herein I take $\mathbf{f} = \mathbf{0}$. I seek the approximation of the form

$$\mathbf{u} = \sum_{\mathbf{L}=-1}^1 \sum_{l=0}^n \frac{\partial^l \mathbf{u}_{\mathbf{L}}}{\partial t^l} \Big|_{t=0} \frac{t^l}{l!} e^{ik\mathbf{L}\cdot\mathbf{x}}, \quad (8)$$

$$p = \sum_{\mathbf{L}=-1}^1 \sum_{l=0}^n \frac{\partial^l p_{\mathbf{L}}}{\partial t^l} \Big|_{t=0} \frac{t^l}{l!} e^{ik\mathbf{L}\cdot\mathbf{x}} \quad (9)$$

to the solution of (1), (2), (3), (4), (5), (6) in light of Theorem 1 and Theorem 2 in the Appendix. Here

$\mathbf{u}_{\mathbf{L}} = \mathbf{u}_{\mathbf{L}}(t)$, $p_{\mathbf{L}} = p_{\mathbf{L}}(t)$, $k = 2\pi$, and $\sum_{\mathbf{L}=-H}^H$ denotes the sum over all $\mathbf{L} \in \mathbb{Z}^3$ with $-H \leq \mathbf{L}_j \leq H$, $1 \leq j \leq 3$.

Herein the smooth divergence-free initial condition \mathbf{u}_0 on \mathbb{R}^3 is chosen to be

$$\mathbf{u}_0 = \sum_{\mathbf{L}=-1}^1 \mathbf{L} \times (\mathbf{L} \times \mathbf{1}) a_{\mathbf{L}} \delta_{|\mathbf{L}|, \sqrt{3}} e^{ik\mathbf{L}\cdot\mathbf{x}} \quad (10)$$

where $\mathbf{1} = (1, 1, 1)$, $\delta_{i,j}$ is the Kronecker delta defined by

$$\delta_{i,j} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}, \quad (11)$$

and $a_{\mathbf{L}}$ are constants that are chosen such that $\mathbf{u}_0 \in \mathbb{R}^3$.

Method 1

Let

$$\mathbf{u} = \sum_{l=0}^n \frac{\partial^l \mathbf{u}}{\partial t^l} \Big|_{t=0} \frac{t^l}{l!}, \quad (12)$$

$$p = \sum_{l=0}^n \frac{\partial^l p}{\partial t^l} \Big|_{t=0} \frac{t^l}{l!}. \quad (13)$$

Substituting (12), (13) into (1) and equating like powers of t in accordance with Theorem 1 yields

$$\frac{\partial^{l+1} \mathbf{u}}{\partial t^{l+1}} \Big|_{t=0} + \sum_{m=0}^l \left(\frac{\partial^{l-m} \mathbf{u}}{\partial t^{l-m}} \Big|_{t=0} \cdot \nabla \right) \frac{\partial^m \mathbf{u}}{\partial t^m} \Big|_{t=0} \binom{l}{m} = \nu \nabla^2 \frac{\partial^l \mathbf{u}}{\partial t^l} \Big|_{t=0} - \nabla \frac{\partial^l p}{\partial t^l} \Big|_{t=0}. \quad (14)$$

Substituting (12) into (2) and equating like powers of t in accordance with Theorem 1 yields

$$\nabla \cdot \frac{\partial^l \mathbf{u}}{\partial t^l} \Big|_{t=0} = 0. \quad (15)$$

Applying $\nabla \times \nabla \times$ to (14) and using the identities

$$\nabla \times \nabla \times \mathbf{a} = \nabla(\nabla \cdot \mathbf{a}) - \nabla^2 \mathbf{a}, \quad (16)$$

$$\nabla \times \nabla a = \mathbf{0} \quad (17)$$

along with (15) gives

$$\nabla^2 \frac{\partial^{l+1} \mathbf{u}}{\partial t^{l+1}} \Big|_{t=0} = \nabla \times \nabla \times \sum_{m=0}^l \left(\frac{\partial^{l-m} \mathbf{u}}{\partial t^{l-m}} \Big|_{t=0} \cdot \nabla \right) \frac{\partial^m \mathbf{u}}{\partial t^m} \Big|_{t=0} \binom{l}{m} + \nu \nabla^4 \frac{\partial^l \mathbf{u}}{\partial t^l} \Big|_{t=0}. \quad (18)$$

Applying the inverse Laplacian ∇^{-2} to (18) gives

$$\frac{\partial^{l+1} \mathbf{u}}{\partial t^{l+1}} \Big|_{t=0} = \nabla^{-2} \nabla \times \nabla \times \sum_{m=0}^l \left(\frac{\partial^{l-m} \mathbf{u}}{\partial t^{l-m}} \Big|_{t=0} \cdot \nabla \right) \frac{\partial^m \mathbf{u}}{\partial t^m} \Big|_{t=0} \binom{l}{m} + \nu \nabla^2 \frac{\partial^l \mathbf{u}}{\partial t^l} \Big|_{t=0} + \Phi_l \quad (19)$$

where Φ_l must satisfy the Laplace equation

$$\nabla^2 \Phi_l = \mathbf{0}. \quad (20)$$

The required solution to (20) is $\Phi_l = \mathbf{0}$ in light of (4), (6). Equation (19) is then solved for $\frac{\partial^{l+1} \mathbf{u}}{\partial t^{l+1}} \Big|_{t=0}$ where $l = 0, 1, \dots, n-1$. Applying $\nabla \cdot$ to (14) and noting (15) yields

$$\nabla^2 \frac{\partial^l p}{\partial t^l} \Big|_{t=0} = -\nabla \cdot \sum_{m=0}^l \left(\frac{\partial^{l-m} \mathbf{u}}{\partial t^{l-m}} \Big|_{t=0} \cdot \nabla \right) \frac{\partial^m \mathbf{u}}{\partial t^m} \Big|_{t=0} \binom{l}{m}. \quad (21)$$

Applying ∇^{-2} to (21) gives

$$\frac{\partial^l p}{\partial t^l} \Big|_{t=0} = -\nabla^{-2} \nabla \cdot \sum_{m=0}^l \left(\frac{\partial^{l-m} \mathbf{u}}{\partial t^{l-m}} \Big|_{t=0} \cdot \nabla \right) \frac{\partial^m \mathbf{u}}{\partial t^m} \Big|_{t=0} \binom{l}{m} + \psi_l \quad (22)$$

where

$$\nabla^2 \psi_l = 0. \quad (23)$$

Arbitrary constant $\psi_l \in \mathbb{R}$ is the solution to (23) in light of (4), (6). Equation (22) is then solved for $\frac{\partial^l p}{\partial t^l} \Big|_{t=0}$ where $l = 0, 1, \dots, n$. After truncating (12), (13) in their modes, expressions for (8), (9) from Method 1 are then known in terms of given functions. Note that for the Fourier series

$$\mathbf{g} = \sum_{\mathbf{L} \neq \mathbf{0}} \mathbf{g}_{\mathbf{L}} e^{i\mathbf{k}_{\mathbf{L}} \cdot \mathbf{x}} \quad (24)$$

where $\sum_{\mathbf{L} \neq \mathbf{0}}$ denotes the sum over all $\mathbf{L} \in \mathbb{Z}^3$ with $\mathbf{L} \neq \mathbf{0}$, the ∇^{-2} operator is defined herein as

$$\nabla^{-2} \sum_{\mathbf{L} \neq \mathbf{0}} \mathbf{g}_{\mathbf{L}} e^{i\mathbf{k}_{\mathbf{L}} \cdot \mathbf{x}} = \sum_{\mathbf{L} \neq \mathbf{0}} \frac{\mathbf{g}_{\mathbf{L}} e^{i\mathbf{k}_{\mathbf{L}} \cdot \mathbf{x}}}{-k^2 |\mathbf{L}|^2}. \quad (25)$$

In Method 1 the assumption of smoothness is only on \mathbf{u}_0 .

Method 2

Let

$$\mathbf{u} = \sum_{\mathbf{L}=-1}^1 \mathbf{u}_{\mathbf{L}} e^{ik\mathbf{L}\cdot\mathbf{x}}, \quad (26)$$

$$p = \sum_{\mathbf{L}=-1}^1 p_{\mathbf{L}} e^{ik\mathbf{L}\cdot\mathbf{x}}. \quad (27)$$

Substituting (26), (27) into (1) and equating like powers of e in accordance with Theorem 2 yields

$$\frac{\partial \mathbf{u}_{\mathbf{L}}}{\partial t} + \sum_{\mathbf{M}} (\mathbf{u}_{\mathbf{L}-\mathbf{M}} \cdot ik\mathbf{M}) \mathbf{u}_{\mathbf{M}} = -\nu k^2 |\mathbf{L}|^2 \mathbf{u}_{\mathbf{L}} - ik\mathbf{L} p_{\mathbf{L}}. \quad (28)$$

Substituting (26) into (2) and equating like powers of e in accordance with Theorem 2 yields

$$\mathbf{L} \cdot \mathbf{u}_{\mathbf{L}} = 0. \quad (29)$$

Applying $\mathbf{L} \times \mathbf{L} \times$ to (28) and noting the vector identity

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c} \cdot \mathbf{a}) \mathbf{b} - (\mathbf{b} \cdot \mathbf{a}) \mathbf{c} \quad (30)$$

along with (29) yields

$$|\mathbf{L}|^2 \frac{\partial \mathbf{u}_{\mathbf{L}}}{\partial t} = \sum_{\mathbf{M}} \mathbf{L} \times (\mathbf{L} \times (\mathbf{u}_{\mathbf{L}-\mathbf{M}} \cdot ik\mathbf{M}) \mathbf{u}_{\mathbf{M}}) - \nu k^2 |\mathbf{L}|^4 \mathbf{u}_{\mathbf{L}}. \quad (31)$$

Equation (31) implies

$$\frac{\partial \mathbf{u}_{\mathbf{L}}}{\partial t} = \sum_{\mathbf{M}} \hat{\mathbf{L}} \times (\hat{\mathbf{L}} \times (\mathbf{u}_{\mathbf{L}-\mathbf{M}} \cdot ik\mathbf{M}) \mathbf{u}_{\mathbf{M}}) - \nu k^2 |\mathbf{L}|^2 \mathbf{u}_{\mathbf{L}} \quad (32)$$

where the right hand side of (32) is $\mathbf{0}$ when $\mathbf{L} = \mathbf{0}$ and $\hat{\mathbf{L}} = \mathbf{L}/|\mathbf{L}|$ is the unit vector in the direction of \mathbf{L} .

Applying $\mathbf{L} \cdot$ to (28) and noting (29) gives

$$ik|\mathbf{L}|^2 p_{\mathbf{L}} = - \sum_{\mathbf{M}} (\mathbf{u}_{\mathbf{L}-\mathbf{M}} \cdot ik\mathbf{M}) (\mathbf{u}_{\mathbf{M}} \cdot \mathbf{L}) \quad (33)$$

implying that

$$p_{\mathbf{L}} = - \sum_{\mathbf{M}} (\mathbf{u}_{\mathbf{L}-\mathbf{M}} \cdot \hat{\mathbf{L}}) (\mathbf{u}_{\mathbf{M}} \cdot \hat{\mathbf{L}}) \quad (34)$$

where $p_{\mathbf{0}} \in \mathbb{R}$ is an arbitrary function of t . Let

$$\mathbf{u}_{\mathbf{L}} = \sum_{l=0}^n \frac{\partial^l \mathbf{u}_{\mathbf{L}}}{\partial t^l} \Big|_{t=0} \frac{t^l}{l!}, \quad (35)$$

$$p_{\mathbf{L}} = \sum_{l=0}^n \frac{\partial^l p_{\mathbf{L}}}{\partial t^l} \Big|_{t=0} \frac{t^l}{l!}. \quad (36)$$

Substituting (35) into (32) and equating like powers of t in accordance with Theorem 1 yields

$$\frac{\partial^{l+1} \mathbf{u}_{\mathbf{L}}}{\partial t^{l+1}} \Big|_{t=0} = \sum_{m=0}^l \sum_{\mathbf{M}} \hat{\mathbf{L}} \times (\hat{\mathbf{L}} \times (\frac{\partial^{l-m} \mathbf{u}_{\mathbf{L}-\mathbf{M}}}{\partial t^{l-m}} \Big|_{t=0} \cdot ik\mathbf{M}) \frac{\partial^m \mathbf{u}_{\mathbf{M}}}{\partial t^m} \Big|_{t=0}) \binom{l}{m} - \nu k^2 |\mathbf{L}|^2 \frac{\partial^l \mathbf{u}_{\mathbf{L}}}{\partial t^l} \Big|_{t=0}. \quad (37)$$

Equation (37) is then solved for $\frac{\partial^{l+1}\mathbf{u}_L}{\partial t^{l+1}}|_{t=0}$ where $l = 0, 1, \dots, n-1$ and $-1 \leq \mathbf{L}_j \leq 1, 1 \leq j \leq 3$. Substituting (35), (36) into (34) and equating like powers of t in accordance with Theorem 1 yields

$$\frac{\partial^l p_L}{\partial t^l}|_{t=0} = - \sum_{m=0}^l \sum_{\mathbf{M}} \left(\frac{\partial^{l-m}\mathbf{u}_{L-\mathbf{M}}}{\partial t^{l-m}}|_{t=0} \cdot \hat{\mathbf{L}} \right) \left(\frac{\partial^m \mathbf{u}_M}{\partial t^m}|_{t=0} \cdot \hat{\mathbf{L}} \right) \binom{l}{m}. \quad (38)$$

Equation (38) is then solved for $\frac{\partial^l p_L}{\partial t^l}|_{t=0}$ where $l = 0, 1, \dots, n$ and $-1 \leq \mathbf{L}_j \leq 1, 1 \leq j \leq 3$. Expressions for (8), (9) from Method 2 are then known in terms of given functions.

With $n = 2$, I found that the approximation (8) found from Method 1 is different to the approximation (8) found from Method 2. The difference occurs at $O(t^2)$. Because of this nonuniqueness at least one of the assumptions used was invalid. The only assumptions used are those required for use of Theorem 1 and Theorem 2. Therefore the only way statement (D) could not be true is if the smoothness of \mathbf{u} can break down at an $\mathbf{x} \in \mathbb{R}^3$ where $t \in \mathbb{C} \setminus \{0\}$ but with $t \neq 0$.

It is found that $(\mathbf{u}(\mathbf{x} - \boldsymbol{\Omega}t, t) + \boldsymbol{\Omega}, p(\mathbf{x} - \boldsymbol{\Omega}t, t))$ is a solution to (1), (2), (3), (4), (5), (6) if $(\mathbf{u}(\mathbf{x}, t), p(\mathbf{x}, t))$ is a solution to (1), (2), (3), (4), (5), (6) where $\boldsymbol{\Omega} \in \mathbb{R}^3$ is a constant. If there exists an $\mathbf{x} = \boldsymbol{\Xi}(t) \in \mathbb{R}^3$ at which the smoothness of $\mathbf{u}(\mathbf{x}, t)$ breaks down where $t \in \mathbb{C} \setminus \{0\}$ then the smoothness of $\mathbf{u}(\mathbf{x} - \boldsymbol{\Omega}t, t) + \boldsymbol{\Omega}$ breaks down at an $\mathbf{x} = \boldsymbol{\Theta}(t) \in \mathbb{R}^3$ with $t \in \mathbb{C} \setminus \{0\}$. It is possible to write $\boldsymbol{\Theta}(t) - \boldsymbol{\Omega}t = \boldsymbol{\Xi}(t)$ and therefore the smoothness of \mathbf{u} can then break down at an $\mathbf{x} \in \mathbb{R}^3$ where $t \in \mathbb{R} \setminus \{0\}$.

For $\nu = 0$, it is found that $(\zeta\mathbf{u}(\mathbf{x}, \zeta t), \zeta^2 p(\mathbf{x}, \zeta t))$ is a solution to (1), (2), (3), (4), (5), (6) if $(\mathbf{u}(\mathbf{x}, t), p(\mathbf{x}, t))$ is a solution to (1), (2), (3), (4), (5), (6) where $\zeta \in \mathbb{R}$ is a constant, so if the smoothness of \mathbf{u} breaks down at $t < 0$ where $\mathbf{u}_0 = \mathbf{U}_0 \in \mathbb{R}^3$ then the smoothness of \mathbf{u} breaks down at $t > 0$ where $\mathbf{u}_0 = -\mathbf{U}_0 \in \mathbb{R}^3$. Therefore statement (D) is true when $\nu > 0$ is replaced with $\nu = 0$.

For $\nu > 0$, when applying Method 1 for $n = 2$ and Method 2 for all $n \in \mathbb{N}$, it is found that the governing equation for \mathbf{u} is effectively

$$\frac{\partial \mathbf{u}}{\partial t} = \nabla^{-2} \nabla \times \nabla \times ((\mathbf{u} \cdot \nabla) \mathbf{u}) + \nu \lambda \mathbf{u} \quad (39)$$

where $\lambda = -3k^2$. Equation (39) implies

$$\frac{\partial}{\partial t} (\mathbf{u} e^{-\nu \lambda t}) = \nabla^{-2} \nabla \times \nabla \times ((\mathbf{u} \cdot \nabla) \mathbf{u}) e^{-\nu \lambda t}. \quad (40)$$

Then a change of variables

$$\tau = e^{\nu \lambda t} - 1, \quad (41)$$

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{v}(\mathbf{x}, \tau) \frac{\partial \tau}{\partial t} \quad (42)$$

yields

$$\frac{\partial \mathbf{v}}{\partial \tau} = \nabla^{-2} \nabla \times \nabla \times ((\mathbf{v} \cdot \nabla) \mathbf{v}). \quad (43)$$

Equation (2) becomes

$$\nabla \cdot \mathbf{v} = 0 \quad (44)$$

and the initial condition (3) becomes

$$\mathbf{v}(\mathbf{x}, 0) = \frac{\mathbf{u}_0}{v\lambda}. \quad (45)$$

Equations (43), (44), (45) define an Euler problem. If the smoothness of \mathbf{v} breaks down at an $\mathbf{x} \in \mathbb{R}^3$ with $\tau \in \mathbb{R} \setminus \{0\}$, then the smoothness of \mathbf{u} can break down at an $\mathbf{x} \in \mathbb{R}^3$ with $t > 0$. Therefore statement (D) is true. \square

Appendix

Theorem 1

Providing that the Maclaurin series

$$\bar{\mathbf{A}} = \sum_{l=0}^n \frac{\partial^l \mathbf{A}}{\partial t^l} \Big|_{t=0} \frac{t^l}{l!} = \sum_{l=0}^n \frac{\partial^l \bar{\mathbf{A}}}{\partial t^l} \Big|_{t=0} \frac{t^l}{l!} \quad (46)$$

of the exact general solution to a Q^{th} order partial differential equation

$$\frac{\partial^Q \mathbf{A}}{\partial t^Q} = \Psi \quad (47)$$

exists, it will solve the coefficients of t^l for all $l = 0, 1, \dots, n - Q$ in (47) with $\mathbf{A} = \bar{\mathbf{A}}$ provided $\Psi|_{\mathbf{A}=\bar{\mathbf{A}}}$ is expandable in Maclaurin series as

$$\Psi|_{\mathbf{A}=\bar{\mathbf{A}}} = \sum_{l=0}^m \frac{\partial^l \Psi|_{\mathbf{A}=\bar{\mathbf{A}}}}{\partial t^l} \Big|_{t=0} \frac{t^l}{l!} \quad (48)$$

where $m \geq n$. Here all of the partial derivatives of \mathbf{A} with respect to t are defined at $t = 0$.

Proof of Theorem 1

Since the Maclaurin series of \mathbf{A} exists and all of the partial derivatives of \mathbf{A} with respect to t are defined at $t = 0$, one can integrate (47) Q times and then substitute the result into (46) to find

$$\bar{\mathbf{A}} = \sum_{l=0}^n \frac{\partial^{l-Q} \Psi}{\partial t^{l-Q}} \Big|_{t=0} \frac{t^l}{l!} = \sum_{l=0}^n \frac{\partial^l \int_Q \Psi dt|_{\mathbf{A}=\bar{\mathbf{A}}}}{\partial t^l} \Big|_{t=0} \frac{t^l}{l!} \quad (49)$$

where $\int_Q \Psi dt$ denotes the Q^{th} integral of Ψ with respect to t . Substituting $\mathbf{A} = \bar{\mathbf{A}}$ into the residual \mathbf{r} of (47) then gives

$$\mathbf{r} = \sum_{l=0}^n \frac{\partial^{l-Q} \Psi|_{\mathbf{A}=\bar{\mathbf{A}}}}{\partial t^{l-Q}} \Big|_{t=0} \frac{t^{l-Q}}{(l-Q)!} - \sum_{l=0}^m \frac{\partial^l \Psi|_{\mathbf{A}=\bar{\mathbf{A}}}}{\partial t^l} \Big|_{t=0} \frac{t^l}{l!} \quad (50)$$

providing $\Psi|_{\mathbf{A}=\bar{\mathbf{A}}}$ is expanded in Maclaurin series as in (48). Collecting like powers of t in (50) yields

$$\mathbf{r} = \sum_{l=0}^{n-Q} \frac{\partial^l \Psi|_{\mathbf{A}=\bar{\mathbf{A}}}}{\partial t^l} \Big|_{t=0} \frac{t^l}{l!} - \sum_{l=0}^m \frac{\partial^l \Psi|_{\mathbf{A}=\bar{\mathbf{A}}}}{\partial t^l} \Big|_{t=0} \frac{t^l}{l!} \quad (51)$$

which shows that Theorem 1 is true. \square

Theorem 2

Providing that the Fourier series

$$\tilde{\mathbf{A}} = \sum_{\mathbf{L}=-N}^N P(\mathbf{A}, e^{ik\mathbf{L}\cdot\mathbf{x}})e^{ik\mathbf{L}\cdot\mathbf{x}} = \sum_{\mathbf{L}=-N}^N P(\tilde{\mathbf{A}}, e^{ik\mathbf{L}\cdot\mathbf{x}})e^{ik\mathbf{L}\cdot\mathbf{x}} \quad (52)$$

of the exact general solution to a Q^{th} order partial differential equation

$$\frac{\partial^Q \mathbf{A}}{\partial t^Q} = \Psi \quad (53)$$

exists, it will solve the coefficients of $e^{ik\mathbf{L}\cdot\mathbf{x}}$ for all $-N \leq \mathbf{L}_j \leq N, 1 \leq j \leq 3$ in (53) with $\mathbf{A} = \tilde{\mathbf{A}}$ provided $\Psi|_{\mathbf{A}=\tilde{\mathbf{A}}}$ is expandable in Fourier series as

$$\Psi|_{\mathbf{A}=\tilde{\mathbf{A}}} = \sum_{\mathbf{L}=-M}^M P(\Psi|_{\mathbf{A}=\tilde{\mathbf{A}}}, e^{ik\mathbf{L}\cdot\mathbf{x}})e^{ik\mathbf{L}\cdot\mathbf{x}} \quad (54)$$

where $M \geq N$. Here \mathbf{A} is spatially periodic and smooth for all $\mathbf{x} \in \mathbb{R}^3$, $k > 0$ is a constant, and $P(\mathbf{h}, e^{ik\mathbf{L}\cdot\mathbf{x}})$ denotes the projection of \mathbf{h} onto $e^{ik\mathbf{L}\cdot\mathbf{x}}$.

Proof of Theorem 2

Since the Fourier series of \mathbf{A} exists where \mathbf{A} is spatially periodic and smooth for all $\mathbf{x} \in \mathbb{R}^3$, one can integrate (53) Q times and then substitute the result into (52) to find

$$\tilde{\mathbf{A}} = \sum_{\mathbf{L}=-N}^N P\left(\int_Q \Psi dt, e^{ik\mathbf{L}\cdot\mathbf{x}}\right)e^{ik\mathbf{L}\cdot\mathbf{x}} = \sum_{\mathbf{L}=-N}^N P\left(\int_Q \Psi dt|_{\mathbf{A}=\tilde{\mathbf{A}}}, e^{ik\mathbf{L}\cdot\mathbf{x}}\right)e^{ik\mathbf{L}\cdot\mathbf{x}}. \quad (55)$$

Substituting $\mathbf{A} = \tilde{\mathbf{A}}$ into the residual \mathbf{r} of (53) then gives

$$\mathbf{r} = \frac{\partial^Q}{\partial t^Q} \sum_{\mathbf{L}=-N}^N P\left(\int_Q \Psi dt|_{\mathbf{A}=\tilde{\mathbf{A}}}, e^{ik\mathbf{L}\cdot\mathbf{x}}\right)e^{ik\mathbf{L}\cdot\mathbf{x}} - \sum_{\mathbf{L}=-M}^M P(\Psi|_{\mathbf{A}=\tilde{\mathbf{A}}}, e^{ik\mathbf{L}\cdot\mathbf{x}})e^{ik\mathbf{L}\cdot\mathbf{x}} \quad (56)$$

providing $\Psi|_{\mathbf{A}=\tilde{\mathbf{A}}}$ is expanded in Fourier series as in (54). Equation (56) can be written as

$$\mathbf{r} = \sum_{\mathbf{L}=-N}^N P(\Psi|_{\mathbf{A}=\tilde{\mathbf{A}}}, e^{ik\mathbf{L}\cdot\mathbf{x}})e^{ik\mathbf{L}\cdot\mathbf{x}} - \sum_{\mathbf{L}=-M}^M P(\Psi|_{\mathbf{A}=\tilde{\mathbf{A}}}, e^{ik\mathbf{L}\cdot\mathbf{x}})e^{ik\mathbf{L}\cdot\mathbf{x}} \quad (57)$$

which shows that Theorem 2 is true. \square

References

- Batchelor, G. K., 1967. *An introduction to fluid dynamics*. Cambridge University Press: Cambridge.
- Fefferman, C. L., 2000. Existence and smoothness of the Navier–Stokes equation. Clay Mathematics Institute: official problem description.