On the Navier–Stokes equations

Daniel Thomas Hayes

January 30, 2018

The problem on the existence and smoothness of the Navier–Stokes equations is solved.

1. Problem description

The Navier–Stokes equations are thought to govern the motion of a fluid in $\mathbb{R}^3$, see [1]. Let $\mathbf{u} = \mathbf{u}(\mathbf{x}, t) \in \mathbb{R}^3$, $p = p(\mathbf{x}, t) \in \mathbb{R}$, and $\mathbf{f} = \mathbf{f}(\mathbf{x}, t) \in \mathbb{R}^3$ be the velocity, pressure, and given externally applied force respectively, each dependent on position $\mathbf{x} \in \mathbb{R}^3$ and time $t \geqslant 0$. The fluid is here assumed to be incompressible with constant viscosity $\nu > 0$ and to fill all of $\mathbb{R}^3$. The Navier–Stokes equations can then be written as

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \nu \nabla^2 \mathbf{u} - \nabla p + \mathbf{f}, \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{2}$$

with initial condition

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0 \tag{3}$$

where $\mathbf{u}_0 = \mathbf{u}_0(\mathbf{x}) \in \mathbb{R}^3$. In these equations $\nabla = (\frac{\partial}{\partial \mathbf{x}_1}, \frac{\partial}{\partial \mathbf{x}_2}, \frac{\partial}{\partial \mathbf{x}_3})$ is the gradient operator and $\nabla^2 = \sum_{i=1}^{3} \frac{\partial^2}{\partial \mathbf{x}_i{}^2}$ is the Laplacian operator. When $\nu = 0$, equations (1), (2), (3) are called the Euler equations. Solutions of (1), (2), (3) are to be found with

$$\mathbf{u}_0(\mathbf{x} + e_j) = \mathbf{u}_0(\mathbf{x}), \quad \mathbf{f}(\mathbf{x} + e_j, t) = \mathbf{f}(\mathbf{x}, t) \;\text{ for }\; 1 \leqslant j \leqslant 3 \tag{4}$$

where $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$. The initial condition $\mathbf{u}_0$ is a given C^∞ divergence-free vector field on $\mathbb{R}^3$ and

$$|\partial_{\mathbf{x}}^{\alpha} \partial_t^{\beta} \mathbf{f}| \leqslant C_{\alpha\beta\gamma}(1 + |t|)^{-\gamma} \;\text{ on }\; \mathbb{R}^3 \times [0, \infty) \;\text{ for any }\; \alpha, \beta, \gamma. \tag{5}$$

A solution of (1), (2), (3) would then be accepted to be physically reasonable if

$$\mathbf{u}(\mathbf{x} + e_j, t) = \mathbf{u}(\mathbf{x}, t), \quad p(\mathbf{x} + e_j, t) = p(\mathbf{x}, t) \;\text{ on }\; \mathbb{R}^3 \times [0, \infty) \;\text{ for }\; 1 \leqslant j \leqslant 3 \tag{6}$$

and

$$\mathbf{u}, p \in C^\infty(\mathbb{R}^3 \times [0, \infty)). \tag{7}$$

I provide a proof of the following statement (D), see [2].

(D) Breakdown of Navier–Stokes Solutions on $\mathbb{R}^3/\mathbb{Z}^3$.

Take $\nu > 0$. Then there exist a smooth, divergence-free vector field $\mathbf{u}_0$ on $\mathbb{R}^3$ and a smooth $\mathbf{f}$ on $\mathbb{R}^3 \times [0, \infty)$, satisfying (4), (5), for which there exist no solutions $(\mathbf{u}, p)$ of (1), (2), (3), (6), (7) on $\mathbb{R}^3 \times [0, \infty)$.

2. Proof of statement (D)

Herein I take $\mathbf{f} = \mathbf{0}$. I seek an approximation of the form

$$\mathbf{u} = \sum_{\mathbf{L}=-\mathbf{1}}^{\mathbf{1}} \sum_{l=0}^{n} \frac{\partial^l \mathbf{u}_\mathbf{L}}{\partial t^l}|_{t=0} \frac{t^l}{l!} e^{ik\mathbf{L}\cdot\mathbf{x}}, \tag{8}$$

$$p = \sum_{\mathbf{L}=-\mathbf{1}}^{\mathbf{1}} \sum_{l=0}^{n-1} \frac{\partial^l p_\mathbf{L}}{\partial t^l}|_{t=0} \frac{t^l}{l!} e^{ik\mathbf{L}\cdot\mathbf{x}} \tag{9}$$

to the solution of (1), (2), (3), (4), (5), (6) in light of Theorem 1 and Theorem 2 in the Appendix. Here $\mathbf{u}_\mathbf{L} = \mathbf{u}_\mathbf{L}(t)$, $p_\mathbf{L} = p_\mathbf{L}(t)$, $i = \sqrt{-1}$, $k = 2\pi$, and $\sum_{\mathbf{L}=-\mathbf{H}}^{\mathbf{H}}$ denotes the sum over all $\mathbf{L} \in \mathbb{Z}^3$ with $-H \leqslant \mathbf{L}_j \leqslant H$. Herein the smooth divergence-free initial condition $\mathbf{u}_0$ on $\mathbb{R}^3$ is chosen to be

$$\mathbf{u}_0 = \sum_{\mathbf{L}=-\mathbf{1}}^{\mathbf{1}} \mathbf{L} \times (\mathbf{L} \times \mathbf{1}) a_\mathbf{L} \delta_{|\mathbf{L}|,\sqrt{3}} e^{ik\mathbf{L}\cdot\mathbf{x}} \tag{10}$$

where $\mathbf{1} = (1, 1, 1)$, $\delta_{i,j}$ is the Kronecker delta defined by

$$\delta_{i,j} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}, \tag{11}$$

and $a_\mathbf{L}$ are constants that are chosen such that $\mathbf{u}_0 \in \mathbb{R}^3$.

Method 1

Let

$$\mathbf{u} = \sum_{l=0}^{n} \frac{\partial^l \mathbf{u}}{\partial t^l}|_{t=0} \frac{t^l}{l!}, \tag{12}$$

$$p = \sum_{l=0}^{n-1} \frac{\partial^l p}{\partial t^l}|_{t=0} \frac{t^l}{l!}. \tag{13}$$

Substituting (12), (13) into (1) and equating like powers of t in accordance with Theorem 1 yields

$$\frac{\partial^{l+1} \mathbf{u}}{\partial t^{l+1}}|_{t=0} + \sum_{m=0}^{l} (\frac{\partial^{l-m} \mathbf{u}}{\partial t^{l-m}}|_{t=0} \cdot \nabla) \frac{\partial^m \mathbf{u}}{\partial t^m}|_{t=0} \binom{l}{m} = \nu \nabla^2 \frac{\partial^l \mathbf{u}}{\partial t^l}|_{t=0} - \nabla \frac{\partial^l p}{\partial t^l}|_{t=0} \tag{14}$$

where

$$\binom{l}{m} = \frac{l!}{m!(l-m)!}. \tag{15}$$

Substituting (12) into (2) and equating like powers of t in accordance with Theorem 1 yields

$$\nabla \cdot \frac{\partial^l \mathbf{u}}{\partial t^l}|_{t=0} = 0. \tag{16}$$

Applying $\nabla \times \nabla \times$ to (14) and using the identities

$$\nabla \times \nabla \times \mathbf{a} = \nabla(\nabla \cdot \mathbf{a}) - \nabla^2 \mathbf{a}, \tag{17}$$

$$\nabla \times \nabla a = \mathbf{0} \tag{18}$$

along with (16) gives

$$\nabla^2 \frac{\partial^{l+1}\mathbf{u}}{\partial t^{l+1}}|_{t=0} = \nabla \times \nabla \times \sum_{m=0}^{l} (\frac{\partial^{l-m}\mathbf{u}}{\partial t^{l-m}}|_{t=0} \cdot \nabla)\frac{\partial^{m}\mathbf{u}}{\partial t^{m}}|_{t=0}\binom{l}{m} + \nu\nabla^4 \frac{\partial^{l}\mathbf{u}}{\partial t^{l}}|_{t=0}. \tag{19}$$

Applying the inverse Laplacian ∇^{-2} to (19) gives

$$\frac{\partial^{l+1}\mathbf{u}}{\partial t^{l+1}}|_{t=0} = \nabla^{-2}\nabla \times \nabla \times \sum_{m=0}^{l} (\frac{\partial^{l-m}\mathbf{u}}{\partial t^{l-m}}|_{t=0} \cdot \nabla)\frac{\partial^{m}\mathbf{u}}{\partial t^{m}}|_{t=0}\binom{l}{m} + \nu\nabla^2 \frac{\partial^{l}\mathbf{u}}{\partial t^{l}}|_{t=0} + \mathbf{\Phi}_l \tag{20}$$

where $\mathbf{\Phi}_l$ must satisfy the Laplace equation

$$\nabla^2 \mathbf{\Phi}_l = \mathbf{0}. \tag{21}$$

The required solution to (21) is $\mathbf{\Phi}_l = \mathbf{0}$ in light of (4), (6). Equation (20) is then solved for $\frac{\partial^{l+1}\mathbf{u}}{\partial t^{l+1}}|_{t=0}$ where $l = 0, 1, \ldots, n-1$. Applying $\nabla\cdot$ to (14) and noting (16) yields

$$\nabla^2 \frac{\partial^{l} p}{\partial t^{l}}|_{t=0} = -\nabla \cdot \sum_{m=0}^{l} (\frac{\partial^{l-m}\mathbf{u}}{\partial t^{l-m}}|_{t=0} \cdot \nabla)\frac{\partial^{m}\mathbf{u}}{\partial t^{m}}|_{t=0}\binom{l}{m}. \tag{22}$$

Applying ∇^{-2} to (22) gives

$$\frac{\partial^{l} p}{\partial t^{l}}|_{t=0} = -\nabla^{-2}\nabla \cdot \sum_{m=0}^{l} (\frac{\partial^{l-m}\mathbf{u}}{\partial t^{l-m}}|_{t=0} \cdot \nabla)\frac{\partial^{m}\mathbf{u}}{\partial t^{m}}|_{t=0}\binom{l}{m} + \psi_l \tag{23}$$

where

$$\nabla^2 \psi_l = 0. \tag{24}$$

Arbitrary constant $\psi_l \in \mathbb{R}$ is the solution to (24) in light of (4), (6). Equation (23) is then solved for $\frac{\partial^l p}{\partial t^l}|_{t=0}$ where $l = 0, 1, \ldots, n-1$. After truncating (12), (13) in their modes, expressions for (8), (9) from Method 1 are then known in terms of given functions.
Note that for the Fourier series

$$\mathbf{g} = \sum_{\mathbf{L} \neq \mathbf{0}} \mathbf{g_L} e^{ik\mathbf{L}\cdot\mathbf{x}} \tag{25}$$

where $\sum_{\mathbf{L}\neq\mathbf{0}}$ denotes the sum over all $\mathbf{L} \in \mathbb{Z}^3$ with $\mathbf{L} \neq \mathbf{0}$, the ∇^{-2} operator is defined herein as

$$\nabla^{-2} \sum_{\mathbf{L} \neq \mathbf{0}} \mathbf{g_L} e^{ik\mathbf{L}\cdot\mathbf{x}} = \sum_{\mathbf{L} \neq \mathbf{0}} \frac{\mathbf{g_L} e^{ik\mathbf{L}\cdot\mathbf{x}}}{-k^2|\mathbf{L}|^2}. \tag{26}$$

The following is the output from the Maple code in the Appendix where $\mathbf{u} = (u, v, w)$ and $\mathbf{x} = (x, y, z)$.

$$
\begin{aligned}
u \;=\;& -4a_{1,-1,-1}e^{ik(x-y-z)} - 2a_{-1,1,-1}e^{ik(-x+y-z)} - 2a_{1,1,-1}e^{ik(x+y-z)} - 2a_{-1,-1,1}e^{ik(-x-y+z)} - 2a_{1,-1,1}e^{ik(x-y+z)} \\
& -4a_{-1,1,1}e^{ik(-x+y+z)} + ((12a_{1,-1,-1}e^{ik(x-y-z)} + 6a_{-1,1,-1}e^{ik(-x+y-z)} + 6a_{1,1,-1}e^{ik(x+y-z)} + 6a_{-1,-1,1}e^{ik(-x-y+z)} \\
& +6a_{1,-1,1}e^{ik(x-y+z)} + 12a_{-1,1,1}e^{ik(-x+y+z)})\nu k^2 + 24i(a_{-1,-1,1}a_{1,-1,-1}e^{-2iky} + a_{-1,1,-1}a_{1,-1,-1}e^{-2ikz} \\
& -a_{-1,1,1}a_{1,1,-1}e^{2iky} - a_{-1,1,1}a_{1,-1,1}e^{2ikz})k)t + ((-18a_{1,-1,-1}e^{ik(x-y-z)} - 9a_{-1,1,-1}e^{ik(-x+y-z)} - 9a_{1,1,-1}e^{ik(x+y-z)} \\
& -9a_{-1,-1,1}e^{ik(-x-y+z)} - 9a_{1,-1,1}e^{ik(x-y+z)} - 18a_{-1,1,1}e^{ik(-x+y+z)})\nu^2k^4 + 120i(-e^{-2iky}a_{-1,-1,1}a_{1,-1,-1} \\
& -e^{-2ikz}a_{-1,1,-1}a_{1,-1,-1} + e^{2iky}a_{-1,1,1}a_{1,1,-1} + e^{2ikz}a_{-1,1,1}a_{1,-1,1})\nu k^3 + (48a_{-1,1,1}a_{1,-1,-1}a_{1,-1,1}e^{ik(x-y+z)} \\
& +\frac{144}{11}a_{-1,1,-1}a_{1,-1,-1}a_{1,1,-1}e^{ik(x+y-3z)} + 48a_{-1,1,1}a_{1,-1,-1}a_{1,1,-1}e^{ik(x+y-z)} + \frac{96}{11}a_{1,-1,-1}a_{1,-1,1}a_{1,1,-1}e^{ik(3x-y-z)} \\
& +\frac{96}{11}a_{-1,-1,1}a_{-1,1,-1}a_{-1,1,1}e^{ik(-3x+y+z)} + 48a_{-1,-1,1}a_{-1,1,1}a_{1,-1,-1}e^{ik(-x-y+z)} \\
& +\frac{144}{11}a_{-1,-1,1}a_{-1,1,1}a_{1,-1,1}e^{ik(-x-y+3z)} + 48a_{-1,1,-1}a_{-1,1,1}a_{1,-1,-1}e^{ik(-x+y-z)} \\
& +\frac{144}{11}a_{-1,1,-1}a_{-1,1,1}a_{1,1,-1}e^{ik(-x+3y-z)} + \frac{144}{11}a_{-1,-1,1}a_{1,-1,-1}a_{1,-1,1}e^{ik(x-3y+z)} \\
& +48a_{-1,1,1}(a_{-1,-1,1}a_{1,1,-1} + a_{-1,1,-1}a_{1,-1,1})e^{ik(-x+y+z)} + 48a^2_{-1,1,1}a_{1,-1,1}e^{ik(-x+y+3z)} \\
& +48a^2_{-1,1,1}a_{1,1,-1}e^{ik(-x+3y+z)} + 48a_{-1,-1,1}a^2_{1,-1,-1}e^{ik(x-3y-z)} + 48a_{-1,1,-1}a^2_{1,-1,-1}e^{ik(x-y-3z)} \\
& +48a_{1,-1,-1}(a_{-1,-1,1}a_{1,1,-1} + a_{-1,1,-1}a_{1,-1,1})e^{ik(x-y-z)} + 48a_{-1,1,1}a^2_{1,-1,1}e^{ik(x-y+3z)} + 48a_{-1,1,1}a^2_{1,1,-1}e^{ik(x+3y-z)} \\
& +48a^2_{-1,-1,1}a_{1,-1,-1}e^{ik(-x-3y+z)} + 48a^2_{-1,1,-1}a_{1,-1,-1}e^{ik(-x+y-3z)})k^2)t^2 + O(t^3),
\end{aligned}
\tag{27}
$$

$$
\begin{aligned}
v \;=\;& -2a_{1,-1,-1}e^{ik(x-y-z)} - 4a_{-1,1,-1}e^{ik(-x+y-z)} - 2a_{1,1,-1}e^{ik(x+y-z)} - 2a_{-1,-1,1}e^{ik(-x-y+z)} - 4a_{1,-1,1}e^{ik(x-y+z)} \\
& -2a_{-1,1,1}e^{ik(-x+y+z)} + ((6a_{1,-1,-1}e^{ik(x-y-z)} + 12a_{-1,1,-1}e^{ik(-x+y-z)} + 6a_{1,1,-1}e^{ik(x+y-z)} + 6a_{-1,-1,1}e^{ik(-x-y+z)} \\
& +12a_{1,-1,1}e^{ik(x-y+z)} + 6a_{-1,1,1}e^{ik(-x+y+z)})\nu k^2 + 24i(a_{-1,1,-1}a_{1,-1,-1}e^{-2ikz} - a_{-1,1,1}a_{1,-1,1}e^{2ikz} \\
& +a_{-1,-1,1}a_{-1,1,-1}e^{-2ikx} - a_{1,-1,1}a_{1,1,-1}e^{2ikx})k)t + ((-9a_{1,-1,-1}e^{ik(x-y-z)} - 18a_{-1,1,-1}e^{ik(-x+y-z)} \\
& -9a_{1,1,-1}e^{ik(x+y-z)} - 9a_{-1,-1,1}e^{ik(-x-y+z)} - 18a_{1,-1,1}e^{ik(x-y+z)} - 9a_{-1,1,1}e^{ik(-x+y+z)})\nu^2k^4 \\
& +120i(-e^{-2ikz}a_{-1,1,-1}a_{1,-1,-1} + e^{2ikz}a_{-1,1,1}a_{1,-1,1} - e^{-2ikx}a_{-1,-1,1}a_{-1,1,-1} + e^{2ikx}a_{1,-1,1}a_{1,1,-1})\nu k^3 \\
& +(48a_{-1,-1,1}a_{-1,1,-1}a_{1,-1,1}e^{ik(-x-y+z)} + 48a_{-1,1,-1}a_{-1,1,1}a_{1,-1,1}e^{ik(-x+y+z)} + 48a_{-1,1,-1}a_{1,-1,-1}a_{1,-1,1}e^{ik(x-y-z)} \\
& +48a_{-1,1,-1}a_{1,-1,1}a_{1,1,-1}e^{ik(x+y-z)} + 48a^2_{-1,-1,1}a_{-1,1,-1}e^{ik(-3x-y+z)} + 48a_{-1,-1,1}a^2_{-1,1,-1}e^{ik(-3x+y-z)} \\
& +48a_{-1,1,-1}(a_{-1,-1,1}a_{1,1,-1} + a_{-1,1,1}a_{1,-1,-1})e^{ik(-x+y-z)} + 48a_{1,-1,1}(a_{-1,-1,1}a_{1,1,-1} + a_{-1,1,1}a_{1,-1,-1})e^{ik(x-y+z)} \\
& +48a^2_{1,-1,1}a_{1,1,-1}e^{ik(3x-y+z)} + 48a_{1,-1,1}a^2_{1,1,-1}e^{ik(3x+y-z)} + \frac{144}{11}a_{-1,1,-1}a_{1,-1,-1}a_{1,1,-1}e^{ik(x+y-3z)} \\
& +\frac{144}{11}a_{1,-1,-1}a_{1,-1,1}a_{1,1,-1}e^{ik(3x-y-z)} + \frac{144}{11}a_{-1,-1,1}a_{-1,1,-1}a_{-1,1,1}e^{ik(-3x+y+z)} \\
& +\frac{144}{11}a_{-1,-1,1}a_{-1,1,1}a_{1,-1,1}e^{ik(-x-y+3z)} + \frac{96}{11}a_{-1,1,-1}a_{-1,1,1}a_{1,1,-1}e^{ik(-x+3y-z)} \\
& +\frac{96}{11}a_{-1,-1,1}a_{1,-1,-1}a_{1,-1,1}e^{ik(x-3y+z)} + 48a^2_{-1,1,1}a_{1,-1,1}e^{ik(-x+y+3z)} + 48a_{-1,1,-1}a^2_{1,-1,-1}e^{ik(x-y-3z)} \\
& +48a_{-1,1,1}a^2_{1,-1,1}e^{ik(x-y+3z)} + 48a^2_{-1,1,-1}a_{1,-1,-1}e^{ik(-x+y-3z)})k^2)t^2 + O(t^3),
\end{aligned}
\tag{28}
$$

$$
\begin{aligned}
w \;=\;& -2a_{1,-1,-1}e^{ik(x-y-z)} - 2a_{-1,1,-1}e^{ik(-x+y-z)} - 4a_{1,1,-1}e^{ik(x+y-z)} - 4a_{-1,-1,1}e^{ik(-x-y+z)} - 2a_{1,-1,1}e^{ik(x-y+z)} \\
& -2a_{-1,1,1}e^{ik(-x+y+z)} + ((6a_{1,-1,-1}e^{ik(x-y-z)} + 6a_{-1,1,-1}e^{ik(-x+y-z)} + 12a_{1,1,-1}e^{ik(x+y-z)} + 12a_{-1,-1,1}e^{ik(-x-y+z)} \\
& +6a_{1,-1,1}e^{ik(x-y+z)} + 6a_{-1,1,1}e^{ik(-x+y+z)})\nu k^2 + 24i(a_{-1,-1,1}a_{1,-1,-1}e^{-2iky} - a_{-1,1,1}a_{1,1,-1}e^{2iky} \\
& +a_{-1,-1,1}a_{-1,1,-1}e^{-2ikx} - a_{1,-1,1}a_{1,1,-1}e^{2ikx})k)t + ((-9a_{1,-1,-1}e^{ik(x-y-z)} - 9a_{-1,1,-1}e^{ik(-x+y-z)} \\
& -18a_{1,1,-1}e^{ik(x+y-z)} - 18a_{-1,-1,1}e^{ik(-x-y+z)} - 9a_{1,-1,1}e^{ik(x-y+z)} - 9a_{-1,1,1}e^{ik(-x+y+z)})\nu^2k^4 \\
& +120i(-e^{-2iky}a_{-1,-1,1}a_{1,-1,-1} + e^{2iky}a_{-1,1,1}a_{1,1,-1} - e^{-2ikx}a_{-1,-1,1}a_{-1,1,-1} + e^{2ikx}a_{1,-1,1}a_{1,1,-1})\nu k^3 \\
& +(48a^2_{-1,-1,1}a_{-1,1,-1}e^{ik(-3x-y+z)} + 48a_{-1,-1,1}a^2_{-1,1,-1}e^{ik(-3x+y-z)} + 48a^2_{1,-1,1}a_{1,1,-1}e^{ik(3x-y+z)} \\
& +48a_{1,-1,1}a^2_{1,1,-1}e^{ik(3x+y-z)} + 48a_{-1,-1,1}(a_{-1,1,-1}a_{1,-1,1} + a_{-1,1,1}a_{1,-1,-1})e^{ik(-x-y+z)} \\
& +48a_{1,1,-1}(a_{-1,1,-1}a_{1,-1,1} + a_{-1,1,1}a_{1,-1,-1})e^{ik(x+y-z)} + 48a_{-1,-1,1}a_{-1,1,-1}a_{1,1,-1}e^{ik(-x+y-z)} \\
& +\frac{96}{11}a_{-1,1,-1}a_{1,-1,-1}a_{1,1,-1}e^{ik(x+y-3z)} + \frac{144}{11}a_{1,-1,-1}a_{1,-1,1}a_{1,1,-1}e^{ik(3x-y-z)} + 48a_{-1,-1,1}a^2_{1,-1,-1}e^{ik(x-3y-z)} \\
& +\frac{144}{11}a_{-1,-1,1}a_{-1,1,-1}a_{-1,1,1}e^{ik(-3x+y+z)} + \frac{96}{11}a_{-1,-1,1}a_{-1,1,1}a_{1,-1,1}e^{ik(-x-y+3z)} + 48a_{-1,1,1}a^2_{1,1,-1}e^{ik(x+3y-z)} \\
& +\frac{144}{11}a_{-1,1,-1}a_{-1,1,1}a_{1,1,-1}e^{ik(-x+3y-z)} + \frac{144}{11}a_{-1,-1,1}a_{1,-1,-1}a_{1,-1,1}e^{ik(x-3y+z)} + 48a^2_{-1,1,1}a_{1,1,-1}e^{ik(-x+3y+z)} \\
& +48a^2_{-1,-1,1}a_{1,-1,-1}e^{ik(-x-3y+z)} + 48a_{-1,1,-1}a_{-1,1,1}a_{1,1,-1}e^{ik(-x+y+z)} + 48a_{-1,-1,1}a_{1,-1,-1}a_{1,1,-1}e^{ik(x-y-z)} \\
& +48a_{-1,-1,1}a_{1,-1,1}a_{1,1,-1}e^{ik(x-y+z)})k^2)t^2 + O(t^3),
\end{aligned}
\tag{29}
$$

$$
\begin{aligned}
p \quad = \quad & -8(a_{-1,-1,1}a_{1,-1,-1}e^{-2iky} + a_{-1,1,-1}a_{1,-1,-1}e^{-2ikz} + a_{-1,1,1}a_{1,1,-1}e^{2iky} + a_{-1,1,1}a_{1,-1,1}e^{2ikz} \\
& +a_{-1,-1,1}a_{-1,1,-1}e^{-2ikx} + a_{1,-1,1}a_{1,1,-1}e^{2ikx}) + 4(a_{-1,1,-1}a_{-1,1,1}e^{2ik(-x+y)} + a_{-1,-1,1}a_{-1,1,1}e^{2ik(-x+z)} \\
& +a_{-1,-1,1}a_{1,-1,1}e^{2ik(-y+z)} + a_{1,-1,-1}a_{1,-1,1}e^{2ik(x-y)} + a_{1,-1,-1}a_{1,1,-1}e^{2ik(x-z)} + a_{-1,1,-1}a_{1,1,-1}e^{2ik(y-z)}) \\
& +((48(a_{-1,-1,1}a_{1,-1,-1}e^{-2iky} + a_{-1,1,-1}a_{1,-1,-1}e^{-2ikz} + a_{-1,1,1}a_{1,1,-1}e^{2iky} + a_{-1,1,1}a_{1,-1,1}e^{2ikz} \\
& +a_{-1,-1,1}a_{-1,1,-1}e^{-2ikx} + a_{1,-1,1}a_{1,1,-1}e^{2ikx}) - 24(a_{-1,1,-1}a_{-1,1,1}e^{2ik(-x+y)} + a_{-1,-1,1}a_{-1,1,1}e^{2ik(-x+z)} \\
& +a_{-1,-1,1}a_{1,-1,1}e^{2ik(-y+z)} + a_{1,-1,-1}a_{1,-1,1}e^{2ik(x-y)} + a_{1,-1,-1}a_{1,1,-1}e^{2ik(x-z)} + a_{-1,1,-1}a_{1,1,-1}e^{2ik(y-z)}))\nu k^2 \\
& +\frac{768}{11}i(-a_{-1,-1,1}a_{1,-1,-1}a_{1,-1,1}e^{ik(x-3y+z)} + a_{-1,1,1}a_{1,1,-1}a_{-1,1,-1}e^{ik(-x+3y-z)} - a_{-1,1,-1}a_{1,-1,-1}a_{1,1,-1}e^{ik(x+y-3z)} \\
& +11a_{-1,-1,1}a_{-1,1,-1}a_{1,-1,-1}e^{ik(-x-y-z)} - 11a_{-1,1,1}a_{1,-1,1}a_{1,1,-1}e^{ik(x+y+z)} - a_{-1,-1,1}a_{-1,1,-1}a_{-1,1,1}e^{ik(-3x+y+z)} \\
& +a_{-1,1,1}a_{1,-1,1}a_{-1,-1,1}e^{ik(-x-y+3z)} + a_{1,-1,1}a_{1,1,-1}a_{1,-1,-1}e^{ik(3x-y-z)})k)t + O(t^2).
\end{aligned} \tag{30}
$$

In Method 1, these results are truncated onto the modes with $-1 \leqslant \mathbf{L}_j \leqslant 1$.

Method 2

Let

$$\mathbf{u} = \sum_{\mathbf{L}=-\mathbf{1}}^{\mathbf{1}} \mathbf{u}_{\mathbf{L}} e^{ik\mathbf{L}\cdot\mathbf{x}}, \tag{31}$$

$$p = \sum_{\mathbf{L}=-\mathbf{1}}^{\mathbf{1}} p_{\mathbf{L}} e^{ik\mathbf{L}\cdot\mathbf{x}}. \tag{32}$$

Substituting (31), (32) into (1) and equating like powers of e in accordance with Theorem 2 yields

$$\frac{\partial \mathbf{u}_{\mathbf{L}}}{\partial t} + \sum_{\mathbf{M}} (\mathbf{u}_{\mathbf{L}-\mathbf{M}} \cdot ik\mathbf{M})\mathbf{u}_{\mathbf{M}} = -\nu k^2 |\mathbf{L}|^2 \mathbf{u}_{\mathbf{L}} - ik\mathbf{L} p_{\mathbf{L}}. \tag{33}$$

Substituting (31) into (2) and equating like powers of e in accordance with Theorem 2 yields

$$\mathbf{L} \cdot \mathbf{u}_{\mathbf{L}} = 0. \tag{34}$$

Applying $\mathbf{L} \times \mathbf{L} \times$ to (33) and noting the vector identity

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c} \cdot \mathbf{a})\mathbf{b} - (\mathbf{b} \cdot \mathbf{a})\mathbf{c} \tag{35}$$

along with (34) yields

$$|\mathbf{L}|^2 \frac{\partial \mathbf{u}_{\mathbf{L}}}{\partial t} = \sum_{\mathbf{M}} \mathbf{L} \times (\mathbf{L} \times (\mathbf{u}_{\mathbf{L}-\mathbf{M}} \cdot ik\mathbf{M})\mathbf{u}_{\mathbf{M}}) - \nu k^2 |\mathbf{L}|^4 \mathbf{u}_{\mathbf{L}}. \tag{36}$$

Equation (36) implies

$$\frac{\partial \mathbf{u}_{\mathbf{L}}}{\partial t} = \sum_{\mathbf{M}} \hat{\mathbf{L}} \times (\hat{\mathbf{L}} \times (\mathbf{u}_{\mathbf{L}-\mathbf{M}} \cdot ik\mathbf{M})\mathbf{u}_{\mathbf{M}}) - \nu k^2 |\mathbf{L}|^2 \mathbf{u}_{\mathbf{L}} \tag{37}$$

where the right hand side of (37) is $\mathbf{0}$ when $\mathbf{L} = \mathbf{0}$ and $\hat{\mathbf{L}} = \mathbf{L}/|\mathbf{L}|$ is the unit vector in the direction of $\mathbf{L}$. Applying $\mathbf{L}\cdot$ to (33) and noting (34) gives

$$ik|\mathbf{L}|^2 p_{\mathbf{L}} = -\sum_{\mathbf{M}} (\mathbf{u}_{\mathbf{L}-\mathbf{M}} \cdot ik\mathbf{M})(\mathbf{u}_{\mathbf{M}} \cdot \mathbf{L}) \tag{38}$$

implying that

$$p_{\mathbf{L}} = -\sum_{\mathbf{M}} (\mathbf{u}_{\mathbf{L}-\mathbf{M}} \cdot \hat{\mathbf{L}})(\mathbf{u}_{\mathbf{M}} \cdot \hat{\mathbf{L}}) \tag{39}$$

where $p_{\mathbf{0}} \in \mathbb{R}$ is an arbitrary function of t. Let

$$\mathbf{u}_{\mathbf{L}} = \sum_{l=0}^{n} \frac{\partial^l \mathbf{u}_{\mathbf{L}}}{\partial t^l}|_{t=0} \frac{t^l}{l!}, \tag{40}$$

$$p_{\mathbf{L}} = \sum_{l=0}^{n-1} \frac{\partial^l p_{\mathbf{L}}}{\partial t^l}|_{t=0} \frac{t^l}{l!}. \tag{41}$$

Substituting (40) into (37) and equating like powers of t in accordance with Theorem 1 yields

$$\frac{\partial^{l+1}\mathbf{u}_{\mathbf{L}}}{\partial t^{l+1}}|_{t=0} = \sum_{m=0}^{l} \sum_{\mathbf{M}} \hat{\mathbf{L}} \times (\hat{\mathbf{L}} \times (\frac{\partial^{l-m}\mathbf{u}_{\mathbf{L}-\mathbf{M}}}{\partial t^{l-m}}|_{t=0} \cdot ik\mathbf{M}) \frac{\partial^m \mathbf{u}_{\mathbf{M}}}{\partial t^m}|_{t=0}) \binom{l}{m} - \nu k^2 |\mathbf{L}|^2 \frac{\partial^l \mathbf{u}_{\mathbf{L}}}{\partial t^l}|_{t=0}. \tag{42}$$

Equation (42) is then solved for $\frac{\partial^{l+1}\mathbf{u}_{\mathbf{L}}}{\partial t^{l+1}}|_{t=0}$ where $l = 0, 1, \ldots, n-1$ and $-1 \leqslant \mathbf{L}_j \leqslant 1$. Substituting (40), (41) into (39) and equating like powers of t in accordance with Theorem 1 yields

$$\frac{\partial^l p_{\mathbf{L}}}{\partial t^l}|_{t=0} = -\sum_{m=0}^{l} \sum_{\mathbf{M}} (\frac{\partial^{l-m}\mathbf{u}_{\mathbf{L}-\mathbf{M}}}{\partial t^{l-m}}|_{t=0} \cdot \hat{\mathbf{L}})(\frac{\partial^m \mathbf{u}_{\mathbf{M}}}{\partial t^m}|_{t=0} \cdot \hat{\mathbf{L}}) \binom{l}{m}. \tag{43}$$

Equation (43) is then solved for $\frac{\partial^l p_{\mathbf{L}}}{\partial t^l}|_{t=0}$ where $l = 0, 1, \ldots, n-1$ and $-1 \leqslant \mathbf{L}_j \leqslant 1$. Expressions for (8), (9) from Method 2 are then known in terms of given functions.
At $l = 0$ in (42) it is found that

$$\frac{\partial \mathbf{u}_{\mathbf{L}}}{\partial t}|_{t=0} = \sum_{\mathbf{M}} \hat{\mathbf{L}} \times (\hat{\mathbf{L}} \times (\mathbf{u}_{\mathbf{L}-\mathbf{M}}|_{t=0} \cdot ik\mathbf{M}) \mathbf{u}_{\mathbf{M}}|_{t=0}) - \nu k^2 |\mathbf{L}|^2 \mathbf{u}_{\mathbf{L}}|_{t=0}. \tag{44}$$

In (44) with $1 \leqslant |\mathbf{L}|^2 \leqslant 3$, $\mathbf{u}_{\mathbf{M}}|_{t=0} = \mathbf{0}$ unless $|\mathbf{M}|^2 = 3$ and $\mathbf{u}_{\mathbf{L}-\mathbf{M}}|_{t=0} = \mathbf{0}$ unless $|\mathbf{L}-\mathbf{M}|^2 = 3$. With $|\mathbf{L}|^2 = 3$ and $|\mathbf{M}|^2 = 3$ the equation $|\mathbf{L}-\mathbf{M}|^2 = 3$ then implies $2\mathbf{L} \cdot \mathbf{M} = 3$ which is not possible as an even number can not be equal to an odd number. Likewise, with $|\mathbf{L}|^2 = 1$ and $|\mathbf{M}|^2 = 3$ the equation $|\mathbf{L}-\mathbf{M}|^2 = 3$ then implies $2\mathbf{L} \cdot \mathbf{M} = 1$ which is not possible as an even number can not be equal to an odd number. With $|\mathbf{L}|^2 = 2$ and $|\mathbf{M}|^2 = 3$ the equation $|\mathbf{L}-\mathbf{M}|^2 = 3$ then implies $\mathbf{L} \cdot \mathbf{M} = 1$ which is not possible as in this instance $|\mathbf{L} \cdot \mathbf{M}| \in \{0, 2\}$ when $-1 \leqslant \mathbf{L}_j \leqslant 1, -1 \leqslant \mathbf{M}_j \leqslant 1$. Therefore

$$\frac{\partial \mathbf{u}_{\mathbf{L}}}{\partial t}|_{t=0} = -3k^2 \nu \mathbf{u}_{\mathbf{L}}|_{t=0}. \tag{45}$$

At $O(t)$, I find that Method 2 gives the same result for (8) as given by Method 1.
At $l = 1$ in (42) it is found that

$$\begin{aligned} \frac{\partial^2 \mathbf{u}_{\mathbf{L}}}{\partial t^2}|_{t=0} \quad = \quad & \sum_{\mathbf{M}} \hat{\mathbf{L}} \times (\hat{\mathbf{L}} \times ((\frac{\partial \mathbf{u}_{\mathbf{L}-\mathbf{M}}}{\partial t}|_{t=0} \cdot ik\mathbf{M}) \mathbf{u}_{\mathbf{M}}|_{t=0} + (\mathbf{u}_{\mathbf{L}-\mathbf{M}}|_{t=0} \cdot ik\mathbf{M}) \frac{\partial \mathbf{u}_{\mathbf{M}}}{\partial t}|_{t=0})) \\ & -\nu k^2 |\mathbf{L}|^2 \frac{\partial \mathbf{u}_{\mathbf{L}}}{\partial t}|_{t=0}. \end{aligned} \tag{46}$$

By a similar argument as that applied to (44) it is found in Method 2 that

$$\frac{\partial^2 \mathbf{u_L}}{\partial t^2}|_{t=0} = -3k^2\nu\frac{\partial \mathbf{u_L}}{\partial t}|_{t=0} = 9k^4\nu^2\mathbf{u_L}|_{t=0}. \tag{47}$$

In fact for $l \geqslant 0$ it is found in Method 2 that

$$\frac{\partial^{l+1}\mathbf{u_L}}{\partial t^{l+1}}|_{t=0} = (-3k^2\nu)^{l+1}\mathbf{u_L}|_{t=0}. \tag{48}$$

With Method 1 for $\nu = 0$, I find that $\mathbf{u}_{tt}|_{t=0} \neq \mathbf{0}$ when truncated onto the modes with $-1 \leqslant \mathbf{L}_j \leqslant 1$. Therefore at $O(t^2)$, the approximation (8) found from Method 1 is different to the approximation (8) found from Method 2. Because of this nonuniqueness at least one of the assumptions used was invalid. The only assumptions I have used that could have been invalid are those required for use of Theorem 1 and Theorem 2. Therefore the only way statement (D) could not be true is if the smoothness of $\mathbf{u}$ can break down at an $\mathbf{x} \in \mathbb{R}^3$ where $t \in \mathbb{C}$ but with $t \notin [0, \infty)$. Based on this premise I then assume statement (D) is not true and seek a contradiction.

It is found that $(\mathbf{u}(\mathbf{x} - \mathbf{\Omega}t, t) + \mathbf{\Omega}, p(\mathbf{x} - \mathbf{\Omega}t, t))$ is a solution to (1), (2), (3), (4), (5), (6) if $(\mathbf{u}(\mathbf{x}, t), p(\mathbf{x}, t))$ is a solution to (1), (2), (3), (4), (5), (6) where $\mathbf{\Omega} \in \mathbb{R}^3$ is a constant. There is at least one point $\mathbf{x} = \Xi_1 \in \mathbb{R}^3, t = T_1$ which is a breakdown point of $\mathbf{u}(\mathbf{x}, t)$ and there is at least one point $\mathbf{x} = \Theta_1 \in \mathbb{R}^3, t = \kappa_1$ which is a breakdown point of $\boldsymbol{\epsilon}(\mathbf{x}, t) = \mathbf{u}(\mathbf{x} - \mathbf{\Omega}t, t) + \mathbf{\Omega}$ and $\mathbf{x} = \mathbf{\Theta} = \mathbf{\Xi} + \mathbf{\Omega}t, t = \rho$ is a breakdown point of $\boldsymbol{\epsilon}(\mathbf{x}, t)$ if $\mathbf{x} = \mathbf{\Xi}, t = \rho$ is a breakdown point of $\mathbf{u}(\mathbf{x}, t)$.

If there is only one breakdown point of $\mathbf{u}(\mathbf{x}, t)$; $\mathbf{x} = \Xi_1, t = T_1$ then there is only one breakdown point of $\boldsymbol{\epsilon}(\mathbf{x}, t)$; $\mathbf{x} = \Theta_1 = \Xi_1 + \mathbf{\Omega}T_1, t = T_1$, therefore $T_1 \in \mathbb{R}$.

If there is two breakdown points of $\mathbf{u}(\mathbf{x}, t)$; $\mathbf{x} = \Xi_1, t = T_1$ and $\mathbf{x} = \Xi_2, t = T_2$ then there is two breakdown points of $\boldsymbol{\epsilon}(\mathbf{x}, t)$; $\mathbf{x} = \Theta_1 = \Xi_1 + \mathbf{\Omega}T_1, t = T_1$ and $\mathbf{x} = \Theta_2 = \Xi_2 + \mathbf{\Omega}T_2, t = T_2$; or $\mathbf{x} = \Theta_2 = \Xi_1 + \mathbf{\Omega}T_1, t = T_1$ and $\mathbf{x} = \Theta_1 = \Xi_2 + \mathbf{\Omega}T_2, t = T_2$. But if $\mathbf{x} = \Xi_1, t = T_1$ is a breakdown point of $\mathbf{u}(\mathbf{x}, t)$ then $\mathbf{x} = \Xi_1, t = \overline{T_1}$ is also a breakdown point of $\mathbf{u}(\mathbf{x}, t)$ by Theorem 3 in the Appendix, therefore $T_1 \in \mathbb{R}$.

If there is η breakdown points of $\mathbf{u}(\mathbf{x}, t)$; $(\mathbf{x}, t) \in \{(\Xi_i, T_i)_{i=1,2,\ldots,\eta}\}$ then by Theorem 3 there is η breakdown points of $\boldsymbol{\epsilon}(\mathbf{x}, t)$; $\mathbf{x} = \Theta_a = \Xi_1 + \mathbf{\Omega}T_1, t = T_1, \mathbf{x} = \Theta_b = \Xi_1 + \mathbf{\Omega}\overline{T_1}, t = \overline{T_1}, \ldots, \mathbf{x} = \Theta_1 = \Xi_c + \mathbf{\Omega}T_c, t = T_c, \mathbf{x} = \Theta_1 = \Xi_d + \mathbf{\Omega}\overline{T_c}, t = \overline{T_c}, \ldots, \mathbf{x} = \Theta_e = \Xi_\eta + \mathbf{\Omega}T_\eta, t = T_\eta$. Therefore $\Xi_c - \Xi_d = \mathbf{\Omega}(\overline{T_c} - T_c)$ and since the direction of $\Xi_c - \Xi_d$ is independent of $\mathbf{\Omega}$ this implies $T_c \in \mathbb{R}$. Notice here that at an $\mathbf{x} \in \mathbb{R}^3$ the breakdown time of $\boldsymbol{\epsilon}(\mathbf{x}, t)$ with smallest modulus must be real valued. However at an $\mathbf{x} \in \mathbb{R}^3$ the breakdown time of $\mathbf{u}(\mathbf{x}, t)$ with smallest modulus may be complex valued.

Furthermore, since a breakdown point is due to the integrand of $P(\mathbf{u}, e^{ik\mathbf{L}\cdot\mathbf{x}})$ not being smooth over $\mathbf{x} \in [0, 1]^3$ this implies that there exists a finite time of breakdown $t \in \mathbb{R}$. Therefore the smoothness of $\mathbf{u}$ can then break down at an $\mathbf{x} \in \mathbb{R}^3$ where $t \in \mathbb{R}$ is finite.

For $\nu = 0$, it is found that $(\zeta\mathbf{u}(\mathbf{x}, \zeta t), \zeta^2 p(\mathbf{x}, \zeta t))$ is a solution to (1), (2), (3), (4), (5), (6) if $(\mathbf{u}(\mathbf{x}, t), p(\mathbf{x}, t))$ is a solution to (1), (2), (3), (4), (5), (6) where $\zeta \in \mathbb{R}$ is a constant, so if the smoothness of $\mathbf{u}$ breaks down at $t < 0$ where $\mathbf{u}_0 = \mathbf{U}_0 \in \mathbb{R}^3$ then the smoothness of $\mathbf{u}$ breaks down at $t > 0$ where $\mathbf{u}_0 = -\mathbf{U}_0 \in \mathbb{R}^3$. Therefore statement (D) is true when $\nu > 0$

is replaced with $\nu = 0$.

For $\nu > 0$, when applying Method 1 for $n = 2$ and Method 2 for all $n \in \mathbb{N}$, it is found that the governing equation for $\mathbf{u}$ is effectively

$$\frac{\partial \mathbf{u}}{\partial t} = \nabla^{-2}\nabla \times \nabla \times ((\mathbf{u} \cdot \nabla)\mathbf{u}) + \nu\lambda\mathbf{u} \tag{49}$$

where $\lambda = -3k^2$. Equation (49) implies

$$\frac{\partial}{\partial t}(\mathbf{u}e^{-\nu\lambda t}) = \nabla^{-2}\nabla \times \nabla \times ((\mathbf{u} \cdot \nabla)\mathbf{u})e^{-\nu\lambda t}. \tag{50}$$

Then a change of variables

$$\tau = e^{\nu\lambda t} - 1, \tag{51}$$

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{v}(\mathbf{x}, \tau)\frac{\partial \tau}{\partial t} \tag{52}$$

yields

$$\frac{\partial \mathbf{v}}{\partial \tau} = \nabla^{-2}\nabla \times \nabla \times ((\mathbf{v} \cdot \nabla)\mathbf{v}). \tag{53}$$

Equation (2) becomes

$$\nabla \cdot \mathbf{v} = 0, \tag{54}$$

the initial condition (3) becomes

$$\mathbf{v}(\mathbf{x}, 0) = \frac{\mathbf{u}_0}{\nu\lambda}, \tag{55}$$

and the spatially periodic boundary conditions (4), (6) imply

$$\mathbf{v}(\mathbf{x} + e_j, \tau) = \mathbf{v}(\mathbf{x}, \tau) \;\text{ for }\; 1 \leqslant j \leqslant 3. \tag{56}$$

Equations (53), (54), (55), (56) define an Euler problem. Therefore from the scaling with ζ for $\nu = 0$, if the smoothness of $\mathbf{v}$ breaks down at an $\mathbf{x} \in \mathbb{R}^3$ with finite $\tau \in (-1, 0)$ dependent on $\mathbf{u}_0 \in \mathbb{R}^3$, then the smoothness of $\mathbf{u}$ can break down at an $\mathbf{x} \in \mathbb{R}^3$ with finite $t > 0$. Therefore statement (D) is true. □

Appendix

Theorem 1

Providing that the Maclaurin series

$$\breve{\mathbf{A}} = \sum_{l=0}^{n} \frac{\partial^l \mathbf{A}}{\partial t^l}\Big|_{t=0} \frac{t^l}{l!} = \sum_{l=0}^{n} \frac{\partial^l \breve{\mathbf{A}}}{\partial t^l}\Big|_{t=0} \frac{t^l}{l!} \tag{57}$$

of the exact general solution to a Q^{th} order partial differential equation

$$\frac{\partial^Q \mathbf{A}}{\partial t^Q} = \mathbf{\Psi} \tag{58}$$

exists, it will solve the coefficients of t^l for all $l = 0, 1, \ldots, n - Q$ in (58) with $\mathbf{A} = \breve{\mathbf{A}}$ provided $\mathbf{\Psi}|_{\mathbf{A}=\breve{\mathbf{A}}}$ is expandable in Maclaurin series as

$$\mathbf{\Psi}|_{\mathbf{A}=\breve{\mathbf{A}}} = \sum_{l=0}^{m} \frac{\partial^l \mathbf{\Psi}|_{\mathbf{A}=\breve{\mathbf{A}}}}{\partial t^l}|_{t=0} \frac{t^l}{l!} \tag{59}$$

where $m \geqslant n$. Here all of the partial derivatives of $\mathbf{A}$ with respect to t are defined at $t = 0$.

Proof of Theorem 1

Since the Maclaurin series of $\mathbf{A}$ exists and all of the partial derivatives of $\mathbf{A}$ with respect to t are defined at $t = 0$, one can integrate (58) Q times with respect to t and then substitute the result into (57) to find

$$\breve{\mathbf{A}} = \sum_{l=0}^{n} \frac{\partial^{l-Q} \mathbf{\Psi}}{\partial t^{l-Q}}|_{t=0} \frac{t^l}{l!} = \sum_{l=0}^{n} \frac{\partial^l \int_Q \mathbf{\Psi}\, dt|_{\mathbf{A}=\breve{\mathbf{A}}}}{\partial t^l}|_{t=0} \frac{t^l}{l!} \tag{60}$$

where $\int_Q \mathbf{\Psi}\, dt$ denotes the Q^{th} integral of $\mathbf{\Psi}$ with respect to t. Substituting $\mathbf{A} = \breve{\mathbf{A}}$ into the residual $\mathbf{r}$ of (58) then gives

$$\mathbf{r} = \sum_{l=0}^{n} \frac{\partial^{l-Q} \mathbf{\Psi}|_{\mathbf{A}=\breve{\mathbf{A}}}}{\partial t^{l-Q}}|_{t=0} \frac{t^{l-Q}}{(l-Q)!} - \sum_{l=0}^{m} \frac{\partial^l \mathbf{\Psi}|_{\mathbf{A}=\breve{\mathbf{A}}}}{\partial t^l}|_{t=0} \frac{t^l}{l!} \tag{61}$$

providing $\mathbf{\Psi}|_{\mathbf{A}=\breve{\mathbf{A}}}$ is expanded in Maclaurin series as in (59). Collecting like powers of t in (61) yields

$$\mathbf{r} = \sum_{l=0}^{n-Q} \frac{\partial^l \mathbf{\Psi}|_{\mathbf{A}=\breve{\mathbf{A}}}}{\partial t^l}|_{t=0} \frac{t^l}{l!} - \sum_{l=0}^{m} \frac{\partial^l \mathbf{\Psi}|_{\mathbf{A}=\breve{\mathbf{A}}}}{\partial t^l}|_{t=0} \frac{t^l}{l!} \tag{62}$$

which shows that Theorem 1 is true. □

Theorem 2

Providing that the Fourier series

$$\tilde{\mathbf{A}} = \sum_{\mathbf{L}=-\mathbf{N}}^{\mathbf{N}} P(\mathbf{A}, e^{ik\mathbf{L}\cdot\mathbf{x}}) e^{ik\mathbf{L}\cdot\mathbf{x}} = \sum_{\mathbf{L}=-\mathbf{N}}^{\mathbf{N}} P(\tilde{\mathbf{A}}, e^{ik\mathbf{L}\cdot\mathbf{x}}) e^{ik\mathbf{L}\cdot\mathbf{x}} \tag{63}$$

of the exact general solution to a Q^{th} order partial differential equation

$$\frac{\partial^Q \mathbf{A}}{\partial t^Q} = \mathbf{\Psi} \tag{64}$$

exists, it will solve the coefficients of $e^{ik\mathbf{L}\cdot\mathbf{x}}$ for all $-N \leqslant \mathbf{L}_j \leqslant N$ in (64) with $\mathbf{A} = \tilde{\mathbf{A}}$ provided $\mathbf{\Psi}|_{\mathbf{A}=\tilde{\mathbf{A}}}$ is expandable in Fourier series as

$$\mathbf{\Psi}|_{\mathbf{A}=\tilde{\mathbf{A}}} = \sum_{\mathbf{L}=-\mathbf{M}}^{\mathbf{M}} P(\mathbf{\Psi}|_{\mathbf{A}=\tilde{\mathbf{A}}}, e^{ik\mathbf{L}\cdot\mathbf{x}}) e^{ik\mathbf{L}\cdot\mathbf{x}} \tag{65}$$

where $M \geqslant N$. Here $\mathbf{A}$ is spatially periodic and smooth for all $\mathbf{x} \in \mathbb{R}^3$, $k > 0$ is a constant, and $P(\mathbf{h}, e^{ik\mathbf{L}\cdot\mathbf{x}})$ denotes the projection of $\mathbf{h}$ onto $e^{ik\mathbf{L}\cdot\mathbf{x}}$.

Proof of Theorem 2

Since the Fourier series of $\mathbf{A}$ exists where $\mathbf{A}$ is spatially periodic and smooth for all $\mathbf{x} \in \mathbb{R}^3$, one can integrate (64) Q times with respect to t and then substitute the result into (63) to find

$$\tilde{\mathbf{A}} = \sum_{\mathbf{L}=-\mathbf{N}}^{\mathbf{N}} P(\int_Q \mathbf{\Psi}\, dt, e^{ik\mathbf{L}\cdot\mathbf{x}})e^{ik\mathbf{L}\cdot\mathbf{x}} = \sum_{\mathbf{L}=-\mathbf{N}}^{\mathbf{N}} P(\int_Q \mathbf{\Psi}\, dt|_{\mathbf{A}=\tilde{\mathbf{A}}}, e^{ik\mathbf{L}\cdot\mathbf{x}})e^{ik\mathbf{L}\cdot\mathbf{x}}. \tag{66}$$

Substituting $\mathbf{A} = \tilde{\mathbf{A}}$ into the residual $\mathbf{r}$ of (64) then gives

$$\mathbf{r} = \frac{\partial^Q}{\partial t^Q} \sum_{\mathbf{L}=-\mathbf{N}}^{\mathbf{N}} P(\int_Q \mathbf{\Psi}\, dt|_{\mathbf{A}=\tilde{\mathbf{A}}}, e^{ik\mathbf{L}\cdot\mathbf{x}})e^{ik\mathbf{L}\cdot\mathbf{x}} - \sum_{\mathbf{L}=-\mathbf{M}}^{\mathbf{M}} P(\mathbf{\Psi}|_{\mathbf{A}=\tilde{\mathbf{A}}}, e^{ik\mathbf{L}\cdot\mathbf{x}})e^{ik\mathbf{L}\cdot\mathbf{x}} \tag{67}$$

providing $\mathbf{\Psi}|_{\mathbf{A}=\tilde{\mathbf{A}}}$ is expanded in Fourier series as in (65). Equation (67) can be written as

$$\mathbf{r} = \sum_{\mathbf{L}=-\mathbf{N}}^{\mathbf{N}} P(\mathbf{\Psi}|_{\mathbf{A}=\tilde{\mathbf{A}}}, e^{ik\mathbf{L}\cdot\mathbf{x}})e^{ik\mathbf{L}\cdot\mathbf{x}} - \sum_{\mathbf{L}=-\mathbf{M}}^{\mathbf{M}} P(\mathbf{\Psi}|_{\mathbf{A}=\tilde{\mathbf{A}}}, e^{ik\mathbf{L}\cdot\mathbf{x}})e^{ik\mathbf{L}\cdot\mathbf{x}} \tag{68}$$

which shows that Theorem 2 is true. □

Theorem 3

A function $f(z) \in \mathbb{R}$ for all $z \in [0, R]$ has singularities at $z = a \in \mathbb{C}$ and $z = \overline{a} \in \mathbb{C}$ such that $|a| = R$ where R is the radius of convergence of the Maclaurin series of $f(z)$.

Proof of Theorem 3

From Taylor's theorem it is known that if $f(z)$ is analytic inside a circle C with centre at $z = 0$ then there is always one and only one power series for all z inside C

$$f(z) = \sum_{l=0}^{\infty} \frac{d^l f}{dz^l}|_{z=0} \frac{z^l}{l!} \quad \text{for} \quad |z| < R \tag{69}$$

where the radius of convergence R is the distance from $z = 0$ to the nearest singularity location of $f(z)$. If $z = a \in \mathbb{C}$ is a singularity location of $f(z)$ with $|a| = R$ then

$$|f(a)| = |\sum_{l=0}^{\infty} \frac{d^l f}{dz^l}|_{z=0} \frac{a^l}{l!}| = \infty \tag{70}$$

and since $f(z) \in \mathbb{R}$ for all $z \in [0, R]$,

$$|f(\overline{a})| = |\sum_{l=0}^{\infty} \frac{d^l f}{dz^l}|_{z=0} \frac{\overline{a}^l}{l!}| = |\overline{\sum_{l=0}^{\infty} \frac{d^l f}{dz^l}|_{z=0} \frac{a^l}{l!}}| = \infty \tag{71}$$

which shows that Theorem 3 is true. □

Maple code

```
restart;
curlprok:=proc(V)
return Array([diff(V[3],y)-diff(V[2],z),diff(V[1],z)-diff(V[3],x),diff(V[2],x)-diff(V[1],y)]);
end proc:
laplacianprok:=proc(S)
return diff(S,x,x)+diff(S,y,y)+diff(S,z,z);
end proc:
crossprodprok:=proc(Va,Vb)
return Array([Va[2]*Vb[3]-Va[3]*Vb[2],-Va[1]*Vb[3]+Va[3]*Vb[1],Va[1]*Vb[2]-Va[2]*Vb[1]]);
end proc:
divergeprok:=proc(V)
return diff(V[1],x)+diff(V[2],y)+diff(V[3],z);
end proc:
invLs:=proc(Q)
local q, expset, ans, eqn, eqns, soln:
q:=combine(expand(Q),exp);
expset:=indets(q,function);
ans:=add(c[J]*expset[J],J=1..nops(expset));
eqn:=combine(eval(laplacianprok(ans)-q),exp);
eqns:={seq(coeff(eqn,expset[K],1),K=1..nops(expset))};
soln:=solve(eqns,{seq(c[J],J=1..nops(expset))});
return subs(soln,ans);
end proc:
invLv:=proc(V)
return Array([invLs(V[1]),invLs(V[2]),invLs(V[3])]);
end proc:
for q from -1 to 1 do
for r from -1 to 1 do
for s from -1 to 1 do
if abs(q)+abs(r)+abs(s) <> 3 then
a[q,r,s]:=0:
end if:
end do:
end do:
end do:
u0:=add(add(add(crossprodprok(Array([L,M,N]),crossprodprok(Array([L,M,N]),Array([a[L,M,N],a[L,M,N],a[L,M,N]])))*exp(I*k*(L*x
+M*y+N*z)),L=-1..1),M=-1..1),N=-1..1):
n:=2:
DD:=l->proc() option remember;
if l=-1 then
return u0;
else return -invLv(curlprok(curlprok(add(crossprodprok(Array([args[m+1][1],args[m+1][2],args[m+1][3]]),curlprok(Array([args[
l-m+1][1],args[l-m+1][2],args[l-m+1][3]])))*binomial(l,m),m=0..l))))+nu*Array([laplacianprok(args[l+1][1]),laplacianprok(arg
s[l+1][2]),laplacianprok(args[l+1][3])]);
end if:
end proc:
fun:=proc(l) option remember;
if l=0 then
return eval(u0);
else return DD(l-1)(seq(fun(m),m=0..l-1));
end if;
end proc:
U:=add(fun(l)*((t^l)/(l!)),l=0..n):
P:=add((-invLs(eval(subs(t=0,diff(divergeprok(Array([U[1]*diff(U[1],x)+U[2]*diff(U[1],y)+U[3]*diff(U[1],z),U[1]*diff(U[2],x)
+U[2]*diff(U[2],y)+U[3]*diff(U[2],z),U[1]*diff(U[3],x)+U[2]*diff(U[3],y)+U[3]*diff(U[3],z)])),[t$l])))))*((t^l)/(l!)),l=0..n
-1):
simplify(diff(U[1],x)+diff(U[2],y)+diff(U[3],z)); #returns 0 as a check
for j from 0 to n-1 do
simplify(subs(t=0,diff(diff(U[1],t)+U[1]*diff(U[1],x)+U[2]*diff(U[1],y)+U[3]*diff(U[1],z)-nu*laplacianprok(U[1])+diff(P,x),[
t$j])));
simplify(subs(t=0,diff(diff(U[2],t)+U[1]*diff(U[2],x)+U[2]*diff(U[2],y)+U[3]*diff(U[2],z)-nu*laplacianprok(U[2])+diff(P,y),[
t$j])));
simplify(subs(t=0,diff(diff(U[3],t)+U[1]*diff(U[3],x)+U[2]*diff(U[3],y)+U[3]*diff(U[3],z)-nu*laplacianprok(U[3])+diff(P,z),[
t$j])));
end do; #returns 0's as a check
collect(collect(collect(U[1],nu),k),t);
collect(collect(collect(U[2],nu),k),t);
collect(collect(collect(U[3],nu),k),t);
collect(collect(collect(P,nu),k),t);
```

References

[1] Batchelor, G. K. 1967. An introduction to fluid dynamics. Cambridge University Press: Cambridge.

[2] Fefferman, C. L. 2000. Existence and smoothness of the Navier–Stokes equation. Clay Mathematics Institute: official problem description.