Thought Experiments About Infinite Sets and Subsets Produce Experimental Artifacts: Relationship to Their Use in Physics

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Abstract

Here, the conclusion in set theory that the size of an infinite set is the same as the size of an infinite subset derived from it is questioned. This is done not to try and invalidate any mathematical results because mathematics is an abstract field and does not necessarily have to accurately describe the physical world but in order to prompt the reexamination of the use of this result in physics, which does have to accurately describe the real, physical world and the relationships between its components. The rationale is as follows. First, it is suggested that thought experiments are still experiments and should follow the rules for good experimental technique, which include the need to study a system in a setting as close as possible to the "natural setting" to try and avoid experimental artifacts. Now, starting with the single set of the positive integers, one wants to compare the total number of integers to the total number of even integers within the "natural setting" of the single original set. The traditional experimental processing method extracts the even integers, puts them into a separate subset and pairs off the subset's and set's members one-to-one with a function. After doing this, no elements are left over, and, therefore, the original set and the subset extracted from it are said to be the same size. However, extracting the evens and putting them into a separate subset dramatically alters the original single set system. This is analogous to a biologist extracting the nucleus from a cell, studying the nucleus and remaining parts of the cell in isolation and assuming that the results obtained are the same as in the original intact cell. They often are not. Does extracting the even integers out into a subset alter the results compared to those that would be obtained in the natural single set system? Yes. In the single set system, the positive integers march lockstep and in-phase with the odd integers from one to infinity, meaning that there is a built-in relationship in this system of one positive integer for every two total integers, which means that there are only one-half as many positive integers as total integers. This is a different result than that obtained after the subset extraction method, which means that the result produced by this method is an experimental artifact. This should be unacceptable in a well done experiment even if it is a thought experiment. It is suggested that this artifact may be related to some of the problems associated with infinities in physics.

Are thought experiments experiments?

From "Webster's New World Dictionary and Thesaurus" (1), an experiment is "a test, trial, action, etc. undertaken to discover or demonstrate something". Based on this definition, then despite being done in the mind, a thought experiment is without a doubt an test or action undertaken to discover something. It is therefore, by definition, an experiment. All experiments should follow the rules for good experimental technique, which include the need to study a system in a setting as close as possible to the natural setting in order to try and avoid experimental artifacts. Most sciences consider results obtained in a system that maintains all the inherent relationships of the natural setting as a more valid representation of that setting than results obtained in an artifact-ridden, non-natural setting. Starting with the single set of all the positive integers, if one wants to determine the number of even integers relative to the total number of all integers within that set, the natural setting is the single set of all the positive integers.

Size Is Relative Because It is Measured with Respect to a Reference Frame

The size of any object, including a set, is measured with respect to a reference frame. This means that the object's size is stated in terms (units) of and is relative to the reference frame with which it's being compared. Two kinds of reference frame are described below.

1. The Reference Frame as a Ruler: Most often, the reference frame can be considered to be a ruler. Thus, the size of something is stated in terms of the unit length of the ruler. For example, suppose one has an object, A, that is six inches long. If one compares A's size with a one-inch reference frame, or ruler, then A is six times as big as the reference frame or 6 inches long. But, if A's size is compared to a one-foot long reference frame, or ruler, then A is only one-half the size of the reference frame or one-half foot long. While the object's absolute size doesn't change, its size relative to the reference frame or ruler does change from six times as big to only one-half as big.

2. The Reference Frame as the Natural or Experimentally Processed State: This meaning of the term reference frame is best explained with an example. Suppose that a compressed spring, S, is contained within a tube containing four other, identical compressed springs positioned end-to-end. This tube of five compressed springs is S's natural setting, or reference frame. It is a physical system with built-in relationships between the springs and between the springs and tube. One wants to know the size of S, within this natural setting, relative to the overall tube, which functions as the reference frame in this example There are two ways of doing this. The first method would be to keep S within the natural setting of the tube and then, as described above, compare its size with the reference frame (e.g., the overall tube) and find out that S's size is one-fifth the size of the tube within this natural reference frame. This method maintains the relationships between the springs and the tube and gives S's size in its natural setting and, therefore, does not artifactually (production of non-natural results due to experimental processing) alter S's size relative to that setting. A second method of measuring S's size would be to first remove it from its natural state of compression within the tube. This allows S to uncompress to, say, five times its original length. Once outside the tube, S is in the experimentally processed setting, or reference frame. Here, one again measures the size of S relative to the reference frame of the overall tube. But here, one finds that S in its uncompressed state now has the same size as the overall tube. The difference between the two methods is that the second process removes S from its natural setting and puts it in a new experimentally processed setting, or reference frame, before measuring it. This method breaks the built-in, inherent relationships between the springs and between the springs and the tube and, therefore, artifactually alters its size relative to the reference frame of the overall tube. If one wants to find S's size in its natural setting, one must use the first method. As will be discussed more fully below, because the traditional method of measuring the size of an infinite subset relative to the single parent infinite set from which it is derived entails removing the subset from the original set and creating a separate subset, this method is analogous to this second, artifact-

producing method of measuring spring S's size.

Thus, an object's size is relative not only to a ruler-reference frame but also to the natural- or experimentally processed-reference frame the object is in. Because the words "finite" and "infinite" both refer to size, whether or not something is finite or infinite is also dependent on the reference frame it is in and on the reference frame the ruler is in.

The Traditional Method of Comparing the Size of an Infinite Subset to Its Parent Infinite Set Creates Artifacts by Its Experimental Processing Methods

In this section, it is shown that the traditional method of comparing the size of an infinite set with an infinite subset derived from it alters the original, natural situation and creates artifacts by its experimental processing methods. This is illustrated using the following example. Suppose that one starts off with a single infinite set, N, of all the positive integers (henceforth, an integer of any type, even or odd, in N will be referred to as an N_i, and an even integer in N will be referred to as an N_e). One wants to compare the total number of even integers with the total number of all integers, and one wants to do this within the natural setting, or reference frame, of the single set N of all the positive integers. The traditional method of doing this size comparison is by splitting the evens out as a separate subset, E, and pairing off the elements of E (henceforth, an even in subset E will be called an E_e) with the elements of N using a function such as $f(N_i)=E_e=2N_i$. By showing that each element of E can be paired off one-to-one with an element of N (e.g., if $N_i=1$, $E_e=2$; if $N_i=2$, $E_e=4$; etc.), with no elements left unpaired, it is shown that E and N are the same size. Splitting E out as a separate set and using a function to pair off each E_e one-to-one with an N_i is the experimental processing used in this method. This means that the experimentally processed setting would consist of two sets: the original set of all N_i and the second, separate set of just E_e. While this method has been widely accepted for many years, it is clear that it is based on extensive experimental processing, which substantially changes the situation from that in the natural setting. Now, does it alter the results compared to those that would be obtained in the natural setting of the single set of all the positive integers and, thus, create an experimental artifact? Unfortunately, as explained next, the answer is yes.

Consider an integer, N_i, in the single, original set N. In this single set system, an intrinsic property of N_i is that there are N_{i-1} integers preceding and accompanying N_i within set N. This can be seen not only by inspection of the set but by the very definition of an integer. An integer is defined based on the counting of elements in a sequential progression of elements. For example, without a preceding and accompanying progression of three other elements, the element 4 would have no meaning. It would just be a symbol on a piece of paper. Therefore, if N_i is 4, this requires the presence of three preceding and accompanying integers in N (1, 2 and 3). This also implies that every even-integer in set N is always accompanied by an accompanying odd integer within the same set. For instance, 4 is always accompanied by 3 in set N. Overall, within the natural milieu of single set N, there is a fixed, intrinsic relationship between each integer N_i and the number of its accompanying integers.

Why are these relationships present in set N? They are there because all the elements are in a single set, and all are marching lockstep, or in phase, towards a single, common, infinite goal. This single, common infinite goal fixes all the elements in phase relative to one another and fixes the relationships between the number of elements present for each integer N_i. This isn't just a local phenomenon in set N; this fixed relationship continues throughout the entire set N.

Does the traditional splitting-out and pairing-off method maintain the fixed relationships between integers and their accompanying elements in set N? Again, the answer is no. For example, if one splits out the integer 4 from set N into the separate subset E, there are now two "4"s present, the original in set N and a copy, or clone, in subset E. These are separate elements in separate sets. The two "4"s look the same, but for the cloned 4 in the subset, there is no requirement that there be three preceding and accompanying elements as in the original set N. This is true for any and all integers split out from set N. What has happened is that by splitting out the even integers into subset E, the tether of the single, common infinity for the even integers and for the other integers in single set N is broken, thereby breaking the fixed, inherent relationship between an even in subset E and its formerly accompanying integers in set N. The evens in subset E are now marching towards a totally separate and independent infinite goal than in set N. In sum, the splitting out of even integers from set N during experimental processing removes a key property and inherent relationship of these integers in the natural setting of the single set: the requirement that each integer be preceded and accompanied by a fixed number of other integers. This is equivalent to the real-world, physical situation described above in which spring S was removed from its tube, allowing it to uncoil, so that its length matched the length of the tube in which it was kept and then assuming that this uncoiled length of spring S is the same as that of coiled S in the original tube setting. An analogous situation in biochemistry would be to remove the nucleus from a cell, study it in isolation, and then assume that all the relationships of the nucleus with the rest of the cell in the original whole cell setting had been maintained in the isolated system. This would be an incorrect assumption.

For an artifact to be a problem, it must cause different results to be obtained than would be obtained in the natural system. This is exactly what happens with the traditional infinite set-subset size comparison method. Because they are not accompanied by their fellow elements as in set N, the infinite number of E_e's in subset E can be paired off one-to-one with the infinite number of N_i's in set N with no elements left over. This provides the result that the sizes of subset E and of set N are the same. However, in the natural setting of set N, every E has an accompanying odd, meaning that for every E, there are two total integers: the E and its accompanying odd. Thus, in the natural setting with all its built-in, inherent relationships intact, there are, in fact, only one-half as many evens as total positive integers. This isn't just a local phenomenon in set N; this fixed relationship and ratio continues throughout the entire set N. So, the traditional infinite set-subset size comparison method alters the results compared to what would be obtained within the original, natural single-set system and thereby creates an experimental artifact. Whether or not this matters in the abstract world of mathematics is debatable, but it most definitely matters in physics, which studies real-world, physical systems that have built-in, inherent relationships between their components.

Suppose one tries to avoid the artifacts produced by the traditional splitting-out (of the subset) and pairing-off method by keeping the even-integers within the original set N and trying to do the pairing off within this set. Does this help avoid the artifacts? Again, the answer is no. Any conventional use of the pairing off function, such as $f(N_i)=E_e=2N_i$, means that the evens within N are being considered separately from the overall integers within N even if they are all contained in the same set. The pairing off function ignores the requirement that for every even integer, there are two total integers: the even and its accompanying odd. That is, with the pairing-off function, an even is always either an E or an N but never both at the same time, as in the original, single-set system. So, with this use of the function, the E_e's are still lacking their preceding and accompanying integers, and the artifact is still created.

In sum, the traditional splitting-out and pairing-off method of parent set-infinite subset size comparison entails extensive experimental processing, which substantially changes the original, single-set system and causes different results to be found compared to what would be found in the original parent set system. This is, by definition, an experimental artifact. While this may be acceptable in mathematics, it should not be acceptable in physics, which is a real science and must follow the rules of good experimental technique, especially in its mathematical foundations.

A Reinterpretation of the Pairing Off Function Can Be Used to Compare the Size of an Infinite Subset with Its Parent Set

If the splitting out and pairing off method for comparing the size of infinite subsets with their parent infinite sets creates artifacts, how can this size comparison be done in such a way as to avoid these alterations? From the above discussion, and again using set N as an example, any method used must avoid considering the even integers separately from all the integers in set N because this breaks the fixed relationship between the even integers and their preceding and accompanying integers in set N. One method that meets these criteria is as follows. If the E_e's in the split out subset E were merged back into single set N and, thus, became the even integers in N once again, this would mean that there would no longer be a separate subset E. All the former E_e's would now be back in a single set with the other integers in N. This means that the E_e on the left side of function $f(N_i)=E_e=2N_i$ would truly merge with the $2N_i$ on the right side of this function. That is, instead of just being an individual, "on-its-own", number E_e, each E_e would become one of two N_i s (e.g., 2 and 1, 4 and 2, 6 and 3; etc.) in an ordered sequence of N_i's within set N. Thought of this way, this means that there are two N_i's (the N_i and its accompanying former E_e) for every even integer, E_e, or, thus, that there are twice as many N_i's as even integers within single set N. This is true not just on a local scale but throughout the entire set of the positive integers. This is the intuitive approach of the layman but is also the more accurate approach due to the avoidance of experimental artifacts as in the traditional pairing off method.

In this interpretation of the pairing off function, the E on the left side of the equation $E=2N_i$ and the N_i on the right side aren't just separate, "on-their-own", individual numbers with no relationship between one another. Instead, they've been merged back into the same, original, single-set system from which they came, and the constant 2 shows the inherent relationship between them within this set. I suggest that this interpretation of the pairing off function is more appropriate for representing the contents of a single physical system where the relationships between the components of the system are present and important and need to be taken into account.

Physical systems contain built-in relationships between their components and these need to be taken into account in studying these systems.

Physics studies physical systems, not abstract ones. The accompanying mathematics is used to describe real physical systems and not abstract ones. In a physical system to be studied, there are built-in, inherent physical relationships between the components of a system. Examples are the tube containing the five compressed springs and the cell containing a nucleus as described above. Also, every moment in time is accompanied, in its march towards the future, by the preceding moment. All locations in space and time have built-in spatial and temporal relationships to other locations. As described above, all of these relationships need to be taken into account in order to accurately describe the system. And yet, physics relies on the mathematics of infinities, which as described above, ignores the relationships involved in the experimental system (the single set of the positive integers) being studied. The mathematics of infinite sets also ignores the relationship between the observer (e.g., the mind of a person that's doing the thought-experimental processing on the set of all positive integers) and the observed (e.g., the original, single set system of all the positive integers). As shown above, the observer and his/her experimental processing dramatically alter the observed single set system but then assumes that the results are the same as what would be found in the unobserved system. Is this rational? I suggest here that this incongruity leads to problems in the use of infinities in physics.

It is not being suggested here that thought experiments in general are unsound, only that thought experiments are experiments and therefore, as in any experimental analysis of a system, that the relationships between the components of the system being studied should be taken into account. This seems to be a reasonable request, especially when the thought experiments are being used to study physical systems. In general, thinking of mathematical constructs as "physical" states with their built-in, physical relationships between components may be beneficial in non-abstract fields like physics.

Conclusion

In conclusion, the traditional thought-experimental method of comparing the size of an infinite subset with the parent infinite set from which its derived entails extensive experimental processing, which alters the result compared to that which would be obtained in the original single set system. It produces an experimental artifact. While poor experimental technique may be acceptable to in the abstract field of mathematics, physics deals with the real, physical world and the relationships between its components, and it should require the following of proper experimental technique, especially in its logical foundations.

Some critiques of the reasoning put forward in this paper and responses to them are as follows.

1. Infinities are just different and weird and don't follow the same intuitive rules that finite sets do.

While it is true that infinities are different and weird, just saying this and thinking that it's a good enough excuse for bad experimental technique is extremely weak reasoning and is an invalid critique. The use of good experimental technique and the scientific method should not change just because one is studying a different and weird area.

2. If one wants to maintain the intuitive result that there are one-half as many even integers as total positive integers, ratios should be used instead of the "counting" method (pairing off). However, while a ratio of 1:2 for the number of evens relative to the total number of positive integers would be found over a finite (local) interval, unfortunately, this ratio becomes undefined over the entire set because the ratio of two infinities is undefined.

I suggest that the ratio of two infinities should not always be undefined. If the two infinities start out in separate sets, then I agree that their ratio is undefined because there is no built-in, intrinsic relationship between the elements in the two sets. However, if the two infinite amounts (e.g., even integers and total positive integers) start out in the same, single, original set where every even integer has a fixed, built-in relationship with its accompanying odd integer and where all the integers maintain this relationship throughout the entire set while marching lockstep towards a single infinite endpoint, then I would suggest that the ratio of these two infinite amounts is not undefined. The ratio is 1:2 throughout the entire infinite set. However, this is true only when the two infinite amounts start out in the same single set. Another way to think about this is that when two infinite amounts start out in a single set, and are of the same size scale (e.g., cardinality) like the even integers and all the positive integers are, their ratio is not undefined but is instead a number of that same size scale as well. That is, the results of calculations involving different infinities depends on the set, or reference frame, each infinity is in. In the real world of physical things studied by physics, it seems fair to say that the components of our physical existence are all in the same set and, therefore, that the ratio of two infinities within our physical existence is not undefined.

3. Infinite subsets are abstract constructs (studied in the mind only) and experimental methodology doesn't apply to mental things. Said another way, thought experiments are not real experiments and thus don't need to follow the rules of good experimental technique.

Why not? If good methodology doesn't apply to doing thought experiments, then the results of all thought experiments, and indeed all reasoning of any type, are potentially incorrect. Additionally, whenever one is methodically trying to figure out what is occurring within a system, one is performing an experiment, whether the system is in the "real" world or in the mind. Furthermore, even when we do "real world" experiments, all knowledge of these experiments is perceived in the mind and so, in a way, even "real world" experiments are abstract constructs. Do the rules of experimental methodology not apply to them either?

4. Infinities aren't found in nature where experiments are done, so the rules of experimental methodology don't apply to them.

It isn't known whether or not infinities occur in nature. Our universe may be infinite in size. Space may be continuous and infinitely divisible. Our universe may be one of an infinite number of universes in a multiverse (multiple universe interpretations of quantum physics). Physics uses infinities in describing many phenomena. Additionally, the rules of experimental methodology should apply to any system being studied, finite or infinite.

In conclusion, suppose a physicist submits a paper about the relationships found in a physical system but studies this system by destroying the built-in, inherent relationships within the system and then ignores the possibility that the results of this experimental processing may have been different than those that would have been obtained in the original system. I suggest that this paper would be, or should be, rejected. And, yet the mathematics of infinities is based on doing just this and is used extensively in physics. I suggest that reworking the mathematics of infinities to more closely align with physical systems would prove fruitful in overcoming some of the difficulties of using infinities in physics.

References

1. Webster's New World Dictionary and Thesaurus, Second Edition, 2002, Wiley Publishing, Inc.