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ON PARALLEL CURVES VIA PARALLEL TRANSPORT FRAME IN EUCLIDEAN 3-SPACE

MUHAMMED T. SARIAYDIN AND VEDAT ASIL

ABSTRACT. In this paper, we study the parallel curve of a space curve according to parallel transport frame. Then, we obtain new results according to some cases of this curve by using parallel transport frame in Euclidean 3-space. Additionally, we give new examples for this characterizations and we illustrate this examples in figures.

1. Introduction

To α is a curve in plane, there exist two curves $\mathcal{P}_+ = \alpha(s) + t\mathbf{e}_2(s)$ and $\mathcal{P}_- = \alpha(s) - t\mathbf{e}_2(s)$ at a given distance t. But these cuves is not easy to characterize in 3-dimensional space. Then, [3] developed a new construction. This construction is carried over the three-dimensional space and as a result, two parallel curves are obtained as well. Additionally study parallel helices in three-dimensional space.

Bishop frame, which is also called alternetive or parallel frame of the curves, was introduced by L.R. Bishop in 1975 by means of parallel vector fields. Recently, many research papers related to this concept have been treated in Euclidean space. For example, in [13] the outhors introduced a new version of Bishop frame and an application to spherical images and in [14] the outhors studied Minkowski space in $\mathbb{E}_1^3$.

In this paper, we obtain some characterizations about parallel curves by using Bishop frame in $\mathbb{E}^3$. Additionally, we give new examples for this characterizations and we illustrate this examples in figures.

2. **Background on parallel curves**

In plane let a smooth curve $\alpha(s) = (x(s), y(s))$, where s is the arc-length and its unit tangent and unit normal vectors are $\mathbf{e}_1(s)$ and $\mathbf{e}_2(s)$,respectively. Then, we get

$$\mathcal{P}_+(S_+) = \alpha(s) + t\mathbf{e}_2(s) \text{ and } \mathcal{P}_-(S_-) = \alpha(s) - t\mathbf{e}_2(s),$$

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where $S_{\pm} = S_{\pm}(s)$ and $S_{\pm}$ denotes the length along $\mathcal{P}\pm$, at the distance t. Determining the length $S_{\pm}$, we can write

$$\frac{dS_{\pm}}{ds} = 1 \pm t\kappa,$$

where κ is the curvature of $\alpha(s)$, [3,11].

Lemma 2.1. *Two curves $\alpha, \beta : I \to \mathbb{E}^3$ are parallel if their velocity vectors $\dot{\alpha}(s)$ and $\dot{\beta}(s)$ are parallel for each s. In this case, if $\alpha(s_0) = \beta(s_0)$ for some one s_0 in I then, $\alpha = \beta$,* [11].

Theorem 2.2. *If $\alpha, \beta : I \to \mathbb{E}^3$ are unit-speed curves such that $\kappa_\alpha = \kappa_\beta$ and $\tau_\alpha = \pm\tau_\beta$ then, α and β are congruent,* [11].

Denote by $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ the moving Frenet–Serret frame along the curve α in the space $\mathbb{E}^3$. For an arbitrary curve α with first and second curvature, κ and τ in the space $\mathbb{E}^3$, the following Frenet–Serret formulae is given

$$\begin{aligned}
\dot{\mathbf{e}}_1 &= \kappa \mathbf{e}_2, \\
\dot{\mathbf{e}}_2 &= -\kappa \mathbf{e}_1 + \tau \mathbf{e}_2, \\
\dot{\mathbf{e}}_3 &= -\tau \mathbf{e}_2,
\end{aligned}$$

where

$$\langle \mathbf{e}_1, \mathbf{e}_1 \rangle = \langle \mathbf{e}_2, \mathbf{e}_2 \rangle = \langle \mathbf{e}_3, \mathbf{e}_3 \rangle = 1,$$

$$\langle \mathbf{e}_1, \mathbf{e}_2 \rangle = \langle \mathbf{e}_1, \mathbf{e}_3 \rangle = \langle \mathbf{e}_2, \mathbf{e}_3 \rangle = 0.$$

Here, curvature functions are defined by $\kappa = \kappa(s) = \|\dot{\mathbf{e}}_1(s)\|$ and $\tau(s) = -\langle \mathbf{e}_2, \dot{\mathbf{e}}_3 \rangle$. Torsion of the curve α is given by the aid of the mixed product

$$\tau = \frac{(\dot{\alpha}, \ddot{\alpha}, \dddot{\alpha})}{\kappa^2}.$$

In the rest of the paper, we suppose everywhere $\kappa \neq 0$ and $\tau \neq 0$.

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. One can express parallel transport of an orthonormal frame along a curve simply by parallel transporting each component of the frame. The tangent vector and any convenient arbitrary basis for the remainder of the frame are used. The Bishop frame is expressed as

$$\begin{aligned}
\dot{\varepsilon}_1 &= \kappa_1 \varepsilon_2 + \kappa_2 \varepsilon_3, \\
\dot{\varepsilon}_2 &= -\kappa_1 \varepsilon_1, \\
\dot{\varepsilon}_2 &= -\kappa_2 \varepsilon_1,
\end{aligned}$$

where

$$\langle \varepsilon_1, \varepsilon_1 \rangle = \langle \varepsilon_2, \varepsilon_2 \rangle = \langle \varepsilon_3, \varepsilon_3 \rangle = 1, \tag{2.1}$$

$$\langle \varepsilon_1, \varepsilon_2 \rangle = \langle \varepsilon_1, \varepsilon_3 \rangle = \langle \varepsilon_2, \varepsilon_3 \rangle = 0. \tag{2.2}$$

Here, we shall call the set $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ as Bishop trihedra and κ_1 and κ_2 as Bishop curvatures. The relation matrix may be expressed as

$$\begin{aligned}\mathbf{e}_1 &= \varepsilon_1,\\ \mathbf{e}_2 &= \cos\theta(s)\varepsilon_2 + \sin\theta(s)\varepsilon_3,\\ \mathbf{e}_3 &= -\sin\theta(s)\varepsilon_2 + \cos\theta(s)\varepsilon_3,\end{aligned}$$

where $\theta(s) = \arctan\frac{\kappa_2}{\kappa_1}$, $\tau(s) = \dot{\theta}(s)$ and $\kappa(s) = \sqrt{\kappa_1^2 + \kappa_2^2}$. Here, Bishop curvatures are defined by

$$\begin{aligned}\kappa_1 &= \kappa(s)\cos\theta(s),\\ \kappa_2 &= \kappa(s)\sin\theta(s).\end{aligned}$$

On the other hand,

$$\begin{aligned}\varepsilon_1 &= \mathbf{e}_1,\\ \varepsilon_2 &= \cos\theta(s)\mathbf{e}_2 - \sin\theta(s)\mathbf{e}_3,\\ \varepsilon_3 &= \sin\theta(s)\mathbf{e}_2 + \cos\theta(s)\mathbf{e}_3.\end{aligned} \tag{2.3}$$

3. Parallel Curves in 3-Dimensional Space

Let a $\alpha : I \to \mathbb{E}^3$ be a regular curve with parametrized by arc-length. Take the derivative of $\alpha(s)$ in accordance with s, we obtain

$$(\alpha(s) - \mathcal{P})^2 = t^2, \tag{3.1}$$

$$\dot{\alpha}(s)(\alpha(s) - \mathcal{P}) = 0, \tag{3.2}$$

$$\ddot{\alpha}(s)(\alpha(s) - \mathcal{P}) + \dot{\alpha}(s)^2 = 0, \tag{3.3}$$

for the point $\mathcal{P}$. Equations (3.1), (3.2) and (3.3) have geometrical interpretation as follows: if (3.1) is regarded a spherical wave consisting of points $\mathcal{P}$ at the distance t from the moving point $\alpha(s)$, then (3.2) is the enveloping surface (employing the intersection of two infinitesimaly close waves) and (3.3) presents the double envelope (the intersection of three close waves, the multiple focus of the waves), [3].

Theorem 3.1. *Let $\alpha : I \to \mathbb{E}^3$ be a regular curve with parametrized by arc-length in 3-dimensional space. If $\mathcal{P}$ is a parallel curve of α, then*

$$\mathcal{P}_{\pm} = \alpha + \left(\frac{1}{\kappa_1} - \frac{2\kappa_2\tan\theta - \kappa_1\kappa_2 C}{2\kappa_1^2\sec^2\theta}\right)\varepsilon_2 \pm \left(\frac{-2\tan\theta + \kappa_1 C}{2\kappa_1\sec^2\theta}\right)\varepsilon_3,$$

where

$$C = \sqrt{4t^2\sec^2\theta - \frac{4}{\kappa_1^2}}.$$

Proof. Considering the (2.2) and (3.2) equations, we obtain

$$\alpha - \mathcal{P} = \mu\varepsilon_2 + \eta\varepsilon_3, \tag{3.4}$$

where μ and η are appropriate coefficients. From (3.3) and equations

$$\dot{\alpha}^2 = \varepsilon_1^2 = 1 \text{ and } \ddot{\alpha} = \dot{\varepsilon}_1 = \kappa_1\varepsilon_2 + \kappa_2\varepsilon_3.$$

Then, we can write

$$(\kappa_1\varepsilon_2+\kappa_2\varepsilon_3)(\mu\varepsilon_2+\eta\varepsilon_3)+1=0,$$

$$\mu\kappa_1+\eta\kappa_2+1=0. \tag{3.5}$$

Also, if we write at (3.4) instead of the left side of (3.1) then, we get

$$\mu^2+\eta^2=t^2. \tag{3.6}$$

Then from (3.5), we have

$$\mu=-\frac{1}{\kappa_1}-\eta\tan\theta. \tag{3.7}$$

Considering together (3.6) and (3.7) equations, we get

$$\eta_1=\frac{-2\tan\theta+\kappa_1 C}{2\kappa_1\sec^2\theta} \text{ and } \eta_2=\frac{-2\tan\theta-\kappa_1 C}{2\kappa_1\sec^2\theta}, \tag{3.8}$$

where

$$C=\sqrt{4t^2\sec^2\theta-\frac{4}{\kappa_1^2}}.$$

In the rest of the paper, we suppose everywhere

$$\eta=\eta_1. \tag{3.9}$$

And so, we get

$$\mu=-\frac{1}{\kappa_1}+\frac{2\kappa_2\tan\theta-\kappa_1\kappa_2 C}{2\kappa_1^2\sec^2\theta}. \tag{3.10}$$

After simple computation, we get

$$\mathcal{P}_\pm=\alpha+\zeta\varepsilon_2\pm\xi\varepsilon_3, \tag{3.11}$$

where

$$\zeta=(\frac{1}{\kappa_1}-\frac{2\kappa_2\tan\theta-\kappa_1\kappa_2 C}{2\kappa_1^2\sec^2\theta}) \text{ and } \xi=(\frac{2\tan\theta-\kappa_1 C}{2\kappa_1\sec^2\theta}).$$

Corallary 3.2. *If* $\kappa_2=0$, *then*

$$\mathcal{P}_\pm=\alpha+\frac{1}{\kappa_1}\varepsilon_2\pm\frac{C}{2}\varepsilon_3,$$

where

$$C=\sqrt{4t^2-\frac{4}{\kappa_1^2}}.$$

Example 3.3. Let us consider a unit speed circular helix in $\mathbb{E}^3$ by

$$\alpha=\alpha(s)=(24\cos\frac{s}{25},24\sin\frac{s}{25},\frac{7s}{25}).$$

One can calculate its Frenet-Serret apparatus as the following

$$\mathbf{e}_1(s)=\frac{1}{25}(-24\sin\frac{s}{25},24\cos\frac{s}{25},7),$$

$$\mathbf{e}_2(s)=(-\cos\frac{s}{25},-\sin\frac{s}{25},0),$$

$$\mathbf{e}_3(s)=\frac{1}{25}(7\sin\frac{s}{25},-7\cos\frac{s}{25},24).$$

Then, the curvatures of α is given by

$$\kappa(s) = \frac{24}{625},$$
$$\tau(s) = \frac{7}{625}.$$

Putting,

$$\theta(s) = \frac{7s}{625},$$

where $\theta(s) = \int_0^s \tau(s)\,ds$, [8]. Then, we can write the Bishop frame from (2.3) by

$$\varepsilon_1(s) = \frac{1}{25}(-24\sin\frac{s}{25}, 24\cos\frac{s}{25}, 7),$$
$$\varepsilon_2(s) = (-\cos\frac{7s}{625}\cos\frac{s}{25} - \frac{7}{25}\sin\frac{7s}{625}\sin\frac{s}{25},$$
$$-\cos\frac{7s}{625}\sin\frac{s}{25} + \frac{7}{25}\sin\frac{7s}{625}\cos\frac{s}{25}, -\frac{24}{25}\sin\frac{7s}{625}),$$
$$\varepsilon_3(s) = (-\sin\frac{7s}{625}\cos\frac{s}{25} + \frac{7}{25}\cos\frac{7s}{625}\sin\frac{s}{25},$$
$$-\sin\frac{7s}{625}\sin\frac{s}{25} - \frac{7}{25}\cos\frac{7s}{625}\cos\frac{s}{25}, +\frac{24}{25}\cos\frac{7s}{625}).$$

Also, the curvatures of α

$$\kappa_1(s) = \frac{24}{625}\cos\frac{7s}{625},$$
$$\kappa_2(s) = \frac{24}{625}\sin\frac{7s}{625}.$$

From (3.9), we get

$$\eta = -\frac{625}{24}\sin\frac{7s}{625} + \frac{1}{2}\cos^2\frac{7s}{625}\sqrt{(4t^2 - \frac{390625}{144})\sec^2\frac{7s}{625}}. \tag{3.12}$$

Therefore, by using (3.7) equation, we have

$$\mu = -\frac{625}{24}\cos\frac{7s}{625} - \frac{1}{2}\cos\frac{7s}{625}\sin\frac{7s}{625}\sqrt{\left(4t^2 - \frac{390625}{144}\right)\sec^2\frac{7s}{625}}. \tag{3.13}$$

After simple computation, we get

$$\mathcal{P}_{\pm} = (24\cos\frac{s}{25}, 24\sin\frac{s}{25}, \frac{7s}{25}) + \zeta\varepsilon_2 \pm \xi\varepsilon_3,$$

where

$$\zeta = \frac{625}{24}\cos\frac{7s}{625} + \frac{1}{2}\cos\frac{7s}{625}\sin\frac{7s}{625}\sqrt{(4t^2 - \frac{390625}{144})\sec^2\frac{7s}{625}}$$

and

$$\xi = -\frac{625}{24}\sin\frac{7s}{625} + \frac{1}{2}\cos^2\frac{7s}{625}\sqrt{(4t^2 - \frac{390625}{144})\sec^2\frac{7s}{625}}.$$

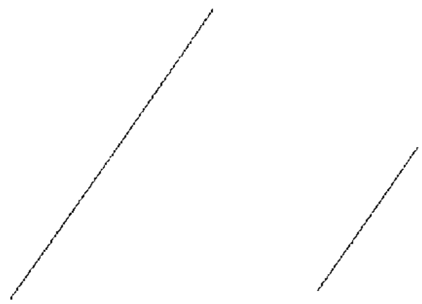

Figure 3.1, $0 \leq s \leq \pi/6$

Example 3.4. Let us consider a unit speed curve in $\mathbb{E}^3$ by, [10],

$$\alpha = \alpha(s) = (\frac{9}{208}\sin 16s - \frac{1}{117}\sin 36s,\ -\frac{9}{208}\cos 16s + \frac{1}{117}\cos 36s, \frac{6}{65}\sin 10s).$$

The transformation matrix for the curve $\alpha = \alpha(s)$ has the form

$$\begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(24\sin 10s) & \sin(24\sin 10s) \\ 0 & -\sin(24\sin 10s) & \cos(24\sin 10s) \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{M}_1 \\ \mathbf{M}_2 \end{bmatrix},$$

where

$$\theta(s) = \int_0^s 24\cos(10s) = \frac{24}{10}\sin 10s.$$

Then, we can give figure of this curve as

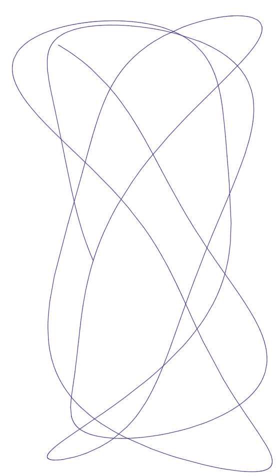

Figure 3.2, $0 \leq s \leq 1$

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Muş Alparslan University, Department of Mathematics,49250, Muş, Turkey, Firat University, Department of Mathematics,23119, Elaziğ, Turkey

E-mail address: talatsarıaydın@gmail.com, vedatasil@gmail.com