n-FOLD FILTERS IN SMARANDACHE RESIDUATED LATTICES, PART (II)

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Abstract. In this paper we introduce the notions of n-fold B_L-Smarandache n-fold B_L-Smarandache fantastic filter and n-fold B_L-Smarandache easy filter in Smarandache residuated lattices and study the relations among them. And we also introduce the notions of n-fold Smarandache n-fold Smarandache fantastic B_L-residuated lattice and n-fold Smarandache easy B_L-residuated lattice and investigate its properties.

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1. Introduction

BL-algebras (basic logic algebras) are the algebraic structures for Hájek basic logic [4], in order to investigate many valued logic by algebraic means. Residuated lattices play an important role in the study of fuzzy logic and filters are basic concepts in residuated lattices and other algebraic structures. A Smarandache structure on a set L means a weak structure W on L such that there exists a proper subset B of L which is embedded with a strong structure S. In [11], Vasantha Kandasamy studied the concept of Smarandache groupoids, subgroupoids, ideal of groupoids and strong Bol groupoids and obtained many interesting results about them. It will be very interesting to study the Smarandache structure in these algebraic structures. A BL-algebra is a weaker structure than residuated lattice, then we can consider in any residuated lattice a weaker structure as BL-algebra.

The concept of Smarandache residuated lattice, Smarandache (positive) implicative filters and Smarandache fantastic filters defined in [1]. In [5, 9] the authors defined the notion of n-fold (positive) implicative filters, n-fold fantastic filters, n-fold obstinate filters in BL-algebras and studied the relation among many type of n-fold filters in BL-algebra. The aim of this paper is to extend this research to Smarandache residuated lattices.

2. **Preliminaries**

[3] A residuated lattice is an algebra $L = (L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ equipped with an order $\leq$ satisfying the following:

(LR_1) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice,

(LR_2) $(L, \odot, 1)$ is a commutative ordered monoid,

(LR_3) $\odot$ and $\rightarrow$ form an adjoint pair i.e, $c \leq a \rightarrow b$ if and only if $a \odot c \leq b$ for all $a, b, c \in L$.

A BL-algebra is a residuated lattice L if satisfying the following identity, for all $a, b \in L$:

(BL_1) $(a \rightarrow b) \vee (b \rightarrow a) = 1$,

(BL_2) $a \wedge b = a \odot (a \rightarrow b)$.

Theorem 2.1. [3, 6, 7, 8] *Let L be a residuated lattice. Then the following properties hold, for all $x, y, z \in L$:*

(lr_1) $1 \rightarrow x = x$, $x \rightarrow x = 1$,

(lr_2) $x \rightarrow y \leq (z \rightarrow x) \rightarrow (z \rightarrow y) \leq z \rightarrow (x \rightarrow y)$,

(lr_3) $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$ *and* $(x \rightarrow y) \odot (y \rightarrow z) \leq x \rightarrow z$,

(lr_4) $x \leq y \Leftrightarrow x \rightarrow y = 1$, $x \leq y \rightarrow x$,

(lr_5) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z) = (x \odot y) \rightarrow z$,

(lr_6) $x \odot (x \rightarrow y) \leq y, x \leq y \rightarrow (x \odot y)$ *and* $y \leq (y \rightarrow x) \rightarrow x$,

(lr_7) If $x \leq y$, *then* $y \rightarrow z \leq x \rightarrow z$, $z \rightarrow x \leq z \rightarrow y$ *and* $y^* \leq x^*$,

(lr_8) $x \leq y$ *and* $z \leq w$ *then* $x \odot z \leq y \odot w$,

(lr_9) $x \leq x^{**}$, $x^{***} = x^*$,

(lr_{10}) $x^* \odot y^* \leq (x \odot y)^*$ *(so,* $(x^*)^n \leq (x^n)^*$ *for every* $n \geq 1$*),*

(lr_{11}) $x^{**} \odot y^{**} \leq (x \odot y)^{**}$ *(so,* $(x^{**})^n \leq (x^n)^{**}$ *for every* $n \geq 1$*).*

The following definitions and theorems are stated from [1].

A Smarandache B_L-residuated lattice is a residuated lattice L in which there exists a proper subset B of L such that:

(1) $0, 1 \in B$ and $|B| > 2$,

(2) B is a BL- algebra under the operations of L.

From now on $L_B = (L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a Smarandache B_L-residuated lattice and $B = (B, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a BL- algebra unless otherwise sepcified. A nonempty subset F of L_B is called a B_L-Smarandache deductive system of L_B if $1 \in F$ and if $x \in F$, $y \in B$, $x \rightarrow y \in F$ imply $y \in F$. A nonempty subset F of L_B is called a B_L-Smarandache filter of L_B if $x, y \in F$ imply $x \odot y \in F$ and if $x \in F$, $y \in B$ and $x \leq y$ imply $y \in F$.

Theorem 2.2. *(i) Let F be a B_L-Smarandache filter of L_B, then F is a B_L-Smarandache deductive system of L_B.*

(ii) Let F be a B_L-Smarandache deductive system of L_B. If $F \subseteq B$, then F is a B_L-Smarandache filter of L_B.

A filter F of a residuated lattice L is called an easy filter if $x^{**} \to (y \to z) \in F$ and $x^{**} \to y \in F$ imply $x^{**} \to z \in F$, for all $x, y, z \in L$, [2].

Definition 2.3. [10] Let F be a subset of Smarandache B_L-residuated lattice L_B and $1 \in F$.

(1) F is called an n-fold B_L-Smarandache positive implicative filter of L_B if $x^n \to (y \to z) \in F$ and $x^n \to y \in F$, then $x^n \to z \in F$, for all $x, y, z \in B$.

(2) F is called an n-fold B_L-Smarandache positive implicative filter of L_B if $x \in F$ and $x \to ((y^n \to z) \to y) \in F$ then $y \in F$, for all $y, z \in B$.

Definition 2.4. [10] Let L_B be a Smarandache B_L-residuated lattice.

(1) L_B is called n-fold Smarandache positive implicative B_L-residuated lattice if $x^{n+1} = x^n$ for all $x \in B$.

(2) L_B is called n-fold Smarandache implicative B_L-residuated lattice if $(x^n)^* \to x = x$, for each $x \in B$.

Theorem 2.5. *Let F be a B_L-Smarandache deductive system of L_B. Then for all $x, y, z \in B$, the following conditions are equivalent:*

(i) F is an n-fold B_L-Smarandache positive implicative filter of L_B,

(ii) $x^n \to x^{2n} \in F$, for all $n \in N$.

Theorem 2.6. *Let F be a B_L-Smarandache deductive system of L_B. The following conditions are equivalent:*

(i) L_B is an n-fold Smarandache positive implicative B_L-residuated lattice,

(ii) every B_L-Smarandache deductive system of L_B is an n-fold B_L-Smarandache positive implicative filter of L_B.

Theorem 2.7. *The following conditions are equivalent for B_L-Smarandache deductive system F of L_B.*

(i) F is an n-fold B_L-Smarandache implicative filter,

(ii) for all $x, y \in B$, $(x^n \to y) \to x \in F$ implies $x \in F$.

Proposition 2.8. *The following conditions are equivalent:*

(i) L_B is an n-fold Smarandache implicative B_L-residuated lattice.

(ii) Every n-fold B_L-Smarandache deductive system F of L_B is an n-fold B_L-Smarandache implicative filter of L_B.

Theorem 2.9. *Let every Smarandache deductive system be a Smarandache filter. Then every n-fold Smarandache implicative filter is n-fold Smarandache positive implicative filter.*

Now, unless mentioned otherwise, $n \geq 1$ will be an integer.

3. n-Fold B_L-Smarandache Fantastic Filters

Definition 3.1. A subset F of L_B is called an n-fold Smarandache fantastic filter of L_B related to B (or briefly n-fold B_L-Smarandache fantastic filter of L_B) if it satisfies in the following conditions:

(i) $1 \in F$

(ii) for all $x, y \in B$, $z \to (y \to x) \in F$ and $z \in F$ imply $((x^n \to y) \to y) \to x \in F$.

Example 3.2. Let $L = \{0, a, b, c, d, 1\}$ be a residuated lattice such that $0 < a < c < d < 1$ and $0 < b < c < d < 1$. We define

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	c	1	1	1
b	c	c	1	1	1	1
c	c	c	c	1	1	1
d	0	a	b	c	1	1
1	0	a	b	c	d	1

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	0	a	a
b	0	0	0	0	b	b
c	0	0	0	0	c	c
d	0	a	b	c	d	d
1	0	a	b	c	d	1

We can see that $L = (L, \wedge, \vee, \odot, \to, 0, 1)$ is a residuated lattice, in which $B = \{0, a, c, 1\}$ is a BL-algebra which properly contained in L. Then L is a Smarandache B_L-residuated lattice. $F = \{d, 1\}$ is an n-fold B_L-Smarandache fantastic filter of L_B, for $n \geq 2$, while it is not 1-fold B_L-Smarandache fantastic filter. Since $a, 0 \in B$ and $d, d \to (0 \to a) = 1 \in F$ but $((a \to 0) \to 0) \to a = c \notin F$.

Theorem 3.3. *Every n-fold B_L-Smarandache fantastic filter of L_B is a B_L-Smarandache deductive system of L_B.*

Proof. Let F be an n-fold B_L-Smarandache fantastic filter and $y, y \to x \in F$. We have $y \to x = y \to (1 \to x) \in F$. Then by Definition 3.1 we get $((x^n \to 1) \to 1) \to x \in F$, i.e. $x \in F$. □

Theorem 3.4. *Let F be a B_L-Smarandache deductive system of L_B. Then the following conditions are equivalent:*
(i) F is an n-fold B_L-Smarandache fantastic filter.
(ii) If $y \to x \in F$ implies $((x^n \to y) \to y) \to x \in F$, for all $x, y \in B$.

Proof. The proof by the Definition 3.1, is clear. □

Proposition 3.5. *Any n-fold B_L-Smarandache fantastic filter is an $(n+1)$-fold B_L-Smarandache fantastic filter.*

Proof. Let F be an n-fold B_L-Smarandache fantastic filte and $y \to x \in F$. Hence $((x^n \to y) \to y) \to x \in F$. By Theorem 2.1 we have $x^{n+1} \leq x^n$. Then $((x^n \to y) \to y) \to x \leq ((x^{n+1} \to y) \to y) \to x$. Therefore $((x^{n+1} \to y) \to y) \to x \in F$, i.e. F is an $(n+1)$-fold B_L-Smarandache fantastic filter. □

Proposition 3.6. *n-fold Smarandache implicative filters are n-fold Smarandache fantastice filters.*

Proof. Assume that F is an n-fold B_L-Smarandache implicative filter. Let $y \to x \in F$, where $x, y \in B$. By Theorem 2.1, we have $x \leq (((x^n \to y) \to y) \to x)$. Then $x^n \leq (((x^n \to y) \to y) \to x)^n$. Hence $(x^n \to y) \geq (((x^n \to$

$y) \to y) \to x)^n \to y$, (I). By (lr_2) we have $y \to x \leq ((x^n \to y) \to y) \to ((x^n \to y) \to x)$, (II). We also have

$$((x^n \to y) \to y) \to ((x^n \to y) \to x) = (x^n \to y) \to (((x^n \to y) \to y) \to x).$$

So, by (II) we get $y \to x \leq (x^n \to y) \to (((x^n \to y) \to y) \to x)$. (III)
Then by (I) we get $((x^n \to y) \to (((x^n \to y) \to y) \to x)) \leq ((((x^n \to y) \to y) \to x)^n \to y) \to ((((x^n \to y) \to y) \to x))$. Hence by (III) we obtain $y \to x \leq ((((x^n \to y) \to y) \to x)^n \to y) \to ((((x^n \to y) \to y) \to x))$. Since $y \to x \in F$, we get $((((x^n \to y) \to y) \to x)^n \to y) \to ((((x^n \to y) \to y) \to x)) \in F$. Then by Theorem 2.7, we obtain $((x^n \to y) \to y) \to x \in F$. Therefore F is an n-fold B_L-Smarandache fantastic filter.

□

In the following example, we show that the converse of Propositon 3.6 is not true in general.

Example 3.7. Let $L = \{0, a, b, 1\}$, where $0 < a < b < 1$. Define $*$ and $\to$ as follows:

$*$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

Then $(L, \wedge, \vee, *, \to, 0, 1)$ is a residuated lattice and $B = \{0, a, 1\}$ is a BL-algebra, which properly contained in L. Then L is a Smarandache B_L-residuated lattice. $F = \{b, 1\}$ is an n-fold B_L-Smarandache fantastic filter of L_B, while it is not 1-fold B_L-Smarandache implicative filter. Since $(a \to 0) \to a \in F$ but $a \notin F$.

Theorem 3.8. *Let every B_L-Smarandache deductive system of L_B be a B_L-Smarandache filter of L_B and F be a B_L-Smarandache deductive system of L_B. Then The following statements are equivalent:*
(i) F is an n-fold B_L-Smarandache fantastic filter and n-fold B_L-Smarandache positive implicative filter of L_B.
(ii) F is an n-fold B_L-Smarandache implicative filter of L_B.

Proof. $(i) \Rightarrow (ii)$ Let $x, y \in B$ such that $(x^n \to y) \to x \in F$. Since F is an n-fold B_L-Smarandache fantastic filter, by the fact that $(x^n \to y) \to x \in F$, we have $((x^n \to (x^n \to y)) \to (x^n \to y)) \to x \in F$. By Theorem 2.1 we get

$$\begin{aligned} x^n \to x^{2n} &\leq (x^{2n} \to y) \to (x^n \to y) \\ &= ((x^n \odot x^n) \to y) \to (x^n \to y) \\ &= (x^n \to (x^n \to y)) \to (x^n \to y). \end{aligned}$$

Hence $(x^n \to x^{2n}) \to x \geq ((x^n \to (x^n \to y)) \to (x^n \to y)) \to x$. Since F is a B_L-Smarandache filter, by the fact that $((x^n \to (x^n \to y)) \to (x^n \to y)) \to x \in F$, we have $(x^n \to x^{2n}) \to x \in F$. Since F is an n-fold B_L-Smarandache positive implicative filter, by Theorem 2.5, we get $x^n \to x^{2n} \in F$ and so $x \in F$. By Theorem 2.7 F is an n-fold B_L-Smarandache implicative filter.

$(ii) \Rightarrow (i)$ By Theorem 2.9 and Propositon 3.6, the proof is clear. □

Definition 3.9. L_B is said to be n-fold Smarandache fantastic B_L-residuated lattice if for all $x, y \in B$, $y \to x = ((x^n \to y) \to y) \to x$.

Example 3.10. (i) Let $L = \{0, a, b, c, d, 1\}$ be a residuated lattice such that $0 < b < a < 1$ and $0 < d < a, c < 1$. We define

$\odot$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	a	b	b	d	0	0
b	b	b	b	0	0	0
c	c	d	0	c	d	0
d	d	0	0	d	0	0
0	0	0	0	0	0	0

$\to$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
0	1	1	1	1	1	1

Then $(L, \wedge, \vee, \odot, \to, 0, 1)$ is a residuated lattice, in which $B = \{0, b, c, 1\}$ is a BL-algebra which properly contained in L. Then L is a Smarandache B_L-residuated lattice. L_B is an n-fold Smarandache fantastic B_L-residuated lattice.

(ii) In Example 3.2, L_B is not a 1-fold Smarandache fantastic B_L-residuated lattice. Since $((a \to 0) \to 0) \to a \neq 0 \to a$.

Proposition 3.11. *The following statements are equivalent:*

(i) L_B is an n-fold Smarandache fantastic B_L-residuated lattice;

(ii) The inequality $(x^n \to y) \to y \leq (y \to x) \to x$ holds for all $x, y \in B$.

Proof. $(i) \Rightarrow (ii)$ Assume that L_B is an n-fold Smarandache fantastic B_L-residuated lattice. Let $x, y \in B$. We have
$((x^n \to y) \to y) \to ((y \to x) \to x) = (y \to x) \to (((x^n \to y) \to y) \to x)$.
By hypothesis $y \to x = ((x^n \to y) \to y) \to x$. Hence $(y \to x) \to (((x^n \to y) \to y) \to x) = 1$. Then we get $((x^n \to y) \to y) \to ((y \to x) \to x) = 1$ or equivalently $((x^n \to y) \to y) \leq ((y \to x) \to x)$.

$(ii) \Rightarrow (i)$ Suppose that the inequality $(x^n \to y) \to y \leq (y \to x) \to x$ holds. Then
$(y \to x) \to (((x^n \to y) \to y) \to x) = ((x^n \to y) \to y) \to ((y \to x) \to x)$, (I).
Since $(x^n \to y) \to y \leq (y \to x) \to x$, we get
$((x^n \to y) \to y) \to ((x^n \to y) \to y) \leq ((x^n \to y) \to y) \to ((y \to x) \to x)$,
that is $((x^n \to y) \to y) \to ((y \to x) \to x) = 1$. Then $(y \to x) \to ((x^n \to y) \to y) \to x) = 1$. Its follows that $(y \to x) \leq ((x^n \to y) \to y) \to x$.

Since $y \leq (x^n \to y) \to y$, we also get $(y \to x) \geq ((x^n \to y) \to y) \to x$. Therefore we obtain $(y \to x) = ((x^n \to y) \to y) \to x$. Hence L_B is an n-fold Smarandache fantastic B_L-residuated lattice.

□

Proposition 3.12. *The following conditions are equivalent:*

(i) L_B is an n-fold Smarandache fantastic B_L-residuated lattice.

(ii) Every B_L-Smarandache deductive system of L_B is an n-fold B_L-Smarandache fantastic filter of L_B.

(iii) $((x^n \to y) \to y) \to ((y \to x) \to x) \in F$, for every B_L-Smarandache deductive system F of L_B.

(iv) $\{1\}$ is an n-fold B_L-Smarandache fantastic filter of L_B.

Proof. $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iv)$ The proofs are clear.

$(iv) \Rightarrow (i)$ Assume that $\{1\}$ is an n-fold B_L-Smarandache fantastic filter. Let $x, y \in B$ and $t = (y \to x) \to x$. By Theorem 2.1, $y \leq t$. So $y \to t = 1$ and by the hypothesis, we have $((t^n \to y) \to y) \to t = 1$, that is $(t^n \to y) \to y \leq t$, (I). On the other hand, $x \leq t$ implies $x^n \leq t^n$, hence $(x^n \to y) \to y \leq (t^n \to y) \to y$. Then by (I) it follows that $(x^n \to y) \to y \leq t = (y \to x) \to x$. Hence by Proposition 3.11, L_B is an n-fold Smarandache fantastic B_L-residuated lattice.

$(i) \Rightarrow (iii)$ By Proposition 3.11, $((x^n \to y) \to y) \to ((y \to x) \to x) = 1 \in F$, for every B_L-Smarandache deductive system F of L_B.

$(iii) \Rightarrow (iv)$ By (iii), $((x^n \to y) \to y) \to ((y \to x) \to x) \in \{1\}$. Hence $((x^n \to y) \to y) \leq ((y \to x) \to x)$. So by Proposition 3.11, L_B is an n-fold Smarandache fantastic B_L-residuated lattice. Let $y \to x \in \{1\}$. By the fact that L_B is an n-fold Smarandache fantastic B_L-residuated lattice we get $((x^n \to y) \to y) \to x \in \{1\}$, i.e. $\{1\}$ is an n-fold B_L-Smarandache fantastic filter of L_B.

□

Corollary 3.13. *Let L_B is an n-fold Smarandache fantastic B_L-residuated lattice and F be a B_L-Smarandache deductive system of L_B. Then $(x^n)^{**} \to x \in F$, for all $x \in B$.*

Proof. In Proposition 3.12, take $y = 0$. Thus the proof is clear. □

Corollary 3.14. *Let every B_L-Smarandache deductive system of L_B be a B_L-Smarandache filter of L_B and F be a B_L-Smarandache deductive system of L_B. Then the following conditions are equivalent:*

(i) L_B is an n-fold Smarandache implicative B_L-residuated lattice.

(ii) L_B is an n-fold Smarandache fantastic B_L-residuated lattice and n-fold Smarandache positive implicative B_L-residuated lattice.

Proof. By Theorem 2.2(i), Propositions 3.12, 2.8, 2.6 and Theorem 3.8, the proof is easy. □

Corollary 3.15. *Let F be a B_L-Smarandache deductive system of L_B. Then the following statements are equivalent:*

(i) F is an n-fold B_L-Smarandache fantastic filter of L_B.
(ii) L_B/F is an n-fold Smarandache fantastic $B_{L/F}$-residuated lattice.

Proof. $(i) \Rightarrow (ii)$ Assume that F is an n-fold B_L-Smarandache fantastic filter of L_B. Let $x, y \in B$ be such that $y/F \to x/F \in \{1/F\}$, then $(y \to x)/F = 1/F$ or equivalently $y \to x \in F$. Since F is an n-fold B_L-Smarandache fantastic filter, we get $((x^n \to y) \to y) \to x \in F$ or equivalently $(((x^n \to y) \to y) \to x)/F = 1/F$, so $((((x/F)^n \to y/F) \to y/F) \to x/F) \in \{1/F\}$, for all $x/F \in B/F$. Hence $\{1/F\}$ is an n-fold $B_{L/F}$-Smarandache fantastic filter of L_B/F, therefore by Proposition 3.12, L_B/F is an n-fold Smarandache fantastic $B_{L/F}$-residuated lattice.

$(ii) \Rightarrow (i)$ Assume that L_B/F is an n-fold Smarandache fantastic $B_{L/F}$-residuated lattice. Let $x, y \in B$ be such that $y \to x \in F$ then $(y \to x)/F = 1/F$ or equivalently $y/F \to x/F \in \{1/F\}$. Since L_B/F is an n-fold Smarandache fantastic $B_{L/F}$-residuated lattice, by Proposition 3.12, $\{1/F\}$ is an n-fold $B_{L/F}$-Smarandache fantastic filter of L_B/F. From this and the fact that $y/F \to x/F \in \{1/F\}$, we have $((((x/F)^n \to y/F) \to y/F) \to x/F) \in \{1/F\}$ or equivalently $(((x^n \to y) \to y) \to x)/F = 1/F$, so $((x^n \to y) \to y) \to x \in F$. Hence F is an n-fold B_L-Smarandache fantastic filter of L_B.

□

Theorem 3.16. *Let F and G be two B_L-Smarandache deductive system of L_B such that $F \subseteq G$. If F is an n-fold B_L-Smarandache fantastic filter, then so is G.*

Proof. Let $x, y \in B$ be such that $y \to x \in G$. Since F is an n-fold B_L-Smarandache fantastic filter, by Corollary 3.15 , L_B/F is an n-fold Smarandache fantastic $B_{L/F}$-residuated lattice. So $((((x/F)^n \to y/F) \to y/F) \to x/F) = y/F \to x/F$, hence $(y \to x) \to (((x^n \to y) \to y) \to x) \in F$, so $(y \to x) \to (((x^n \to y) \to y) \to x) \in G$. By the fact that $y \to x \in G$ we get $(((x^n \to y) \to y) \to x) \in G$. Hence G is an n-fold B_L-Smarandache fantastic filter of L_B. □

4. n-Fold B_L-Smarandache Easy Filters

Definition 4.1. A subset F of L_B is called an n-fold Smarandache easy filter of L_B related to B (or briefly n-fold B_L-Smarandache easy filter of L_B) if $(x^{**})^n \to (y \to z) \in F$ and $(x^{**})^n \to y \in F$ imply $(x^{**})^n \to z \in F$, for all $x, y, z \in B$.

Example 4.2. In Example 3.2, $F = \{d, 1\}$ is an n-fold B_L-Smarandache easy filter, for $n \geq 2$. F is not a 1-fold B_L-Smarandache easy filter. Since $a^{**} \to (c \to a) = 1 \in F$ and $a^{**} \to c = 1 \in F$ but $a^{**} \to a = c \notin F$.

Theorem 4.3. *Every n-fold B_L-Smarandache easy filter of L_B is a B_L-Smarandache deductive system of L_B.*

Proof. Let F be an n-fold B_L-Smarandache easy filter of L_B. Suppose $z \in B$, such that $y, y \to z \in F$. We have $(1^{**})^n \to y, (1^{**})^n \to (y \to z) \in F$, these imply $z = (1^{**})^n \to z \in F$. Hence F is a B_L-Smarandache deductive system of L_B. □

Theorem 4.4. *Every n-fold B_L-Smarandache positive implicative filter of L_B is an n-fold B_L-Smarandache easy filter of L_B.*

Proof. Let F be an n-fold B_L-Smarandache positive implicative filter of L_B and $x, y, z \in B$ such that $(x^{**})^n \to (y \to z) \in F$ and $(x^{**})^n \to y \in F$. So by Definition 2.3(i), $(x^{**})^n \to z \in F$. So F is an n-fold B_L-Smarandache easy filter of L_B. □

Example 4.5. Let $L = \{0, a, b, c, 1\}$, where $0 < a < b < c < 1$. Define $\odot$ and $\to$ as follows:

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	a	a	b
c	0	a	a	b	c
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	c	1	1	1
c	0	b	c	1	1
1	0	a	b	c	1

We can see that $L = (L, \wedge, \vee, \odot, \to, 0, 1)$ is a residuated lattice, in which $B = \{0, a, 1\}$ is a BL-algebra which properly contained in L. Then L is a Smarandache B_L-residuated lattice. $F = \{1\}$ is an n-fold B_L-Smarandache easy filter of L_B, while is not 1-fold and 2-fold B_L-Smarandache positive implicative filter. Since $c^1 \to (c \to c^2) = 1 \in F$, but $c^1 \to c^2 = c \notin F$. Also $c^2 \to (c \to c^4) = 1 \in F$ and $c^2 \to c = 1 \in F$, but $c^2 \to c^4 = c \notin F$.

Theorem 4.6. *Let F be a B_L-Smarandache filter of L_B. The following conditions are equivalent:*

(i) F is an n-fold B_L-Smarandache easy filter of L_B;

(ii) $(x^{**})^n \to (y \to z) \in F$ implies $((x^{**})^n \to y) \to ((x^{**})^n \to z) \in F$, for all $x, y, z \in B$;

(iii) $(x^{**})^n \to ((x^{**})^n \to z) \in F$ implies $(x^{**})^n \to z \in F$, for all $x, z \in B$;

(iv) $(x^{**})^n \to (x^{**})^{2n} \in F$, for all $x \in B$.

Proof. $(i) \Rightarrow (ii)$ Let F be an n-fold B_L-Smarandache easy filter and $x, y, z \in B$ such that $(x^{**})^n \to (y \to z) \in F$. By Theorem 2.1 we have $(x^{**})^n \to ((x^{**})^n \to (((x^{**})^n \to y) \to z)) = (x^{**})^n \to (((x^{**})^n \to y) \to ((x^{**})^n \to z)) \geq (x^{**})^n \to (y \to z)$. Hence $(x^{**})^n \to ((x^{**})^n \to (((x^{**})^n \to y) \to z)) \in F$. From $(x^{**})^n \to (x^{**})^n = 1 \in F$ and the hypothesis we get $(x^{**})^n \to (((x^{**})^n \to y) \to z) \in F$. By Theorem 2.1, we obtain $((x^{**})^n \to y) \to ((x^{**})^n \to z) \in F$.

$(ii) \Rightarrow (iii)$ Take $y = (x^{**})^n$, hence the proof is clear.

$(iii) \Rightarrow (i)$ Let $(x^{**})^n \to (y \to z) \in F$ and $(x^{**})^n \to y \in F$. By Theorem 2.1 we have $(x^{**})^n \to (y \to z) = y \to ((x^{**})^n \to z)$. So $((x^{**})^n \to y) \to ((x^{**})^n \to ((x^{**})^n \to z)) \in F$. Since $(x^{**})^n \to y \in F$ and F is an n-fold B_L-Smarandache filter, $((x^{**})^n \to ((x^{**})^n \to z)) \in F$. By Theorem 2.1 we have $(x^{**})^n \to ((x^{**})^n \to z)) = (x^{**})^{2n} \to z \in F$. By (ii), $(x^{**})^n \to (x^{**})^{2n} \in F$. So $((x^{**})^n \to (x^{**})^{2n}) \odot ((x^{**})^{2n} \to z) \in F$. By Theorem 2.1, $((x^{**})^n \to (x^{**})^{2n}) \odot ((x^{**})^{2n} \to z) \leq (x^{**})^n \to z$. So $(x^{**})^n \to z \in F$. Therefore F is an n-fold B_L-Smarandache easy filter.

$(iii) \Rightarrow (iv)$ For $x \in B$, we have $(x^{**})^n \to ((x^{**})^n \to (x^{**})^{2n}) = (x^{**})^{2n} \to (x^{**})^{2n} = 1 \in F$. So by (iii), $(x^{**})^n \to (x^{**})^{2n} \in F$.

$(iv) \Rightarrow (iii)$ Let $x, y \in B$ such that $x^{**})^n \to ((x^{**})^n \to y \in F$, so $(x^{**})^{2n} \to y \in F$. By (iv), $(x^{**})^n \to (x^{**})^{2n} \in F$. By Theorem 2.1, $(x^{**})^n \to (x^{**})^{2n} \leq ((x^{**})^{2n} \to y) \to ((x^{**})^n \to y)$. Since F is a B_L-Smarandache filter, $((x^{**})^{2n} \to y) \to ((x^{**})^n \to y) \in F$, so $(x^{**})^n \to y \in F$. Hence F is an n-fold B_L-Smarandache easy filter. □

Theorem 4.7. *Let F and G be B_L-Smarandache filters of L_B and $F \subseteq G$. If F is an n-fold B_L-Smarandache easy filter of L_B, then G is is an n-fold B_L-Smarandache easy filter of L_B.*

Proof. Follows from Theorem 4.6 $(iv) \Leftrightarrow (i)$.

Proposition 4.8. *For a Smarandache B_L-residuated lattice L_B, the following conditions are equivalent:*

(i) $\{1\}$ is an n-fold B_L-Smarandache easy filter of L_B;

(ii) Every B_L-Smarandache filter of L_B is an n-fold B_L-Smarandache easy filter of L_B;

(iii) $(x^{**})^n = (x^{**})^{2n}$, for all $x \in B$.

Proof. $(i) \Leftrightarrow (ii)$ The proof is clear, by Theorem 4.7.

$(i) \Leftrightarrow (iv)$ By Theorem 4.6, we have $(x^{**})^n \to (x^{**})^{2n} \in \{1\} \Leftrightarrow (x^{**})^n \to (x^{**})^{2n} = 1 \Leftrightarrow (x^{**})^n \leq (x^{**})^{2n} \Leftrightarrow (x^{**})^n = (x^{**})^{2n}$, for all $x \in B$. □

Corollary 4.9. *If $\{1\}$ is an n-fold B_L-Smarandache easy filter of L_B, then $(x^2)^* = x^*$, for all $x \in B$.*

Proof. By Proposition 4.8, $(x^{**})^n = (x^{**})^{2n}$, for all $x \in B$ and $n \in N$. Let $n = 1$. Then $x^{**} = (x^{**})^2$, (I). We know $(x^{**})^n \leq (x^n)^{**}$, for all $n \in N$. Let $n = 2$. Then $(x^{**})^2 \leq (x^2)^{**}$. Then by (I), we get $x^{**} \leq (x^2)^{**}$. By Theorem 2.1, we get $(x^2)^* \leq x^*$, (II). By Theorem 2.1, we know $x^2 \leq x$, so $x^* \leq (x^2)^*$. Therefore by (II), $(x^2)^* = x^*$. □

Definition 4.10. L_B is said to be n-fold Smarandache easy B_L-residuated lattice if for all $x \in B$, $(x^{**})^n = (x^{**})^{2n}$.

Example 4.11. (i) Consider the Smarandache B_L-residuated lattice L in Example 3.10(i). Clearly L_B is an n-fold Smarandache easy B_L-residuated lattice.

(ii) Consider the Smarandache B_L-residuated lattice L in Example 3.2. Clearly L_B is not a 1-fold Smarandache easy B_L-residuated lattice, since $c = (a^{**})^1 \neq (a^{**})^2 = 0$.

Corollary 4.12. *For a Smarandache B_L-residuated lattice L_B, the following conditions are equivalent:*
(i) $\{1\}$ is an n-fold B_L-Smarandache easy filter of L_B;
(ii) Every B_L-Smarandache filter of L_B is an n-fold B_L-Smarandache easy filter of L_B;
(iii) L_B is an n-fold Smarandache easy B_L-residuated lattice.

Proof. By Proposition 4.8, the proof is clear. □

Corollary 4.13. *Let F be a B_L-Smarandache deductive system of L_B. Then the following statements are equivalent:*
(i) F is an n-fold B_L-Smarandache easy filter of L_B (in short n-fold B_L-SEF);
(ii) L_B/F is an n-fold Smarandache easy $B_{L/F}$-residuated lattice (in short n-fold B_L-SERL).

Proof. Let F be a B_L-Smarandache deductive system of L_B. By Theorem 4.6 we get:

$$
\begin{aligned}
F \text{ is an } n\text{-fold } B_L\text{-SEF} \quad &\Leftrightarrow \quad (x^{**})^n \to (x^{**})^{2n} \in F,\ \forall\, x \in B,\\
&\Leftrightarrow \quad ((x^{**})^n \to (x^{**})^{2n})/F = 1/F,\ \forall\, x/F \in B/F,\\
&\Leftrightarrow \quad (x^{**}/F)^n \to (x^{**}/F)^{2n} = 1/F,\ \forall\, x/F \in B/F,\\
&\Leftrightarrow \quad ((x/F)^{**})^n \leq ((x/F)^{**})^{2n},\ \forall\, x/F \in B/F,\\
&\Leftrightarrow \quad ((x/F)^{**})^n = ((x/F)^{**})^{2n},\ \forall\, x/F \in B/F,\\
&\Leftrightarrow \quad L_B/F \ \text{ is an } n\text{-fold } B_{L/F}\text{-SERL}.
\end{aligned}
$$

□

By Corollary 4.12 and 4.9, we have the following corollary.

Corollary 4.14. *Let L_B be an n-fold Smarandache easy B_L-residuated lattice. Then $(x^2)^* = x^*$, for all $x \in B$.*

Proposition 4.15. *Every n-fold Smarandache positive implicative B_L-residuated lattice is an n-fold Smarandache easy B_L-residuated lattice.*

Proof. Let L_B be an n-fold Smarandache positive implicative B_L-residuated lattice. By Theorem 2.6, every B_L-Smarandache deductive system of L_B is an n-fold B_L-Smarandache positive implicative filter of L_B. Hence by Theorem 4.4, every B_L-Smarandache deductive system of L_B is an n-fold B_L-Smarandache easy filter of L_B. Since every B_L-Smarandache filter is a B_L-Smarandache deductive system, then every B_L-Smarandache filter is an n-fold B_L-Smarandache easy filter of L_B.Therefore by Corollary 4.12, L_B is an n-fold Smarandache easy B_L-residuated lattice. □

Theorem 4.16. *For a Smarandache B_L-residuated lattice L_B, the following conditions are equivalent:*

(i) L_B is an n-fold Smarandache easy B_L-residuated lattice;

(ii) $(x^{**})^n \leq y \to z$ implies $(x^{**})^n \to y \leq (x^{**})^n \to z$, for all $x, y, z \in B$;

(iii) $(x^{**})^n \leq (x^{**})^n \to z$ implies $(x^{**})^n \leq z$, for all $x, z \in B$.

Proof. $(i) \Rightarrow (ii)$ Let L_B be an n-fold Smarandache easy B_L-residuated lattice. By Corollary 4.12, $\{1\}$ is an n-fold B_L-Smarandache easy filter of L_B. Let $(x^{**})^n \leq y \to z$, for all $x, y, z \in B$. So $(x^{**})^n \to (y \to z) \in \{1\}$.Then by Theorem 4.6, we get $((x^{**})^n \to y) \to ((x^{**})^n \to z) \in \{1\}$. And so $(x^{**})^n \to y \leq (x^{**})^n \to z$.

$(ii) \Rightarrow (iii)$ Take $y = (x^{**})^n$, hence the proof is clear.

$(iii) \Rightarrow (i)$ Let $(x^{**})^n \to ((x^{**})^n \to z) \in \{1\}$. So $(x^{**})^n \leq (x^{**})^n \to z$. Hence by (iii), $(x^{**})^n \leq z$, for all $x, z \in B$. Then $(x^{**})^n \to z \in \{1\}$, for all $x, z \in B$. By Theorem 4.6, $\{1\}$ is an n-fold B_L-Smarandache easy filter of L_B. Therefore by Corollary 4.12, L_B is an n-fold Smarandache easy B_L-residuated lattice. □

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