On the General Solution of Initial Value Problems of Ordinary Differential Equations Using the Method of Iterated Integrals.[1]

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Abstract

Our goal is to give a very simple, effective and intuitive algorithm for the solution of initial value problem of ODEs of 1st and arbitrary higher order with general i.e. constant, variable or nonlinear coefficients and the systems of these ordinary differential equations. We find an expansion of the differential equation/function to get an infinite series containing iterated integrals evaluated solely at initial values of the dependent variables in the ordinary differential equation. Our series represents the true series expansion of the closed form solution of the ordinary differential equation. The method can also be used easily for general 2nd and higher order ordinary differential equations. We explain with examples the steps to solution of initial value problems of 1st order ordinary differential equations and later follow with more examples for linear and non-linear 2nd order and third order ODEs. We have given mathematica code for the solution of nth order ordinary differential equations and show with examples how to use the general code on 1st, 2nd and third order ordinary differential equations. We also give mathematica code for the solution of systems of a large number of general ordinary differential equations each of arbitrary order. We give an example showing the ease with which our method calculates the solution of system of three non-linear second order ordinary differential equations.

[1] The basic research was done by author in April 2016 and first appeared in public at the web site: https://forum.wilmott.com/viewtopic.php?f=4&t=99702

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1. Introduction to the Method of Iterated Integrals

Method of iterated integrals is a technique to evaluate integrals of functions that depend on several dependent variables and one (or possibly more) independent variable. Usually such integrals are very difficult to solve since they require complete knowledge of simultaneous evolution of the dependent variables with respect to the independent variable. We try to make the evaluations of these integrals simpler by expanding the integral in the form of an infinite series where we need only initial values of the dependent variables in the series to evaluate the integral. Thus we are saved from having to determine the future evolution of the dependent variables while evaluating the integrals of complicated functions. We suppose that dependence structure that defines the relationship between the dependent variables and the independent variables is given to us usually in the form of an ordinary differential equation. When the solution of the ordinary differential equation can be written in the form of an integral, we can use the method of iterated integrals to evaluate the solution integral in order to solve the very ordinary differential equation defining the relationship between the dependent variables and the independent variable.

In the method of iterated integrals we rely on a trick of writing the integrand function as a sum of two terms, in the first one of which the dependent variables in the original integrand function are evaluated at initial values of the dependent variables and the second term (which is an integral) integrates the evolution of derivatives of the original integrand function with respect to dependent variables along the independent variable. So we have the first term which can readily be calculated since it depends solely on the initial values of the dependent variables and a second term that is an integral depending on the future evolution of the dependent variable and its derivatives. We apply this trick again to the integrand function inside the integral in the second term and convert it again, just like in the first step into two terms the first one which is evaluated at initial values of the dependent variables and the second one integrating the evolution of the derivatives. Repeated applications of this trick convert the second term inside the integral in each previous step into terms depending solely on integrals calculated at initial values of the dependent variables and we get an infinite series of the terms containing iterated integrals evaluated solely at initial values of the dependent variables.

To start the discussion, we try to find the solution to General initial value problem associated with first order Ordinary Differential Equations given as

$$\frac{dy(t)}{dt} = f(t, y(t)),\ t \geq 0,\ y(t_0) = y_0 \quad (1)$$

Integrating Equation (1), we get the formal solution to the initial value problem as

$$y(t) = y(t_0) + \int_{t_0}^{t} f(\tau, y(\tau))d\tau \quad (2)$$

If only we could exactly solve the formal integral in equation (2), we would find our desired solution. Obviously $f(\tau, y(\tau))$ cannot in general be directly and analytically integrated in the above integral. In what follows, we will try to solve this formal solution integral in equation (2) using the method of iterated integrals.

However we can write integrand function in the formal integral in equation (2) as another function of initial value of dependent variable(s) $y(\tau) = y(t_0)$ and a second formal integral that integrates the derivative of the integrand function in equation (2) with respect to dependent variable $y(\tau)$ along the evolution of the independent variable so that sum of both terms is equated with integrand function in formal integral in equation (2). We can for example write as

$$f(\tau, y(\tau)) = f(\tau, y(t_0)) + \int_{t_0}^{\tau} \frac{\partial f(\tau, y(s))}{\partial y} \frac{dy(s)}{ds} ds \qquad \text{Equation (3)}$$

$$= f(\tau, y(t_0)) + \int_{t_0}^{\tau} \frac{\partial f(\tau, y(s))}{\partial y} f(s, y(s))ds \qquad \text{Equation (4)}$$

We have used equation (1) with equation (3) to get equation (4).

The above equation (3) is a special case analogue (where one dependent variable depends on another independent variable) of the following equation from the function of two variables that can very easily be proven.

$$f(\tau, y_1) = f(\tau, y_0) + \int_{y_0}^{y_1} \frac{\partial f(\tau, y)}{\partial y} dy \quad \text{Equation (3A)}$$

As an aside, we see wide applications of equation (3) and equation (3A) in solution of many difficult problems in mathematics and its applications.

Equation (2) can now be written after substituting equation (4) in it as

$$y(t) = y(t_0) + \int_{t_0}^{t} f\big(\tau, y(t_0)\big)d\tau + \int_{t_0}^{t}\int_{t_0}^{\tau} \frac{\partial f(\tau,y(s))}{\partial y} f\big(s, y(s)\big)ds \ d\tau$$

Equation(5)

In Equation (5), we notice that first order integral in second term could mostly be easily evaluated since all instances of the dependent variable are evaluated at initial time $\tau = t_0$. The second order integral in third term has to be taken in formal sense as this integral depends upon future evolution of the dependent variable and its derivatives and this second order integral could not, in general, be easily evaluated.

However the term inside the second order formal integral in equation (5) can again similarly be written in terms of initial values of the dependent variable $y(t_0)$ and an appropriate formal integral exactly like what we did for integrand function in equation (2)

$$\frac{\partial f\big(\tau, y(s)\big)}{\partial y} f\big(s, y(s)\big)$$

$$= \frac{\partial f\big(\tau, y(t_0)\big)}{\partial y} f\big(s, y(t_0)\big) + \int_{t_0}^{s} \left(\frac{\partial}{\partial y}\left[\frac{\partial f\big(\tau, y(v)\big)}{\partial y} f\big(s, y(v)\big)\right]\frac{dy(v)}{dv}\right)dv$$

$$= \frac{\partial f(\tau,y(t_0))}{\partial y} f\big(s, y(t_0)\big) + \int_{t_0}^{s} \left(\frac{\partial}{\partial y}\left[\frac{\partial f(\tau,y(v))}{\partial y} f\big(s, y(v)\big)\right] f\big(v, y(v)\big)\right)dv$$

Equation (6)

Substituting Equation(6) in the second order formal integral in equation (5), we can write as

$$y(t) = y(t_0) + \int_{t_0}^{t} f\big(\tau, y(t_0)\big)d\tau + \int_{t_0}^{t}\int_{t_0}^{\tau} \frac{\partial f(\tau,y(s))}{\partial y} f\big(s, y(s)\big)ds \, d\tau = y(t_0) + \int_{t_0}^{t} f\big(\tau, y(t_0)\big)d\tau + \int_{t_0}^{t}\int_{t_0}^{\tau} \frac{\partial f(\tau,y(t_0))}{\partial y} f\big(s, y(t_0)\big)ds \, d\tau + \int_{t_0}^{t}\int_{t_0}^{\tau}\int_{t_0}^{s} \left(\frac{\partial}{\partial y}\left[\frac{\partial f(\tau,y(v))}{\partial y} f\big(s, y(v)\big)\right] f\big(v, y(v)\big)\right)dv \, ds \, d\tau$$

Equation (7)

In the equation (7), the first order and second order integrals can easily be evaluated at initial values $y(t_0)$ while the third integral has to be taken in the formal sense and we can continue to expand these integrals in a series of iterated integral terms evaluated at initial values of the dependent variables $y(\tau) = y(t_0)$

Equation (7) is a formal solution to Equation (1) while equation (5) and equation (2) are also formal solutions to the equation (1). The formal solution in equation (2) is very difficult to evaluate as such. However the first term in the formal solution in equation (5) can easily be calculated as the dependent variable is evaluated at its initial value while the last integral has to be taken in the formal sense and would be difficult to evaluate. Again the first two terms in the formal solution in Equation (7) can be easily evaluated as the dependent variable there is calculated at y(t0) and the last third term has to be considered in the formal sense. We can find infinite number of formal solutions equivalent to Equation(2) and equivalent to each other where n number of terms (n is an arbitrary positive integer) have been evaluated at $y(t_0)$ and final n+1 th term has to be understood in formal sense and could not easily be evaluated in general. We hope to choose a formal solution in which enough number of terms have been evaluated at y(t0) so that the last term which is considered in formal sense and cannot be evaluated has negligible value. Of course, like many other series, this may always not be the case.

It is interesting to notice intuitively how the upper limits of the inside integrals go to time in each outer integral so as the higher indirect derivative in inner integrals make a contribution to lower indirect derivatives only till running time in successive outer integral.

2. Solution to First Order ODEs

We find the solution to General first order Ordinary Differential Equations given as

$$\frac{dy}{dt} = f(t, y), \qquad t \geq 0, \qquad y(t_0) = y_0$$

Equation (8)

We suppose that function f in the equation above is sufficiently well behaved and satisfies the usual regularity conditions.
Integrating Equation (8), we get the formal solution of the ordinary differential equation as

$$y(t) = y(t_0) + \int_{t_0}^{t} f\big(\tau, y(\tau)\big)d\tau \qquad \text{Equation (9)}$$

Following the earlier discussion in section (1) and the equation (7), we write the first four terms of the solution as a series in the Equation (10) below. The first three terms in the series are evaluated at initial values of the dependent variable $y(\tau) = y(t_0)$, while the last term is an integral containing forward values with respect to the dependent variable and can similarly be expanded to arbitrary order in the form of terms containing only the initial values of the dependent variable. When integrating these iterated integrals, we have to be careful because independent variable in the integrals changes in each of the iterated integrals when the first derivative of the Ordinary differential equation with respect to the dependent variable $y(\tau)$ has explicit dependence on the independent variable.

$$y(t) = y(t_0) + \int_{t_0}^{t} f\big(\tau, y(t_0)\big)d\tau + \int_{t_0}^{t}\int_{t_0}^{\tau}\left(\frac{\partial}{\partial y}\big[f\big(\tau, y(t_0)\big)\big]f\big(s, y(t_0)\big)\right)ds\ d\tau$$

$$+ \int_{t_0}^{t}\int_{t_0}^{\tau}\int_{t_0}^{s}\left(\frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}\big[f\big(\tau, y(t_0)\big)\big]f\big(s, y(t_0)\big)\right]f\big(u, y(t_0)\big)\right)du\ ds\ d\tau$$

$$+ \int_{t_0}^{t}\int_{t_0}^{\tau}\int_{t_0}^{s}\int_{t_0}^{u}\left(\frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}\big[f\big(\tau, y(t_0)\big)\big]f\big(s, y(t_0)\big)\right]f\big(u, y(t_0)\big)\right]f\big(v, y(t_0)\big)\right)dv\, du\, ds\, d\tau$$

$$+ \text{ higher order terms}$$

Equation (10)

We can continue to expand the above equation to any higher order containing terms depending on initial values of the dependent variable.

When the first order ordinary differential function $f\big(t, y(t)\big)$ has no explicit dependence on t, we can write equation (10) above as

$$y(t) = y(t_0) + \int_{t_0}^{t} f\big(y(t_0)\big)d\tau + \int_{t_0}^{t}\int_{t_0}^{\tau}\left(\frac{\partial}{\partial y}\big[f\big(y(t_0)\big)\big]f\big(y(t_0)\big)\right)ds\ d\tau$$

$$+ \int_{t_0}^{t}\int_{t_0}^{\tau}\int_{t_0}^{s}\left(\frac{\partial}{\partial y}\left[\left(\frac{\partial}{\partial y}\big[f\big(y(t_0)\big)\big]\right)f\big(y(t_0)\big)\right]f\big(y(t_0)\big)\right)du\ ds\ d\tau$$

$$+\int_{t_0}^{t}\int_{t_0}^{\tau}\int_{t_0}^{s}\int_{t_0}^{u}\left(\frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}[f(y(t_0))]f(y(t_0))\right]f(y(t_0))\right]f(y(t_0))\right)dv\,du\,ds\,d\tau$$
$$+ \text{higher order terms}$$

Equation (11)

Since all coefficients in the first three terms are evaluated at $y(\tau) = y(t_0)$, we can write the simple solution to Equations (7) and (8) expanded as

$$y(t) = y(t_0) + y(t_0)\int_{t_0}^{t} d\tau + \left(\frac{\partial}{\partial y}[f(y(t_0))]f(y(t_0))\right)\int_{t_0}^{t}\int_{t_0}^{\tau} ds\,d\tau$$

$$+\left(\frac{\partial}{\partial y}\left[\left(\frac{\partial}{\partial y}[f(y(t_0))]\right)f(y(t_0))\right]f(y(t_0))\right)\int_{t_0}^{t}\int_{t_0}^{\tau}\int_{t_0}^{s} du\,ds\,d\tau$$

$$+\left(\frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}[f(y(t_0))]f(y(t_0))\right]f(y(t_0))\right]f(y(t_0))\right)\int_{t_0}^{t}\int_{t_0}^{\tau}\int_{t_0}^{s}\int_{t_0}^{u} dv\,du\,ds\,d\tau$$
$$+ \text{higher order terms}$$

Equation (12)

Example 1.

Use Equation (11) to solve the initial value problem

$\frac{dy}{dt} = f(y(t)) = y,\ \ t_0 = 0,\ \ y(t_0) = y_0$ Equation (13)

Evaluating second term on RHS in equation (12) having one integral

$$f(y(t_0))\int_{t_0}^{t} d\tau = y(t_0)\int_{0}^{t} d\tau = y_0 t$$

Evaluating third term on RHS in equation (12) having two iterated integrals

$$\left(\frac{\partial}{\partial y}[f(y(t_0))]f(y(t_0))\right)\int_{t_0}^{t}\int_{t_0}^{\tau} ds\,d\tau = y(t_0)\int_{0}^{t}\int_{0}^{\tau} ds\,d\tau = y_0\frac{t^2}{2}$$

Evaluating fourth term on RHS in equation (12) having three iterated integrals

$$\left(\frac{\partial}{\partial y}\left[\left(\frac{\partial}{\partial y}[f(y(t_0))]\right)f(y(t_0))\right]f(y(t_0))\right)\int_{t_0}^{t}\int_{t_0}^{\tau}\int_{t_0}^{s}du\,ds\,d\tau = y(t_0)\int_{t_0}^{t}\int_{t_0}^{\tau}\int_{t_0}^{s}du\,ds\,d\tau = y_0\frac{t^3}{6}$$

Putting these coefficients in equation (12), we get the expansion for exponential function multiplied by the initial values for the first three terms as

$$y(t) = y_0 + y_0 t + \frac{1}{2}y_0 t^2 +.\frac{1}{6}y_0 t^3 + \cdots$$

Example 2.

Solve the initial value problem

$\frac{dy}{dt} = f\big(y(t)\big) = 1 + y^2, \quad y(0) = y_0$ Equation (14)

Evaluating second term on RHS in equation (12) having one integral

$$f(y(t_0))\int_{t_0}^{t}d\tau = (1 + {y_0}^2)\int_{0}^{t}d\tau = y_0 t$$

Evaluating third term on RHS in equation (12) having two integral

$$\left(\frac{\partial}{\partial y}[f(y(t_0))]f(y(t_0))\right)\int_{t_0}^{t}\int_{t_0}^{\tau}ds\,d\tau = 2y_0(1 + {y_0}^2)\int_{0}^{t}\int_{0}^{\tau}ds\,d\tau = \frac{1}{2}\big(2y_0(1 + {y_0}^2)\big)t^2$$

Evaluating fourth term on RHS in equation (12) having three integral

$$\left(\frac{\partial}{\partial y}\left[\left(\frac{\partial}{\partial y}[f(y(t_0))]\right)f(y(t_0))\right]f(y(t_0))\right)\int_{t_0}^{t}\int_{t_0}^{\tau}\int_{t_0}^{s}du\,ds\,d\tau$$

$$= 2(1 + 3{y_0}^2)(1 + {y_0}^2)\int_{0}^{t}\int_{0}^{\tau}\int_{0}^{s}du\,ds\,d\tau = 2(1 + 3{y_0}^2)(1 + {y_0}^2)\frac{t^3}{6}$$

Evaluating fifth term on RHS in equation (12) having four iterated integrals

$$\frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}\left[\left(\frac{\partial}{\partial y}\big[f\big(y(t_0)\big)\big]\right)f\big(y(t_0)\big)\right]f\big(y(t_0)\big)\right]f\big(y(t_0)\big)\int_{t_0}^{t}\int_{t_0}^{\tau}\int_{t_0}^{s}\int_{t_0}^{u}dv\;du\,ds\,d\tau$$

$$= 8(2y_0 + 3{y_0}^3)(1 + {y_0}^2)\int_{0}^{t}\int_{0}^{\tau}\int_{0}^{s}\int_{0}^{u}dv\;du\,ds\,d\tau$$

$$=\frac{1}{24}\big(8(2y_0 + 3{y_0}^3)(1 + {y_0}^2)\big)t^4$$

Similarly Evaluating sixth term having five iterated integrals following the last term in equation (12)

$$\left(\frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}\left[\left(\frac{\partial}{\partial y}[f(y(t_0))]\right)f(y(t_0))\right]f(y(t_0))\right]f(y(t_0))\right]f(y(t_0))\right)\int_{t_0}^{t}\int_{t_0}^{\tau}\int_{t_0}^{s}\int_{t_0}^{u}\int_{t_0}^{v} dw\, dv\ dv\ du\, ds\, d\tau$$

$$= 8(2+15\,{y_0}^2 + 15\,{y_0}^4)(1+{y_0}^2)\int_0^t\int_0^\tau\int_0^s\int_0^u\int_0^v dw\, dv\ dv\ du\, ds\, d\tau$$

$$= \frac{1}{120}\left(8(2+15\,{y_0}^2 + 15\,{y_0}^4)(1+{y_0}^2)\right)t^5$$

Putting these coefficients in Equation (12), we get the expansion for integral of differential function and the first five terms are given as

$$y(t) = y_0 + (1+{y_0}^2)t + \frac{1}{2}\left(2y_0(1+{y_0}^2)\right)t^2 + \frac{1}{6}\left(2(1+3{y_0}^2)(1+{y_0}^2)\right)t^3 + \frac{1}{24}\left(8(2y_0+3{y_0}^3)(1+{y_0}^2)\right)t^4 + \frac{1}{120}\left(8(2+15\,{y_0}^2+15\,{y_0}^4)(1+{y_0}^2)\right)t^5 + \cdots$$

Equation (14A)

We can recognize a simpler problem whose solution is known if we recast the initial value problem as

$$\frac{dy}{dt} = f(y(t)) = 1 + y^2, \quad y(0) = y_0 = 0$$

We can recognize that $\frac{dy}{dt} = 1 + (tan(t))^2 = (\sec(t))^2 = \frac{d}{dt}[\tan(t)]$

Resulting in solution $y(t) = \tan(t) = t + \frac{t^3}{3} + \frac{2t^5}{15} + \cdots$

Which can be verified by substituting $y_0 = 0$ in Equation (14A).

Example 3

Solve the initial value problem

$\frac{dy}{dt} = f(y(t)) = \frac{y(t)}{\sqrt{t}}, \quad y(0) = y_0,\ t_0 = 0$ Equation (15)

Using Equation (16), we calculate the values of coefficients in terms of initial values.

Evaluating the first term in equation (10) having one integral

$$\int_{t_0}^{t} f(\tau, y(t_0))d\tau = \int_{0}^{t} \frac{y_0}{\sqrt{\tau}} d\tau = 2y_0\sqrt{t}$$

Evaluating the second term in equation (10) having two iterated integrals

$$\int_{t_0}^{t}\int_{t_0}^{\tau} \frac{\partial}{\partial y}[f(\tau, y(t_0))] f(s, y(t_0))ds\ d\tau = \int_{0}^{t}\int_{0}^{\tau} \frac{1}{\sqrt{\tau}} \frac{y_0}{\sqrt{s}}\ ds\ d\tau$$
$$= 2y_0 t$$

Evaluating the Third term in equation (10) having three iterated integrals

$$\int_{t_0}^{t}\int_{t_0}^{\tau}\int_{t_0}^{s} \frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}[f(\tau, y(t_0))] f(s, y(t_0))\right] f(u, y(t_0))du\ ds\ d\tau$$
$$= \int_{0}^{t}\int_{0}^{\tau}\int_{0}^{s} \frac{1}{\sqrt{\tau}} \frac{1}{\sqrt{s}} \frac{y_0}{\sqrt{u}} du\ ds\ d\tau\ = \frac{4}{3} y_0 t^{1.5}$$

Evaluating the fourth term in continuation of equation (10) having four iterated integrals

$$\int_{t_0}^{t}\int_{t_0}^{\tau}\int_{t_0}^{s}\int_{t_0}^{u} \frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}[f(\tau, y(t_0))] f(s, y(t_0))\right] f(u, y(t_0))\right] f(v, y(t_0))dv\, du\, ds\, d\tau$$

$$= \int_{0}^{t}\int_{0}^{\tau}\int_{0}^{s}\int_{0}^{u} \frac{1}{\sqrt{\tau}} \frac{1}{\sqrt{s}} \frac{1}{\sqrt{u}} \frac{y_0}{\sqrt{v}} dv\, du\, ds\, d\tau\ = \frac{2}{3} y_0 t^2$$

So we calculate the first few terms of the solution as

$$y(t) = y_0 + 2y_0\sqrt{t} + 2y_0 t + \frac{4}{3} y_0 t^{1.5} + \frac{2}{3} y_0 t^2 + \cdots$$

Where the analytic solution is known to be

$$y(t) = y_0 \exp(2\sqrt{t})$$

We can see that our series solution we found is indeed Taylor expansion of the closed form solution above.

Mathematica Program for Symbolic Computing of the Solution of First Order Ordinary Differential Equations.

Here we give the mathematica code for a general nth order ordinary differential equation and show it in a few examples how to use it for symbolic solution of 1^{st} order ordinary differential equations. We write the mathematica notebook commands in the serial order.

Example 4:

We solve the following initial value problem using mathematica code

$$\frac{dy(t)}{dt} = t\,y(t)^2 - t\,y(t), \quad y(t_0) = .5, \qquad t_0 = 0;$$

Equation (16)

1. Clear[t, y, Zi, Wi, Yi, Y, y0, i, i1, p, n, ZAns];
2. $n = 1$;
3. Array[$y, n + 1$,0];
4. $y[1] := ty[0]$^2 $- ty[0]$;
5. Array[y0, n, 0];
6. y0[0] = .5;
7. $p = 10$;
8. $Y :=$ y0[0];
9. For[$k = 1, k < n, k + +, Y = Y +$ y0[k] $*$ t^k/k!]
10. Array[Zi, $n + p - 1$,0];
11. Array[Wi, $n + p - 1$,0];
12. Array[Yi, $n + p - 1$,0];
13. Array[ti, $n + p - 1$,0];
14. Zi[n] = ($y[n]$/. $t \rightarrow$ ti[n]);
15. For[$i = n, i < p + n - 1, i + +$, (Zi[$i + 1$] = 0;
    For[$k = 0, k < n, k + +$, (Zi[$i + 1$] = Zi[$i + 1$] +
    D[Zi[i], $y[k]$] $*$ ($y[k + 1]$/. $t \rightarrow$ ti[$i + 1$]));]);];

16. For[$i = n, i \leq p + n - 1, i + +$, (Wi[$i$] = Zi[$i$] ; For[$k = 0, k < n, k + +$, (Wi[$i$] =
    (Wi[i]/. $y[k] \rightarrow$ y0[k]))];)];

17. For[$i = n, i \leq p + n - 1, i++$, (Yi[$i$] = Wi[$i$]; For[i1 = i, i1 ≥ 1, i1 − −, (Yi[i1 − 1] = Integrate[Yi[i1], {ti[i1],0, ti[i1 − 1]}])]; $Y = Y$ + (Yi[0]/. ti[0] → t);)];
18. ZAns = Collect[Y, t, Simplify]//PolynomialForm[#, TraditionalOrder → False]&

Here is the Output of the program which completely agrees with the series expansion of the closed form solution of equation (16).

Y= $0.5 - 0.125t^2 + 0.0026041666666666665t^6 - 0.000065104166666666667t^{10} + 0.0000016469804067460 31t^{14} - 4.171273905974426 \times 10^{-8}t^{18}$

Here is the line by line explanation of the mathematica code.

Command 1 is used to clear the variables from memory so previously stored values would not affect outcome of our program.

Command 2 describes the order of the ordinary differential equation. In the case of first order ODEs, n=1.

In command 3, we define the array of n+1 variables representing derivatives of various orders of the dependent variable in the nth order ODE. Here y[0] describes y[t] while y[1] describes the first derivative $\frac{dy(t)}{dt}$ of dependent variable while y[n] describes the nth order derivative of dependent variable with respect to the independent variable.
In command 4, we specify the ODE to solve. In our case this is ODE in Equation().
In command 5, we define the array of n initial values for the nth order ODE.
In command 6, we specify the numerical initial values of the array defined in previous command 5.
In command 7, we specify the number of iterated integral terms to include in the series solution to the nth order ODE.
In commands 8 and 9, we assign initial values to the solution variable.
In commands 10 to 13, we define the array variables for symbolic computations in the following loops.
In commands 14 and 15, we find the terms inside the integrals before integration is carried out.
In command 16, we assign numerical initial values to the integrands for various iterated integral terms calculated in command 15.
In command 17, we carry out the integrations and continue to add the results of each iterated integral term to the solution variable Y.

In command 18, we simplify and arrange the solution in ascending powers of independent variable t.

3. Solution to Second Order ODEs

We find the solution to General second order Ordinary Differential Equations given as

$$\frac{d^2y}{dt^2} = f(t, y, y'), \qquad t \geq 0, \qquad y(t_0) = y_0 \quad y'(t_0) = y'_0$$

Equation (17)

We suppose that function f in the equation above is sufficiently well behaved and satisfies the usual regularity conditions.

Integrating Equation (18), we get the formal solution

$$y(t) = y(t_0) + y'(t_0)(t - t_0) + \int_{t_0}^{t}\int_{t_0}^{s} f\big(\tau, y(\tau), y'(\tau)\big)d\tau\, ds$$

Equation (18)

The differential function in the second order ODE takes three arguments in general. We evaluate the dependent variables in the original integrand function in Equation (18) at initial values of the dependent variables and add more (integral) terms that integrate the evolution of derivatives of the original integrand function with respect to dependent variables along the independent variable so that sum of all the terms equates with the integrand in equation (18). We expand the integrand function in equation (18) as

$$f\big(\tau, y(\tau), y'(\tau)\big) = f\big(\tau, y(t_0), y'(t_0)\big)$$
$$+ \int_{t_0}^{\tau} \frac{\partial f\big(\tau, y(u), y'(u)\big)}{\partial y}\frac{dy(u)}{du}du + \int_{t_0}^{\tau} \frac{\partial f\big(\tau, y(u), y'(u)\big)}{\partial y'}\frac{dy'(u)}{du}du$$

Equation (19)

We repeat the above expansion trick on the integrands of both integrals on the right hand side of equation (19). Expanding the first term, we get

$$
\begin{aligned}
\frac{\partial f\big(\tau, y(u), y'(u)\big)}{\partial y}\frac{dy(u)}{du}
&= \frac{\partial f\big(\tau, y(t_0), y'(t_0)\big)}{\partial y}\frac{dy(t_0)}{du} \\
&+ \int_{t_0}^{u} \frac{\partial}{\partial y}\left[\frac{\partial f\big(\tau, y(v), y'(v)\big)}{\partial y}\frac{dy(u, y(v), y'(v))}{du}\right]\frac{dy(v)}{dv}\, dv \\
&+ \int_{t_0}^{u} \frac{\partial}{\partial y'}\left[\frac{\partial f\big(\tau, y(v), y'(v)\big)}{\partial y}\frac{dy(u, y(v), y'(v))}{du}\right]\frac{dy'(v)}{dv}\, dv
\end{aligned}
$$

Equation (20)

last of the formal first order integrals in equation (19) can further be expanded as

$$
\begin{aligned}
\frac{\partial f\big(\tau, y(u), y'(u)\big)}{\partial y'}\frac{dy'(u)}{du} = \frac{\partial f\big(\tau, y(u), y'(u)\big)}{\partial y'} f\big(u, y(u), y'(u)\big)
&= \frac{\partial f\big(\tau, y(t_0), y'(t_0)\big)}{\partial y'} f\big(u, y(t_0), y'(t_0)\big) \\
&+ \int_{t_0}^{u} \frac{\partial}{\partial y}\left[\frac{\partial f\big(\tau, y(v), y'(v)\big)}{\partial y} f\big(u, y(v), y'(v)\big)\right]\frac{dy(v)}{dv}\, dv \\
&+ \int_{t_0}^{u} \frac{\partial}{\partial y'}\left[\frac{\partial f\big(\tau, y(v), y'(v)\big)}{\partial y} f\big(u, y(v), y'(v)\big)\right]\frac{dy'(v)}{dv}\, dv
\end{aligned}
$$

Equation (21)

We substitute equations (20) and (21) in equation (19) and later substitute equation (19) in equation (18) to get the new equation as

$$y(t) = y(t_0) + y'(t_0)(t - t_0) + \int_{t_0}^{t}\int_{t_0}^{s} f\big(\tau, y(t_0), y'(t_0)\big) d\tau\, ds$$
$$+ \int_{t_0}^{t}\int_{t_0}^{s}\int_{t_0}^{\tau} \frac{\partial f\big(\tau, y(t_0), y'(t_0)\big)}{\partial y}\frac{dy(t_0)}{du} du\, d\tau\, ds$$
$$+ \int_{t_0}^{t}\int_{t_0}^{s}\int_{t_0}^{\tau} \frac{\partial f\big(\tau, y(t_0), y'(t_0)\big)}{\partial y'} f\big(u, y(t_0), y'(t_0)\big) du\, d\tau\, ds$$
$$+ \int_{t_0}^{t}\int_{t_0}^{s}\int_{t_0}^{\tau}\int_{t_0}^{u} \frac{\partial}{\partial y}\left[\frac{\partial f\big(\tau, y(v), y'(v)\big)}{\partial y}\frac{dy(t_0)}{du}\right]\frac{dy(v)}{dv}\, dv$$
$$+ \int_{t_0}^{t}\int_{t_0}^{s}\int_{t_0}^{\tau}\int_{t_0}^{u} \frac{\partial}{\partial y'}\left[\frac{\partial f\big(\tau, y(v), y'(v)\big)}{\partial y}\frac{dy(t_0)}{du}\right]\frac{dy'(v)}{dv}\, dv$$
$$+ \int_{t_0}^{t}\int_{t_0}^{s}\int_{t_0}^{\tau}\int_{t_0}^{u} \frac{\partial}{\partial y}\left[\frac{\partial f\big(\tau, y(v), y'(v)\big)}{\partial y}\frac{dy'(u, y(v), y'(v))}{du}\right]\frac{dy(v)}{dv}\, dv$$
$$+ \int_{t_0}^{t}\int_{t_0}^{s}\int_{t_0}^{\tau}\int_{t_0}^{u} \frac{\partial}{\partial y'}\left[\frac{\partial f\big(\tau, y(v), y'(v)\big)}{\partial y}\frac{dy'(u, y(v), y'(v))}{du}\right]\frac{dy'(v)}{dv}\, dv$$

Equation (22)

Substituting initial values and also expanding one level further, we get the expression

$$y(t) = y_0 + y_0'(t - t_0) +$$
$$= \int_{t_0}^{t}\int_{t_0}^{s} f\big(\tau, y(t_0), y'(t_0)\big) d\tau\, ds$$
$$+ \int_{t_0}^{t}\int_{t_0}^{s}\int_{t_0}^{\tau}\Bigg(\frac{\partial}{\partial y}\big[f\big(\tau, y(t_0), y'(t_0)\big)\big]y'(t_0)$$
$$+ \frac{\partial}{\partial y'}\big[f\big(\tau, y(t_0), y'(t_0)\big)\big] f\big(u, y(t_0), y'(t_0)\big)\Bigg) du\, d\tau\, ds$$
$$+ \int_{t_0}^{t}\int_{t_0}^{s}\int_{t_0}^{\tau}\int_{t_0}^{u}\Bigg(\frac{\partial}{\partial y}\bigg[\frac{\partial}{\partial y}\big[f\big(\tau, y(t_0), y'(t_0)\big)\big]y'(t_0)$$
$$+ \frac{\partial}{\partial y'}\big[f\big(\tau, y(t_0), y'(t_0)\big)\big] f\big(u, y(t_0), y'(t_0)\big)\bigg]\, y'(t_0)$$
$$+ \frac{\partial}{\partial y'}\bigg[\frac{\partial}{\partial y}\big[f\big(\tau, y(t_0), y'(t_0)\big)\big]y'(t_0)$$
$$+ \frac{\partial}{\partial y'}\big[f\big(\tau, y(t_0), y'(t_0)\big)\big] f\big(u, y(t_0), y'(t_0)\big)\bigg]\, f\big(v, y(t_0), y'(t_0)\big)\Bigg)\, dv du\, d\tau\, ds$$

Equation Continued on next page.

$$+\int_{t_0}^{t}\int_{t_0}^{s}\int_{t_0}^{\tau}\int_{t_0}^{u}\int_{t_0}^{v}\Bigg(\frac{\partial}{\partial y}\Bigg[\frac{\partial}{\partial y}\Bigg[\frac{\partial}{\partial y}\big[f\big(\tau, y(t_0), y'(t_0)\big)\big]y'(t_0)$$

$$+\frac{\partial}{\partial y'}\big[f\big(\tau, y(t_0), y(t_0)\big)\big]f\big(u, y(t_0), y'(t_0)\big)\Bigg]$$

$$+\frac{\partial}{\partial y'}\Bigg[\frac{\partial}{\partial y}\big[f\big(\tau, y(t_0), y'(t_0)\big)\big]y'(t_0)$$

$$+\frac{\partial}{\partial y'}\big[f\big(\tau, y(t_0), y'(t_0)\big)\big]f\big(u, y(t_0), y'(t_0)\big)\Bigg]f\big(v, y(t_0), y'(t_0)\big)\Bigg]y'(t_0)$$

$$+\frac{\partial}{\partial y'}\Bigg[\frac{\partial}{\partial y}\Bigg[\frac{\partial}{\partial y}\big[f\big(\tau, y(t_0), y'(t_0)\big)\big]y'(t_0) + \frac{\partial}{\partial y'}\big[f\big(\tau, y(t_0), y'(t_0)\big)\big]f\big(u, y(t_0), y'(t_0)\big)\Bigg]y'(t_0)$$

$$+\frac{\partial}{\partial y'}\Bigg[\frac{\partial}{\partial y}\big[f\big(\tau, y(t_0), y'(t_0)\big)\big]y'(t_0)$$

$$+\frac{\partial}{\partial y'}\big[f\big(\tau, y(t_0), y'(t_0)\big)\big]f\big(u, y(t_0), y'(t_0)\big)\Bigg]f\big(v, y(t_0), y'(t_0)\big)\Bigg]f\big(w, y(t_0), y'(t_0)\big)\Bigg)dw\,dv\,du\,d\tau\,ds$$

+ higher order terms

Equation (23)

Example 5

Consider the 2nd order initial value problem given as

$$y'' = f\big(\tau, y(\tau), y'(\tau)\big) = -2y' - y, \qquad y(0) = y_0 = 1, \qquad y'(0) = y_0' = 1$$

Equation (24)

Evaluating the second order iterated integral term in equation (23) having two iterated integrals

$$\int_{t_0}^{t}\int_{t_0}^{s} f\big(\tau, y(t_0), y'(t_0)\big)d\tau\, ds = (-2y_0' - y_0)\frac{t^2}{2} = \frac{-3\,t^2}{2}$$

Evaluating the third order iterated integral term in equation (23) having three iterated integrals

$$\int_{t_0}^{t}\int_{t_0}^{s}\int_{t_0}^{\tau} \frac{\partial}{\partial y}\big[f\big(\tau, y(t_0), y'(t_0)\big)\big]y'(t_0) + \frac{\partial}{\partial y'}\big[f\big(\tau, y(t_0), y'(t_0)\big)\big]f\big(u, y(t_0), y'(t_0)\big)du\, d\tau\, ds$$

$$= -y_0'\frac{t^3}{6} - 2(-2\,y_0' - y_0)\frac{t^3}{6} = \frac{4t^3}{3}$$

Evaluating the fourth order iterated integral term in equation (23) having four iterated integrals

$$\int_{t_0}^{t}\int_{t_0}^{s}\int_{t_0}^{\tau}\int_{t_0}^{u} \left[\frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}\big[f\big(\tau, y(t_0), y'(t_0)\big)\big]y'(t_0) + \frac{\partial}{\partial y'}\big[f\big(\tau, y(t_0), y'(t_0)\big)\big]f\big(u, y(t_0), y'(t_0)\big)\right]y'(t_0)\right.$$

$$+ \frac{\partial}{\partial y'}\left[\frac{\partial}{\partial y}\big[f\big(\tau, y(t_0), y'(t_0)\big)\big]y'(t_0)\right.$$

$$\left.\left. + \frac{\partial}{\partial y'}\big[f\big(\tau, y(t_0), y'(t_0)\big)\big]f\big(u, y(t_0), y'(t_0)\big)\right] f\big(v, y(t_0), y'(t_0)\big)\right] dv\, du\, d\tau\, ds$$

$$= 0 + 2y_0'\frac{t^4}{24} - (-2\,y_0' - y_0)\frac{t^4}{24} + 4(-2\,y_0' - y_0)\frac{t^4}{24} = \frac{-7\,t^4}{24}$$

We now get the solution as

$$y(t) = 1 + t - \frac{3\,t^2}{2} + \frac{4t^3}{3} - \frac{7\,t^4}{24} + \cdots$$

Example 6

Consider the 2nd order initial value problem given as

$$y'' = f\big(\tau, y(\tau), y'(\tau)\big) = -2\, y\, y', \qquad y(0) = y_0 = 0, \qquad y'(0) = y_0' = 1$$

Equation (25)

From the general solution expansion given in Equation (23), we calculate the coefficients of the first few terms as

$$\int_{t_0}^{t}\int_{t_0}^{s} f(\tau, y_0, y_0')d\tau\, ds = (-2y_0'y_0)\frac{t^2}{2} = 0$$

Evaluating the third order iterated integral term in equation (23) having three iterated integrals

$$\int_{t_0}^{t}\int_{t_0}^{s}\int_{t_0}^{\tau} \quad \frac{\partial}{\partial y}[f(\tau, y_0, y_0')]y'(t_0) + \frac{\partial}{\partial y'}[f(\tau, y_0, y_0')]f(u, y_0, y_0')du\, d\tau\, ds$$
$$= (-2y_0'^2)\frac{t^3}{6} + (-2y_0(-2y_0'y_0))\frac{t^3}{6} = -\frac{t^3}{3}$$

Evaluating the fourth order iterated integral term in equation (23) having four iterated integrals

$$\int_{t_0}^{t}\int_{t_0}^{s}\int_{t_0}^{\tau}\int_{t_0}^{u} \quad \left[\frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}[f(\tau, y_0, y_0')]y_0' + \frac{\partial}{\partial y'}[f(\tau, y_0, y_0')]f(u, y_0, y_0')\right]y_0' + \frac{\partial}{\partial y'}\left[\frac{\partial}{\partial y}[f(\tau, y_0, y_0')]y_0' + \frac{\partial}{\partial y'}[f(\tau, y_0, y_0')]f(u, y_0, y_0')\right]f(v, y_0, y_0')\right] dv\, du\, d\tau\, ds = 0 + (8y_0y_0'^2)\ \frac{t^4}{24}$$

Evaluating the fifth order iterated integral term in equation (23) having five iterated integrals

$$\int_{t_0}^{t}\int_{t_0}^{s}\int_{t_0}^{\tau}\int_{t_0}^{u}\int_{t_0}^{v} \left[\frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}[f(\tau, y_0, y_0')]y_0' + \frac{\partial}{\partial y'}[f(\tau, y_0, y_0')]f(u, y_0, y_0')\right]y_0'\right.\right.$$

$$+ \frac{\partial}{\partial y'}\left[\frac{\partial}{\partial y}[f(\tau, y_0, y_0')]y_0' + \frac{\partial}{\partial y'}[f(\tau, y_0, y_0')]f(u, y_0, y_0')\right]f(v, y_0, y_0')\Bigg]y_0'$$

$$+ \frac{\partial}{\partial y'}\left[\frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}[f(\tau, y_0, y_0')]y_0' + \frac{\partial}{\partial y'}[f(\tau, y_0, y_0')]f(u, y_0, y_0')\right]y_0' y\right.$$

$$+ \frac{\partial}{\partial y'}\left[\frac{\partial}{\partial y}[f(\tau, y_0, y_0')]y_0'\right.$$

$$\left.\left.\left. + \frac{\partial}{\partial y'}[f(\tau, y_0, y_0')]f(u, y_0, y_0')\right]f(v, y_0, y_0')\right]f\big(w, y(t_0), y'(t_0)\big)\right] dw\, dv\, du\, d\tau\, ds$$

$$= 0 + (8y_0'^3)\,\frac{t^5}{120} + (8y_0'^3)\,\frac{t^5}{120} + (-24{y_0}^2 y_0'^2)\,\frac{t^5}{120} + 0 + (16y_0y_0'(-2y_0'y_0))\,\frac{t^5}{120}$$

$$+ (16y_0y_0'(-2y_0'y_0))\,\frac{t^5}{120} + (-8{y_0}^3(-2y_0'y_0))\,\frac{t^5}{120} = \frac{2\,t^5}{15}$$

By continuing the expansion for another two terms, we can verify the next term in expansion as $\frac{-17\,t^7}{315}$

And we get the first significant terms of the solution as

$y(t) = t - \frac{t^3}{3} + \frac{2\,t^5}{15} - \frac{17\,t^7}{315} + \cdots ..$

Mathematica Program for Symbolic Computing of the Solution of Second Order Ordinary Differential Equations.

Here we give the mathematica code for a general nth order ordinary differential equation and use it in examples for symbolic solution of 2nd order ordinary differential equations. We write the mathematica notebook commands in the serial order.

Example 7

We consider the 2^{nd} order initial value problem of Poisson-Boltzmann Ordinary Differential Equation given as

$$y'' = -\frac{2}{x}y' + e^{y}, \qquad y(0) = y_0 = 0, \qquad y'(0) = y'_0 = 0$$

Equation (26)

Before we proceed towards application of method of iterated integrals, we have to transform the equation into suitable form. We multiply both sides of the equation by x^2 and use the transformation $x = e^t$ to get the new ODE suitable for application of our method. The transformed ODE is given as

The initial limits $x = 0$ for independent variable change to $t = -\infty$.

$$y''(t) = -y'(t) - e^{2t}e^{y}, \qquad y(-\infty) = y_0 = 0, \qquad y'(-\infty) = y'_0 = 0$$

Equation (27)

We have used the following mathematica code to solve the transformed Ordinary differential equation.

The code format used to solve the above transformed equation (27) is exactly the same as we used for the solution of first order ordinary differential equation in example 4. Since the general code is the same, we will only comment on the differences between the code in example 4 and example 7 here. In command 2, n =2 for 2^{nd} order ordinary differential equations. Command 4 that specifies the ODE is of course unique to each ODE and takes input from transformed equation (27) above. Commands 6 and 7 are different since they specify the unique initial values. Rest of the program is the same but in command 18, we have specified lower integration limit to -∞ as in our transformed problem and later we replaced the transformed variable back to log(x) in the same command 18.

```
1. Clear[t, y, Zi, Wi, Yi, Y, y0, i, i1, p, n, ZAns];
2. n = 2;
3. Array[y, n + 1,0];
4. y[2]: = −y[1] − Exp[2t]Exp[y[0]]
5.  Array[y0, n, 0];
6.  y0[0] = 0;
7. y0[1] = 0;
8. p = 11;
9. Y: = y0[0];
10. For[k = 1, k < n, k + +, Y = Y + y0[k] ∗ t^k/k!]
11. Array[Zi, n + p − 1,0];
12. Array[Wi, n + p − 1,0];
13. Array[Yi, n + p − 1,0];
14. Array[ti, n + p − 1,0];
15. Zi[n] = (y[n]/. t → ti[n]);
16. For[i = n, i < p + n − 1, i + +, (Zi[i + 1] = 0; For[k = 0, k < n, k + +, (Zi[i + 1] =
Zi[i + 1] + D[Zi[i], y[k]] ∗ (y[k + 1]/. t → ti[i + 1])); ]); ];
17. For[i = n, i ≤ p + n − 1, i + +, (Wi[i] = Zi[i]; For[k = 0, k < n, k + +, (Wi[i] =
(Wi[i]/. y[k] → y0[k]))]; )];
18. For[i = n, i ≤ p + n − 1, i + +, (Yi[i] = Wi[i]; For[i1 = i, i1 ≥ 1, i1 − −, (Yi[i1 − 1] =
Integrate[Yi[i1], {ti[i1], −Infinity, ti[i1 − 1]}])]; Y = Y + (Yi[0]/. ti[0] → Log[x]); )];
19.  ZAns = Collect[Y, t, Simplify]//PolynomialForm[#, TraditionalOrder → False]&//N
```

Here is the output of the program.

$$y(x) = -0.166748046875x^2 + 0.00837397575378418x^4 - 0.0005844226971466246x^6 + 0.00007462808134133924x^8 - 0.000014826597990813078x^{10} + 0.0000012715657552083333x^{12}$$

Adding a few more integral terms in the expansion would continue to add accuracy.

The true solution to eighth power is given as

$$y(x) = -0.16666666666666666x^2 + 0.008333333333333333x^4 - 0.0005291005291005291x^6 + 0.000009798157946306094x^8$$

Example 8

Consider the 2nd order initial value problem of Ordinary Differential Equation given as

$$y'' = (y')^2 - y^2 + e^t, \qquad y(0) = y_0 = 1, \qquad y'(0) = y_0' = 1$$

Equation (28)

We have used the following mathematica code to solve the above non-linear Ordinary differential equation.

```
1. Clear[t, y, Zi, Wi, Yi, Y, y0, i, i1, p, n, ZAns];
2. n = 2;
3. Array[y, n + 1,0];
4. y[2]: = y[1]^2 − y[0]^2 + Exp[t]
5.  Array[y0, n, 0];
6.  y0[0] = 1;
7. y0[1] = 1;
8. p = 10;
9. Y: = y0[0];
10. For[k = 1, k < n, k + +, Y = Y + y0[k] ∗ t^k/k!]
11. Array[Zi, n + p − 1,0];
12. Array[Wi, n + p − 1,0];
13. Array[Yi, n + p − 1,0];
14. Array[ti, n + p − 1,0];
15. Zi[n] = (y[n]/. t → ti[n]);
16. For[i = n, i < p + n − 1, i + +, (Zi[i + 1] = 0; For[k = 0, k < n, k + +, (Zi[i + 1] =
Zi[i + 1] + D[Zi[i], y[k]] ∗ (y[k + 1]/. t → ti[i + 1])); ]); ];
17. For[i = n, i ≤ p + n − 1, i + +, (Wi[i] = Zi[i]; For[k = 0, k < n, k + +, (Wi[i] =
(Wi[i]/. y[k] → y0[k]))]; )];
18. For[i = n, i ≤ p + n − 1, i + +, (Yi[i] = Wi[i]; For[i1 = i, i1 ≥ 1, i1 − −, (Yi[i1 − 1] =
Integrate[Yi[i1], {ti[i1], 0, ti[i1 − 1]}])]; Y = Y + (Yi[0]/. ti[0] → t); )];
19.  ZAns = Collect[Y, t, Simplify]//PolynomialForm[#, TraditionalOrder → False]&
```

The long answer we get is given below:

$$Y(t) = \frac{-4207309314083 + 4209642036000e^{t} + 6494418000e^{2t} - 9303688000e^{3t} + 555217875e^{4t} + 61298208e^{5t}}{139968000}$$
$$+ \left(-\frac{61754916511}{9331200} - \frac{71864e^{t}}{3} + \frac{29543e^{2t}}{48} - \frac{7117e^{3t}}{243} + \frac{5465e^{4t}}{13824}\right)t + \left(\frac{32404819}{38880} + \frac{130171e^{t}}{18} - \frac{3797e^{2t}}{32}\right.$$
$$\left.- \frac{4091e^{3t}}{972} - \frac{677e^{4t}}{1152}\right)t^2 + \frac{(11159501 - 27474336e^{t} + 1920024e^{2t} - 85152e^{3t})t^3}{15552}$$
$$+ \frac{(369949 + 107676e^{t} - 1404e^{2t} - 368e^{3t})t^4}{1296} + \frac{(195749 - 64584e^{t} + 15426e^{2t})t^5}{2160} + \left(\frac{5957}{270} - \frac{176e^{t}}{15} + \frac{97e^{2t}}{144}\right)t^6$$
$$+ \left(\frac{529}{140} - \frac{739e^{t}}{630}\right)t^7 + \frac{(-262 + 257e^{t})t^8}{2520} - \frac{3803t^9}{9072} - \frac{1627t^{10}}{10080} - \frac{28927t^{11}}{1247400}$$

Using Mathematica, we expand each of the exponentials in the body of above solution expression as their own series in t and then simplify to get the answer as

$$Y(t) = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} + \frac{t^5}{120} + \frac{t^6}{720} + \frac{t^7}{5040} + \frac{t^8}{40320} + \frac{t^9}{362880} + \frac{585721t^{10}}{3628800} - \frac{1627t^{10}}{10080} - \frac{28927t^{11}}{1247400}$$

We recognize the first nine terms as the expansion of e^t which is the solution of the above differential equation with given initial values. The three highest order terms would also change to exponential series solution if we increased the order of expansion but, as always with the series, we would have to be careful with some of the highest order terms since they could change further when the expansion order would increase.

5. Solution to nth Order Ordinary Differential Equations

The solution to nth order ordinary differential equations is very simple and follows intuitive from our approach to the solution of second order differential equations. Exactly following the logic for the solution of second order ODEs, we write the integral equation of Nth derivative. Here all derivatives of order less than Order N present in the integral equation would be dependent variables.

Mathematica Program for Symbolic Computing of the Solution of Third Order Ordinary Differential Equations.

Here is the initial value problem for the third order ordinary differential equation where we use our general n-order ordinary differential equation solution code.

Example 9

The IVP is stated as

$$y''' = -(t^2 - 2t + 5)y'' - (t - 8)y' + 4y,$$

$$y(0) = y_0 = 1, \quad y'(0) = y_0' = 0, \quad y''(0) = y_0'' = 0$$

Equation (29)

Here is the mathematica code in text format following which the solution is given.

```
1. Clear[t, y, Zi, Wi, Yi, Y, y0, i, i1, m, n, ZAns];
2. n = 3;
3. Array[y, n + 1, 0];
4. y[3] := -(t^2 - 2 t + 5) y[2] - (t - 8) y[1] + 4 y[0];
5. Array[y0, n, 0];
6. y0[0] = 1;
7. y0[1] = 0;
8. y0[2] = 0;
9. p = 10;
10. Y := y0[0];
11. For[k = 1, k < n, k++, Y = Y + y0[k]*t^k/k!]
```

```
12. Array[Zi, n + p - 1, 0];
13. Array[Wi, n + p - 1, 0];
14. Array[Yi, n + p - 1, 0];
15. Array[ti, n + p - 1, 0];
16. Zi[n] = (y[n] /. t -> ti[n]);
17. For[i = n, i < p + n - 1, i++, (Zi[i + 1] = 0;
   For[k = 0, k < n, k++, (Zi[i + 1] = Zi[i + 1] + D[Zi[i], y[k]]*(y[k + 1] /. t -> ti[i +
1]));]);];
18. For[i = n, i <= p + n - 1, i++, (Wi[i] = Zi[i];
For[k = 0, k < n, k++, (Wi[i] = (Wi[i] /. y[k] -> y0[k]))];)];
19. For[i = n, i <= p + n - 1, i++, (Yi[i] = Wi[i]; For[i1 = i, i1 >= 1,i1--, (Yi[i1 - 1] =
Integrate[Yi[i1], {ti[i1], 0, ti[i1 - 1]}])];
  Y = Y + (Yi[0] /. ti[0] -> t);)];
20. ZAns = Collect[Y, t, Simplify] // PolynomialForm[#, TraditionalOrder -> False] &
```

The first ten terms of the solution are given as

$$Y(t)=1+\frac{2t^3}{3}-\frac{5t^4}{6}+\frac{37t^5}{30}-\frac{13t^6}{9}+\frac{488t^7}{315}-\frac{15217t^8}{10080}+\frac{3527t^9}{2592}-\frac{346613t^{10}}{302400}$$

Example 10

The non-linear IVP is stated as

$$y''' = .5yy''$$

$$y(0) = 0, \qquad y'(0) = 0, \qquad y''(0) = y_0''$$

Equation (29)

Here is the mathematica code for the solution of initial value problem in equation (29) in text format following which the solution is given.

```
1. ClearAll[Z, t, y, Zi, Wi, Yi, Y, y0, i, i1, p, n, ZAns]
2. n = 3;
3. Array[y,n+1,0];
4. y[3]: = .5y[0]y[2];
5. Array[y0, n, 0];
6. y0[0] = 0;
7. y0[1] = 0;
```

```
8.  p=15;
9. Y:=y0[0];
10. For[k = 1, k < n, k + +, Y = Y + y0[k] * t^k/k!]
11. Array[Zi,n+p-1,0];
12. Array[Zi, n + p - 1, 0];
13. Array[Wi,n+p-1,0];
14. Array[Yi, n + p - 1, 0];
15. Array[ti, n + p - 1, 0];
16. Zi[n] = (y[n] /. t -> ti[n]);
17. For[i = n, i < p + n – 1, i + +, (Zi[i + 1] = 0; For[k = 0, k < n, k + +, (Zi[i + 1] = Zi[i +
1] + D[Zi[i], y[k]] ∗ (y[k + 1] /. t → ti[i + 1])); ]); ];
18. For[i = n, i ≤ p + n – 1, i + +, (Wi[i] = Zi[i]; For[k = 0, k < n, k + +, (Wi[i] =
(Wi[i] /. y[k] → y0[k]))]; )];
19. For[i = n, i ≤ p + n – 1, i + +, (Yi[i] = Wi[i]; For[i1 = i, i1 ≥ 1, i1 – –, (Yi[i1 – 1] =
Integrate[Yi[i1], {ti[i1],0, ti[i1 – 1]}])]; Y = Y + (Yi[0] /. ti[0] → t); )];
 20. ZAns = Collect[Y, t, Simplify] // PolynomialForm[#, TraditionalOrder -> False] &
```

The solution is given in terms of arbitrary values of y0[2] which agrees with the series solution of the Ordinary differential equation (29). y0[2] is defined as the initial value of the second derivative in the third order ODE problem.

$Y(t) = 0. + \frac{1}{2}t^2 y0[2] + 0.004166666666666667 t^5 y0[2]^2 + 0.00006820436507936508 t^8 y0[2]^3 + 0.0000011743175805 6758 t^{11} y0[2]^4 + 1.999996187124163 \times 10^{-8} t^{14} y0[2]^5 + 3.353661721712716 \times 10^{-10} t^{17} y0[2]^6$

6. Solution to System of Differential Equations using the method of Iterated Integrals

Let us write the problem of a system of two first order differential equations with initial conditions. Here we do not present any theoretical considerations regarding existence of solutions, stability of the system or any other theoretical considerations. We merely try to present a problem that can solve large non-linear systems that are sufficiently well-behaved. Let us consider a system of two first order equations given below.

$$y' = f(\tau, y(\tau), u(\tau)), \quad y(t_0) = y_0$$

$$u' = g(\tau, y(\tau), u(\tau)), \quad u(t_0) = u_0 \qquad \text{Equation (30)}$$

Just following the logic of the method of iterated integrals as given for the first and second order ODEs, I will write the first few iterated integrals terms of the solution to each dependent variable in the system of two first order differential equations. Since in general each variable in the system of two first order equations depends upon both variables in the system, we have to differentiate each of the differential equations with respect to both variables and then convert everything into derivative with respect to independent variable using both differential equations posed in the problem while evaluating each dependent variable in the solution of iterated integrals conveniently at its initial value.

$$\begin{aligned} y(t) = y(t_0) &+ \int_{t_0}^{t} f(\tau, y(t_0), u(t_0))\, d\tau \\ &+ \int_{t_0}^{t}\int_{t_0}^{\tau} \Big(\frac{\partial}{\partial y}[f(\tau, y(t_0), u(t_0))] f(s, y(t_0), u(t_0)) \\ &+ \frac{\partial}{\partial u}[f(\tau, y(t_0), u(t_0))] g(s, y(t_0), u(t_0))\Big)\, ds\, d\tau \\ &+ \int_{t_0}^{t}\int_{t_0}^{\tau}\int_{t_0}^{s} \Big(\frac{\partial}{\partial y}[\frac{\partial}{\partial y}[f(\tau, y(t_0), u(t_0))] f(s, y(t_0), u(t_0)) \\ &+ \frac{\partial}{\partial u}[f(\tau, y(t_0), u(t_0))] g(s, y(t_0), u(t_0))] f(v, y(t_0), u(t_0))\Big)\, dv\, ds\, d\tau \\ &+ \int_{t_0}^{t}\int_{t_0}^{\tau}\int_{t_0}^{s} \Big(\frac{\partial}{\partial u}[\frac{\partial}{\partial y}[f(\tau, y(t_0), u(t_0))] f(s, y(t_0), u(t_0)) \\ &+ \frac{\partial}{\partial u}[f(\tau, y(t_0), u(t_0))] g(s, y(t_0), u(t_0))] g(v, y(t_0), u(t_0))\Big)\, dv\, ds\, d\tau \\ &+ \text{higher order terms} \end{aligned}$$

$$
\begin{aligned}
u(t) = u(t_0) &+ \int_{t_0}^{t} \mathrm{g}(\tau, \mathrm{y}(t_0), \mathrm{u}(t_0))\, d\tau \\
&+ \int_{t_0}^{t}\int_{t_0}^{\tau} \Big(\frac{\partial}{\partial y}[\mathrm{g}(\tau, \mathrm{y}(t_0), \mathrm{u}(t_0))]\, f(s, y(t_0), u(t_0)) \\
&+ \frac{\partial}{\partial u}[\mathrm{g}(\tau, \mathrm{y}(t_0), \mathrm{u}(t_0))]\, \mathrm{g}(s, \mathrm{y}(t_0), \mathrm{u}(t_0))\Big)\, ds\, d\tau \\
&+ \int_{t_0}^{t}\int_{t_0}^{\tau}\int_{t_0}^{s} \Big(\frac{\partial}{\partial y}[\frac{\partial}{\partial y}[g(\tau, y(t_0), u(t_0))]f(s, y(t_0), u(t_0)) \\
&+ \frac{\partial}{\partial u}[g(\tau, u(t_0))]g(s, y(t_0), u(t_0))]f(v, y(t_0), u(t_0))\Big)\, dv\, ds\, d\tau \\
&+ \int_{t_0}^{t}\int_{t_0}^{\tau}\int_{t_0}^{s} \Big(\frac{\partial}{\partial u}[\frac{\partial}{\partial y}[g(\tau, y(t_0), u(t_0))]f(s, y(t_0), u(t_0)) \\
&+ \frac{\partial}{\partial u}[g(\tau, y(t_0), u(t_0))]g(s, y(t_0), u(t_0))]g(v, y(t_0), u(t_0))\Big)\, dv\, ds\, d\tau \\
&+ \text{higherOrderIntegralterms}
\end{aligned}
$$

Equation (31)

Systems of large number of equations and systems of higher order differential equations can also be very easily solved by simple algorithms using symbolic computing as we give a general program for the solution of systems of m equations each having their own order of the highest derivative.

Mathematica Program for Symbolic Computing of the Solution of Systems of Arbitrary Order Ordinary Differential Equations.

Let us try to solve a system of three second order non-linear ordinary differential equations using our very general mathematica code. The system of ordinary differential equations is given as

Example 11

$$
\begin{aligned}
y_1'' &= -.5(y_1')^2 - y_1', \qquad y_1(0) = 0, \qquad y_1'(0) = 2 \\
y_2'' &= -.5(y_2')^2 - y_1'y_2' - y_2', \qquad y_2(0) = 0, \qquad y_2'(0) = -3 \\
y_3'' &= -(y_3')^2 - y_1'y_3' - y_2'y_3' - y_3', \qquad y_3(0) = 0, \qquad y_3'(0) = -2
\end{aligned}
$$

Equation (32)

The problem without particular initial values is discussed as example 1 in [2] ,where the solution is given as

$$y_1 = 2\ln\left[\frac{c1 + c2\, e^{-t}}{2}\right]$$

$$y_2 = 2\ln\left[\frac{c3 + c4\, e^{-t}}{c1 + c2\, e^{-t}}\right]$$

$$y_3 = \ln\left[\frac{2(c5 + c6\, e^{-t})}{c3 + c4\, e^{-t}}\right]$$

Equations (33)

Here is the mathematica code for systems of ordinary differential equations of arbitrary order in text.

```
1. Clear[t, y, Zi, Wi, Yi, Y, y0, i, i1, l, m, n, p, j, k, p, ZAns]
2. m = 3;
3. Array[n, m, 0];
4. n[0] = 2;
5. n[1] = 2;
6. n[2] = 2;
7. nmax = 2;
8. Array[y, {m, nmax + 1}, {0, 0}];
9. y[0, 2] := -.5 y[0, 1]^2 - y[0, 1];
10. y[1, 2] := -.5 y[1, 1]^2 - y[1, 1] y[0, 1] - y[1, 1];
11. y[2, 2] := -y[2, 1]^2 - y[1, 1] y[2, 1] - y[0, 1] y[2, 1] - y[2, 1];
12. Array[y0, {m, nmax}, {0, 0}];
13. y0[0, 0] = 0;
14. y0[0, 1] = 2;
15. y0[1, 0] = 0;
16. y0[1, 1] = -3;
17. y0[2, 0] = 0;
18. y0[2, 1] = -2;
19. Array[Y, m, 0];
20. For[j = 0, j < m, j++, (Y[j] := y0[j, 0];
For[k = 1, k < n[j], k++, Y[j] = Y[j] + y0[j, k]*t^k/k!];)]
21. p = 10;
22. Array[Zi, {m, nmax + p - 1}, {0, 0}];
23. Array[Wi, {m, nmax + p - 1}, {0, 0}];
24. Array[Yi, {m, nmax + p - 1}, {0, 0}];
25. Array[ti, nmax + p - 1, 0];
26. For[j = 0, j < m, j++, Zi[j, n[j]] = (y[j, n[j]] /. t -> ti[n[j]]);]
27. For[j = 0, j < m, j++, (For[i = n[j], i < p + n[j] - 1, i++, (Zi[j, i + 1] = 0;
   For[k = 0, k < m, k++, (For[l = 0, l < n[k], l++,
  (Zi[j, i + 1] = Zi[j, i + 1] + D[Zi[j, i], y[k, l]]*(y[k, l + 1] /. t -> ti[i + 1]))])];)])]
```

```
28. For[j = 0, j < m, j++, (For[i = n[j], i <= p + n[j] - 1, i++, (Wi[j, i] = Zi[j, i];
   For[k = 0, k < m, k++, For[l = 0, l < n[k], l++,
     (Wi[j, i] = (Wi[j, i] /. y[k, l] -> y0[k, l]))]];)])]
29. For[j = 0, j < m, j++, (For[i = n[j], i <= p + n[j] - 1, i++, (Yi[j, i] = Wi[j, i];
   For[i1 = i, i1 >= 1,  i1--, (Yi[j, i1 - 1] =  Integrate[Yi[j, i1], {ti[i1], 0, ti[i1 - 1]}])];
   Y[j] = Y[j] + (Yi[j, 0] /. ti[0] -> t);)])]
30.For[j = 0, j < m, j++,
(ZAns[j] = Collect[Y[j], t, Simplify] // PolynomialForm[#, TraditionalOrder -> False] &)]
```

Here is the output result of the above code.

Y[0]= $2t - 2.\,t^2 + 2.\,t^3 - 2.1666666666666665t^4 + 2.5t^5 - 3.0055555555555555t^6 + 3.716666666666667t^7 - 4.691765873015872t^8 + 6.016699735449735t^9 - 7.812236552028217t^{10} + 10.246068121693122t^{11}$

Y[1]= $-3t + 2.25t^2 - 2.\,t^3 + 2.15625t^4 - 2.5t^5 + 3.00625t^6 - 3.716666666666667t^7 + 4.691713169642856t^8 - 6.016699735449735t^9 + 7.812240823412698t^{10} - 10.246068121693122t^{11}$

Y[2]= $2t - 2t^2 - 2.5t^3 - 3.6666666666666665t^4 - 5.75t^5 - 9.380555555555555t^6 - 15.739583333333332t^7 - 26.96051587301587t^8 - 46.91410383597883t^9 - 82.6557881393298t^{10} - 147.0983837632275t^{11}$

Which agrees with the series expansion of the solution equations (33) for the choice of constants chosen to match the initial conditions.

Explanation of the Mathematica code for the solution of systems of arbitrary order of ordinary differential equations.

Here is the line by line explanation of the mathematica code.
Command 1 is used to clear the variables from memory so previously stored values would not affect outcome of our program.
Command 2 describes the number of equations in the system of the ordinary differential equations of arbitrary order. In the case of our chosen example with three equations, m=3.
Command 3 defines an array that would be used to store the order of each ordinary differential equations. For m equations, this array has length m. n[0] stores the order of first differential equation. n[1] stores the order of second differential equation so till n[m-1] representing order of the last equation in system.

Commands 4-6 specify the order of each ordinary differential equation in the system one by one.
Command 7 describes the highest order of all the ordinary differential equations in the system.
Command 8 defines a two dimensional array of variables that would specify the equations in the system. The first index of each array would specify which ordinary differential equation the variable inherently belongs and the second index would specify the order of the derivative associated with the variable. Y[0,2] would specify that the variable is a second order derivative of the variable associated with the first ordinary differential equation in the system as indexing starts at zero.
Commands 9-11 define the three ordinary differential equations in our system.
Command 12 defines the array of initial values of the derivatives associated with each ordinary differential equation in the system. This array would be populated in the following commands 13-18.
Command 19 defines an m-dimensional array that would be used to store the calculated solution of each of the ordinary differential equation in the system of m equations.
Command 20 initialize the solution array with initial values of the derivatives of each equation.
Command 21 defines p as the number of iterated integral terms to solve for the series solution to each equation.
In commands 22 to 24, we define the array variables for symbolic computations in the following loops.
In command 25, we define the array that would be used as a variable of integration changing in each loop. This array of variables of integration represents the variable of integration used in the integration with respect to independent variable in the ordinary differential equations.
In command 26, we assign first iterated integral towards the computation of the solution of each equation in the system (of order m).
In command 27, we calculate the integrand for (iterated integrations to be done later) each of the iterated integrals term one by one.
In command 28, we assign numerical initial values to dependent variables in the integrands for various iterated integral terms calculated in command 27.
In command 29, we do the iterated integrations for the solution of each iterated integrals term.

In command 30, we simplify the solution and arrange it in ascending powers of the independent variable.

References:

[1] Nonlinear Ordinary Differential Equations: Analytical Approximation and Numerical Methods by Martin Hermann, Masoud Saravi

[2] http://file.scirp.org/pdf/AM_2014021310154871.pdf

[3] Nonlinear Ordinary Differential Equations and Their Applications by P. L. Sachdev