Luhn Primes of Order ω

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Abstract

A prime p, that added to its reverse results in a new prime is called a *Luhn prime* . The number 229 is a *Luhn prime* because $229 + 922 = 1151$ and 1151 is also a prime .

We extend the definition of *Luhn primes*, present some of the of their properties and ways to determine them .

1 Introduction

The reversed of a positive integer number is the number obtained by arranging the digits in reverse order . I.E. the reversed of the positive integer number $\overline{d_1 d_2 \ldots d_m}$ is $\overline{d_m d_{m-1} \ldots d_1}$, where $d_k \in \{0, 1, 2, \ldots, 9\}$, $k = 1, 2, \ldots, m$. We denote by $R(n)$ the reversed of $n \in \mathbb{N}$, A004086 . Let us denote the set of prime numbers by $\mathbb{P}$.

In 2013 Luhn, [1], observed that 229 is the smallest prime number that added to its reversed results in a new prime number, [6], A061783 . We denote the set of numbers that have this property by $\mathbb{L}$ and refer to them as Luhn prime numbers . We use the expression $\mathbb{L}$ numbers for numbers belonging to the set of numbers $\mathbb{L}$.

Using a Mathcad program that verifies all natural numbers up to 10^7 we showed in [5] that there are 50598 $\mathbb{L}$ numbers . Later Chai Wah Wu A061783 determined the 3010506 $\mathbb{L}$ numbers smaller then 10^9 .

In this article we try to answer questions regarding the set of $\mathbb{L}$. There are 27, 1078, 50598, and 3010506 $\mathbb{L}$ numbers smaller then 10^3, 10^5, 10^7, and respectively 10^9.

Definition 1. The numbers $p \in \mathbb{P}$ that satisfy the condition $p^\omega + R(p)^\omega \in \mathbb{P}$ are called *Luhn primes of order* ω .

We denote Luhn primes of order ω by $\mathbb{L}_\omega$. Thus, using definition 1 we state that the set of numbers, defined by Luhn, are $\mathbb{L}_1$.

Remark 2. A prime $p \in \mathbb{P}$ belongs to $\mathbb{L}_\omega$ only if its first decimal is 2, 4, 6, or 8 . This condition is necessary since $q = p^\omega + R(p)^\omega$ will be an odd number only if p's first decimal is an even digit . This observation drastically reduced the search space for the $\mathbb{L}_\omega$ numbers to 44.(4)% of all considered primes .

The sequences of integers that use the notion "reverse" are: "*numbers n of the form $k+reverse(k)$ for at least one k*", A067030, "*emirp numbers*", A006567 and "*read n backwards*" A004086 .

2 Very Probably Primes

The *IsPrime* command is a probabilistic primality testing routine . It returns false if n is shown to be composite within one *strong pseudo-primality* Miller–Rabin test and one Lucas test . It returns true otherwise . If *IsPrime* returns true, n is very probably prime, [6, 10, 12, 13, 11, 4] . No counterexample is known and it has been conjectured that such a counterexample must be hundreds of digits long .

We call very probably primes those natural numbers $n \in \mathbb{N}$ that *IsPrime* Mathcad function returns 1 to .

We denote this set of numbers by $\mathrm{VP}\mathbb{P}$. Is is proven that up to 10^{16} Miller–Rabin and Lucas tests are equivalent to a deterministic algorithm, [7] . Even more so, there are no known counter example for the two tests . As a consequence we define:

Definition 3. For any $n \in \mathbb{N}$

- If $n < 10^{16}$ and $IsPrime(n) = 1 \Rightarrow n \in \mathbb{P}$;
- If $n \geq 10^{16}$ and $IsPrime(n) = 1 \Rightarrow n \in \mathrm{VP}\mathbb{P}$;
- If $IsPrime(n) = 0 \Rightarrow n \notin \mathbb{P}$.

3 Luhn primes of order 0

Theorem 4. *$\mathbb{L}_0$ is equivalent to the set of primes $\mathbb{P}$.*

Proof. For any $p \in \mathbb{P}$ we have that $p^0 + R(p)^0 = 2 \in \mathbb{P}$. Thus, according to the Definition 1, $p \in \mathbb{L}_0$, namely $\mathbb{P} \subset \mathbb{L}_0$. If $\ell \in \mathbb{L}_0$ then, by definition $\ell \in \mathbb{P}$, thus $\mathbb{L}_0 \subset \mathbb{P}$. Therefor $\mathbb{L}_0 \equiv \mathbb{P}$. □

4 Luhn primes of order 1

We present a theorem proving his assertion Chai Wah Wu, A061783 .

Theorem 5. *All $\mathbb{L}_1$ numbers have an odd number of digits .*

Proof. If p a prime has $2n$ digits $d_k \in \{0, 1, 2, \ldots, 9\}$, then

$$\begin{aligned} p + R(p) = \overline{d_1 d_2 d_3 \ldots d_{2n}} + \overline{d_{2n} d_{2n-1} d_{2n-2} \ldots d_1} \\ = (10^{2n-1} d_1 + 10^{2n-2} d_2 + 10^{2n-3} d_3 + \ldots + d_{2n}) \\ + (10^{2n-1} d_{2n} + 10^{2n-2} d_{2n-1} + 10^{2n-3} d_{2n-2} + \ldots + d_1) \\ = (10^{2n-1} + 1)(d_1 + d_{2n}) + (10^{2n-2} + 10)(d_2 + d_{2n-1}) \\ + (10^{2n-3} + 10^2)(d_3 + d_{2n-2}) + \ldots + (10^n + 10^{n-1})(d_n + d_{n+1}). \end{aligned}$$

A positive integer number is divisible by 11 if the difference between the sum of even placed digits and the sum of the odd placed digits is a multiple of 11 .

All numbers $10^{2n-1} + 1$, $10^{2n-2} + 10$, $10^{2n-3} + 10^2$, $\ldots$, $10^n + 10^{n-1}$ are divisible by 11 because the difference between the sums of all even placed digits and all odd placed digits is 0, and 0 is divisible by 11 .

Consequently, $p + R(p) \notin \mathbb{P}$, for any $n \in \mathbb{N}$. Therefore all primes with even number of digits can not be in $\mathbb{L}_1$. ☐

Remark 6. Theorem 5 allows us to reduce the search space for $\mathbb{L}_1$ number to the set of primes that have an odd number of digits .

Using remark 2 we conclude that the search space is formed by prime numbers that start with with 2, 4, 6, or 8 and must have an odd number of digits .

Using a Mathcad program we selected the 1078 $\mathbb{L}_1$ numbers out of the 9592 primes smaller then 10^5.

Table 1: Primes p such that $p + R(p) = q \in \mathbb{P}$

q	p, p, $\ldots$;
383	241;
443	271;
463	281;
787	443, 641;
827	463, 661;
887	691;
929	613, 811;
1009	257, 653;
1049	277, 673;
1069	683, 881;
1151	229, 823;
1171	239;
1231	269, 467, 863;
1373	439;

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q	p, p, . . . ;
1453	479, 677;
1493	499;
1777	839;
30203	20101;
30403	20201;
31013	20011, 21001;
32213	20611, 21601;
32413	21701;
32423	21211;
33023	20521, 22501;
33223	21611;
34033	21521, 22511;
34843	20441;
35053	20051, 22031, 23021;
36263	20161, 23131, 24121, 25111;
36653	21841, 25801;
37273	25121, 26111;
37463	21751, 22741, 26701;
37663	21851, 23831, 24821, 26801;
38083	25031, 26021, 27011, 28001;
38273	21661, 22651, 24631, 25621;
38873	21961, 27901;
39293	22171, 24151, 28111, 29101;
39883	20981, 22961, 25931, 26921, 28901;
40093	22571, 23561, 24551, 25541, 29501;
40493	23761, 25741, 26731, 28711;
40693	21881, 22871, 24851, 25841;
40903	21491, 22481, 27431, 29411;
41113	22091, 23081, 24071, 28031, 29021;
41513	22291, 25261, 26251, 27241, 29221;
41903	21991, 23971, 25951, 28921;
42323	24181, 25171, 26161, 29131;
42923	24481, 25471;
43133	24091, 27061, 28051;
44533	28751, 29741;
44543	27271, 29251;
45343	26681, 28661;
45553	27281;
45943	26981, 28961;

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q	p, p, ...;
45953	27481;
46153	26591,27581, 28571;
47363	27691, 29671;
47563	27791;
48383	29191;
70207	60101;
71807	40903, 60901;
72227	40123, 41113;
72617	40813, 60811;
72817	41903;
73037	41023, 42013, 43003, 62011;
73237	43103, 61121;
73637	60331, 62311;
74047	42023, 43013, 60041, 61031;
76667	63331, 66301;
77267	42643, 43633, 44623, 45613, 60661, 61651, 64621, 66601;
77477	41263, 60271, 61261, 63241, 64231;
77867	41953, 42943, 43933, 60961, 64921;
78277	44633, 67601;
78487	40283, 45233, 66221, 67211;
78877	40973, 42953, 43943, 47903, 61961, 65921, 67901;
78887	40483, 42463, 45433, 61471, 67411;
79087	40583, 44543, 45533, 46523, 47513, 67511, 68501;
79687	40883, 42863, 43853, 44843, 45833, 61871, 62861, 65831, 66821;
79697	42373, 45343, 48313, 61381, 63361, 68311;
80107	42083, 45053, 48023, 61091, 62081, 66041, 69011;
80897	40993, 41983, 43963, 44953, 45943, 46933, 61981, 62971, 64951, 66931;
81307	42683, 46643, 48623, 49613, 63671, 64661, 65651, 67631;
81517	42293, 43283, 44273, 45263, 49223, 63281, 64271, 69221;
81707	41893, 45853, 48823, 65851, 66841, 68821;
82727	44383, 47353, 49333, 63391, 64381, 65371, 66361;
83137	45083, 46073, 49043, 64091, 66071, 67061;
83537	44293, 46273, 66271, 67261;
83737	47363, 48353, 65381, 68351, 69341;
84137	46573, 47563, 64591, 65581, 66571;
84347	46183, 48163, 68161, 69151;
84737	44893, 49843, 64891, 65881;
84947	48463;
85147	48563;

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q	$p, p, \ldots;$
86357	48673, 49663, 69661;
86767	48383, 67391, 69371;
87767	48883, 67891, 68881;
87977	68491, 69481;
88177	48593;
89387	69691;
91009	20507;
91019	60013, 81001;
92219	61603, 80611;
92419	20717, 61703, 81701;
93229	20627, 21617, 60623, 61613, 62603, 80621, 81611, 82601;
93239	60133, 63103, 83101;
93629	21817, 22807, 61813;
94049	20047, 22027, 23017, 24007, 81031, 82021;
94439	21727, 22717, 60733, 61723, 63703, 83701;
94649	20347, 60343, 61333, 62323, 63313, 64303, 80341, 81331, 83311;
94849	23417, 24407, 60443, 62423, 64403, 82421, 84401;
96059	24517, 61543, 62533, 64513, 82531;
96259	21647, 22637, 23627, 61643, 62633, 64613, 80651, 83621, 85601;
96269	21157, 22147, 25117, 26107, 61153, 62143, 64123, 66103, 82141, 84121;
96469	22247, 61253, 64223, 65213, 82241, 83231, 84221, 86201;
97879	20477, 21467, 23447, 26417, 27407, 61463, 63443, 64433, 65423, 66413, 80471, 84431;
98479	21767, 23747, 26717, 60773, 62753, 63743, 66713, 81761, 84731, 86711, 87701;
98689	21377, 22367, 23357, 28307, 60383, 63353, 81371, 82361, 85331, 88301;
99089	21577, 22567, 23557, 24547, 25537, 62563, 66523, 82561, 85531, 87511;
99289	26627, 27617, 28607, 61673, 65633, 80681, 81671;
99689	20887, 23857, 24847, 27817, 28807, 63853, 85831, 87811, 88801;
100109	25057, 28027, 29017, 63073, 64063, 65053, 67033, 68023, 83071, 84061;
100699	20897, 22877, 25847, 27827, 28817, 62873, 63863, 64853, 65843, 68813, 88811;
101119	23087, 24077, 29027, 65063, 67043, 85061, 87041, 89021;
101719	22397, 25367, 26357, 29327, 64373, 67343, 85361, 86351;
102329	23197, 29137, 65173, 67153, 84181, 86161, 87151;
102929	23497, 27457, 28447, 29437, 63493, 64483, 66463, 67453, 68443, 84481, 86461, 89431;
103529	63793, 64783, 66763, 68743, 83791, 87751, 88741;
104149	25097, 27077, 66083, 67073, 85091, 87071, 89051;

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q	p, p, . . . ;
104549	27277, 65293, 67273, 88261;
105359	29167, 69163, 87181;
106759	29867, 67883, 87881;
108179	28597, 29587, 88591;
108379	28697, 89681;
109789	89891;
111211	80603;
111611	20809, 80803;
112031	20029, 21019, 41017, 81013, 82003;
112831	21419, 22409, 42407, 40427;
113041	40037, 42017, 81023, 82013, 83003;
114041	21529, 23509;
114451	20249, 22229, 42227, 44207, 81233, 82223;
114641	80833, 82813;
115061	22039, 23029, 24019, 41047, 44017, 45007, 81043, 83023;
115861	24419, 25409, 42437, 43427, 44417, 83423;
116461	20759, 22739, 42737, 45707, 84713, 85703;
116471	20269, 24229, 25219, 26209, 41257, 43237, 80263, 83233, 84223, 85213;
117071	21559, 22549, 23539, 46507, 81553, 84523, 85513;
117281	21169, 22159, 26119, 27109, 40177, 42157, 45127, 80173, 81163, 82153, 86113, 87103;
117671	21859, 25819, 40867, 45817, 46807, 80863, 81853, 83833;
117881	20479, 24439, 27409, 41467, 42457, 45427, 47407, 80473, 81463, 83443, 86413, 87403;
118081	21569, 23549, 27509, 40577, 42557, 44537, 47507, 81563, 84533, 85523;
118681	20879, 22859, 27809, 45827, 46817, 47807, 83843, 86813, 87803;
118691	20389, 21379, 22369, 25339, 28309, 40387, 45337, 46327,47317, 81373, 85333, 86323, 87313;
118891	22469, 23459, 25439, 28409, 40487, 42467, 43457, 47417, 48407, 82463, 84443, 86423;
119101	21089, 22079, 28019, 29009, 43067, 48017, 81083, 82073, 83063, 84053, 89003;
119291	22669, 25639, 42667, 44647, 80683, 83653, 87613;
119701	20399, 23369, 24359, 25349, 26339, 27329, 28319, 41387, 44357, 46337, 49307, 82373, 87323, 89303;
119891	25939, 27919, 28909, 42967, 47917, 48907, 81973, 82963, 85933, 86923, 88903;
120511	23279, 26249, 27239, 28229, 44267, 47237, 81293, 83273, 84263, 86243, 88223, 89213;

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q	$p, p, \ldots;$
121321	23189, 24179, 25169, 29129, 42197, 47147, 82193, 89123;
121711	23879, 26849, 29819, 41897, 44867, 47837, 82883, 83873, 85853, 86843, 87833;
121721	24379, 42397, 48337, 82393, 83383, 85363, 86353;
121921	25469, 26459, 27449, 28439, 29429, 43487, 46457, 48437, 82493, 86453, 87443;
122131	23099, 27059, 44087, 45077, 47057, 49037, 83093;
122321	22699, 23689, 29629, 42697, 45667, 49627, 84673, 87643;
122921	24979, 25969, 26959, 43987, 46957, 47947, 49927, 83983, 87943, 89923;
123341	25189, 48157;
123731	23899, 24889, 44887, 46867, 47857, 48847, 87853, 88843, 89833;
123931	24989, 28949, 43997, 44987, 48947, 49937;
123941	24499, 26479, 44497, 46477, 89443;
124351	26189, 27179, 45197, 46187, 49157, 85193, 86183, 89153;
124541	24799, 28759, 44797, 48757, 49747, 84793;
124951	26489, 27479, 45497, 87473, 88463;
125551	25799, 27779, 29759, 47777, 48767, 49757, 85793, 86783, 89753;
126761	28879, 88873;
126961	28979, 46997, 86993, 89963;
127781	29389, 48397;
128591	49297, 89293;
128981	29989, 88993, 89983;
131023	60017, 61007;
131213	40609, 60607;
131413	40709;
132233	40129, 60127;
132623	40819, 41809;
132833	40429, 42409, 60427, 61417;
133033	40529, 41519, 42509, 60527, 62507;
133633	40829;
133843	62417;
134053	41039, 43019, 64007;
134243	40639, 43609, 60637, 61627, 62617, 63607;
134443	40739, 41729, 42719 ,60737;
135463	42239, 60257, 64217;
136273	40169, 44129, 45119, 60167, 66107;
136463	40759, 60757, 63727, 64717, 65707;
137273	41659, 42649, 61657, 64627, 65617;
137483	41269, 46219, 63247, 64237;

Continued on next page

q	p, p, . . . ;
137873	41959, 64927;
138283	41669, 43649, 46619, 47609, 61667, 63647, 66617, 67607;
138493	40289, 44249, 46229, 67217, 68207;
138683	40879, 42859, 44839, 46819, 47809, 65827, 67807;
138883	41969, 44939, 46919, 61967, 64937, 65927;
138893	41479, 44449, 45439, 47419, 48409, 62467, 65437;
139303	41189, 42179, 44159, 47129, 48119, 49109, 64157, 65147, 66137;
139493	43759, 64747;
139703	41389, 42379, 60397, 63367, 66337;
142123	44579, 45569, 46559, 48539, 49529, 62597, 63587, 64577, 67547;
142733	43399, 44389, 49339, 63397, 69337;
143333	47659, 48649, 49639, 63697, 65677;
143743	45389, 66377;
145753	46889, 48869, 69857;
145963	46499, 48479, 68477, 69467;
146173	68087;
146563	48779, 66797, 68777, 69767;
147583	48299;
147773	48889, 69877;
148783	68897;
149393	69697;
149993	49999, 69997;
171617	80809;
171827	81409;
172027	81509;
174257	80149, 82129;
174457	81239, 83219;
175067	81049, 82039, 85009;
175267	82139, 85109;
176467	81749, 84719;
176677	80369, 81359, 82349, 83339;
177487	80279, 84239, 85229;
177677	85819;
177887	83449, 85429;
178487	80779, 81769, 82759, 86719;
178697	84349;
178897	80489, 82469, 83459, 84449, 85439;
179107	84059, 85049, 88019, 89009;
179497	80789, 86729, 87719;

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q	$p, p, \ldots;$
179897	80989, 86929;
180307	81689, 84659, 87629;
180317	81199, 82189, 85159, 88129, 89119;
180907	83969, 86939, 88919, 89909;
181717	81899, 82889, 84869, 89819;
181927	82499, 85469;
182537	83299, 86269;
182927	84979, 86959;
184157	88069;
184957	88469, 89459;
185167	88079, 89069;
185557	89759;
185567	89269;
185767	86399, 88379;
185957	85999, 88969, 89959;
187177	88589;
187387	89189;
187987	88499;
189797	89899.

Let the function $L_1 : \mathbb{P} \to \mathbb{N}$, $L_1(x) = x + R(x)$, where $R(x)$ is reverse the function . Then, Definition 1, $p \in \mathbb{L}_1$ it is equivalent to the statement $L_1(p) \in \mathbb{P}$. Therefor

– if $p \in \mathbb{L}_1$ then, according to Definition 1 $p \in \mathbb{P}$ and $p + R(p) \in \mathbb{P}$, i.e. $L_1(p) \in \mathbb{P}$;

– if $L_1(p) \in \mathbb{P}$ it follows that $p \in \mathbb{P}$ and $p + R(p) \in \mathbb{P}$, i.e. $p \in \mathbb{L}_1$.

We consider the function $L_1^2(x) = L_1(L_1(x))$. If $L_1^2(x) \in \mathbb{P}$, then it follows that $L_1(x) \in \mathbb{L}_1 \subset \mathbb{P}$ and $x \in \mathbb{L}_1$. We denote $\mathbb{L}_1^2 = \{p \mid L_1^2(p) \in \mathbb{P}\}$. Then for $p < 10^5$ we have 25 $\mathbb{L}_1^2$ numbers: 271, 281, 21491, 21991, 22091, 22481, 23081, 23971, 24071, 25951, 26681, 27271, 27431, 27691, 27791, 28031, 28661, 28921, 29021, 29191, 29251, 29411, 29671 .

Table 2: Table $\mathbb{L}_1^2$ numbers for $p < 10^5$

$p \in \mathbb{P}$	$L_1(p) \in \mathbb{P}$	$L_1^2(p) \in \mathbb{P}$
271	443	787
281	463	827
21491	40903	71807

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$p \in \mathbb{P}$	$L_1(p) \in \mathbb{P}$	$L_1^2(p) \in \mathbb{P}$
21991	41903	72817
22091	41113	72227
22481	40903	71807
23081	41113	72227
23971	41903	72817
24071	41113	72227
25951	41903	72817
26681	45343	79697
26981	45943	80897
27271	44543	79087
27431	40903	71807
27691	47363	83737
27791	47563	84137
28031	41113	72227
28661	45343	79697
28921	41903	72817
28961	45943	80897
29021	41113	72227
29191	48383	86767
29251	44543	79087
29411	40903	71807
29671	47363	83737

Therefore up to 10^5, respectively 10^7 we have 25, respectively 610 $\mathbb{L}_1^2$ numbers .

Using Chai Wah Wu's sequence of numbers $\mathbb{L}_1$ smaller 10^9, A061783, we determined that $299176991 \in \mathbb{L}_1^2$, because $299176991 \in \mathbb{P}$, $L_1(299176991) = 498848983 \in \mathbb{P}$ and

$$L_1^2(299176991) = L_1(L_1(299176991)) = L_1(498848983) = 888697877 \in \mathbb{P} \ .$$

For any $p \in \mathbb{L}_1^2$ by definition we like $L_1^2(p) \in \mathbb{P}$ from which it follows that $L_1(p) \in \mathbb{L}_1$ from which it follows that $p \in \mathbb{L}_1$.

Remark 7.

- Table 1 contains all $\mathbb{L}_1$ numbers up to 10^5;
- Primes 443, 463, 40903, 41113, 41903, 44543, 45343, 45943, 47363, 47563, 48383 $\in L(\mathbb{L}_1^2)$;
- Primes $q = 787,\ 827,\ 929,\ \ldots$ have two additive decomposition form $q = p + R(p)$, where $p \in \mathbb{L}_1$, see Table 1;

- Primes $q = 1231,\ 35053,\ 37463,\ \dots$ have three additive decompositions form $q = p + R(p)$, where $p \in \mathbb{L}_1$, see Table 1;
- Primes $q = 36263,\ 37663,\ 38083,\ \dots$ have four additive decompositions form $q = p + R(p)$, where $p \in \mathbb{L}_1$, see Table 1;
- Primes $q = 39883,\ 40093,\ 41113,\ \dots$ have five additive decompositions form $q = p + R(p)$, where $p \in \mathbb{L}_1$, see Table 1;
- Primes $q = 114451,\ 115861,\ 116461,\ \dots$ six two additive decompositions form $q = p + R(p)$, where $p \in \mathbb{L}_1$, see Table 1;
- Primes $q = 78877,\ 79087,\ 80107,\ \dots$ have seven additive decompositions form $q = p + R(p)$, where $p \in \mathbb{L}_1$, see Table 1;
- Primes $q = 77267$, 81307, 81517, 93229, 101119, 101719, 115061, 112321, 122321, 124351, 138283 have eight additive decompositions form $q = p + R(p)$, where $p \in \mathbb{L}_1$, see Table 1;
- Primes $q = 79687$, 94649, 96259, 99689, 123731, 125551, 139303, 142123 have nine additive decompositions form $q = p + R(p)$, where $p \in \mathbb{L}_1$, see Table 1;
- Primes $q = 79687$, 94649, 96259, 99689, 123731, 125551, 139303, 142123 have ten additive decompositions form $q = p + R(p)$, where $p \in \mathbb{L}_1$, see Table 1;
- Primes $q = 98479$, 100699, 119101, 119891, 121711, 121921 have eleven additive decompositions form $q = p + R(p)$, where $p \in \mathbb{L}_1$, see Table 1;
- Primes $q = 97879$, 102929, 117281, 117881, 118891, 120511 have twelve additive decompositions form $q = p + R(p)$, where $p \in \mathbb{L}_1$, see Table 1;
- The number $q \in \mathbb{P}$, $q = 118691$ have thirteen additive decomposition form $q = p + R(p)$, where $p \in \mathbb{L}_1$, see Table 1;
- The number prim, $q = 119701$ have fourteen additive decomposition form $q = p + R(p)$, where $p \in \mathbb{L}_1$, see Table 1 .

Open problems:

1. Is $\mathbb{L}_1$ a finite set?
2. Is $\mathbb{L}_1^2$ a finite set?
3. If $p < 10^{2n+1}$, $n \in \mathbb{N}$, then $Card(\mathbb{L}_1)$?
4. How many primes q have two additive decompositions $p + R(p) = q$, where $p \in \mathbb{L}_1$ (like 787, 827, 929, ...)?

5. How many primes q have three additive decompositions $p + R(p)$, where $p \in \mathbb{L}_1$ (like 1231, 35053, 37463, ...)?

6. How many primes q have four, five, six, ... additive decompositions $p + R(p)$, where $p \in \mathbb{L}_1$?

5 Luhn primes of order 2

Next we propose a mechanism that reduced the search space for $\mathbb{L}_2$.

Theorem 8. *The prime numbers $q = p^2 + R(p)^2$, where $p \in \mathbb{P}$, must have 3, 5 or 7 as last digit .*

Proof. All prime numbers except 2 and 5 end with 1, 3, 7 or 9 as their last digit . As a consequence p^2 must end with either 1 or 9 because $1^1 = 1$, $3^2 = 9$, $7^2 = 49$ and $9^2 = 81$.

If we have a prime number $q = p^2 + R(p)^2$ $(\neq 2)$, then its $R(p)$ must be an even number, namely p's first digit must be 2, 4, 6 or 8 . Therefore $R(p)^2$ must have 4 or 6 as its last digit since $2^2 = 4$, $4^2 = 16$, $6^2 = 36$ and $8^2 = 64$.

As a consequence, the prime number $q = p^2 + R(p)^2$ must have 7, 5 or 3 as its last digit because $4 + 1 = 5$, $6 + 1 = 7$, $4 + 9 = 13$, $6 + 9 = 15$. □

Remark 9. When searching for $\mathbb{L}_2$ numbers, if $p \in \mathbb{P}$ and its first digit is 2, 4, 6 , and 8 then $q = p^2 + R(p)^2$ will have 3, 5, or 7 as its last digit . Since all numbers ending in 5 are divisible by 5 we can exclude them from our primality test .

Remark 10. The functions determine the first and last digit of the number are:

$$\begin{aligned} f(x) &:= trunc\left(x \cdot 10^{-floor(\log(x))}\right) , \\ \ell(x) &:= x - Trunc(x, 10) , \end{aligned}$$

in syntax Mathcad .

We used a program to determine elements in $\mathbb{L}_2$. The $\mathbb{L}_2$ numbers smaller then 10^5, out of the 9592 primes are: 23, 41, 227, 233, 283, 401, 409, 419, 421, 461, 491, 499, 823, 827, 857, 877, 2003, 2083, 2267, 2437, 2557, 2593, 2617, 2633, 2677, 2857, 2887, 2957, 4001, 4021, 4051, 4079, 4129, 4211, 4231, 4391, 4409, 4451, 4481, 4519, 4591, 4621, 4639, 4651, 4871, 6091, 6301, 6329, 6379, 6521, 6529, 6551, 6781, 6871, 6911, 8117, 8243, 8273, 8317, 8377, 8543, 8647, 8713, 8807, 8863, 8963, 20023, 20483, 20693, 20753, 20963, 20983, 21107, 21157, 21163, 21383, 21433, 21563, 21587, 21683, 21727, 21757, 21803, 21863, 21937, 21997, 22003, 22027, 22063, 22133, 22147, 22193, 22273, 22367, 22643, 22697, 22717, 22787, 22993, 23057, 23063, 23117, 23227, 23327, 23473, 23557, 23603, 23887, 24317, 24527, 24533, 24547, 24623, 24877, 24907, 25087, 25237, 25243, 25453, 25523, 25693, 25703, 25717, 25943, 26053, 26177, 26183, 26203, 26237, 26357, 26407, 26513, 26633, 26687, 26987, 27043, 27107, 27397, 27583, 27803, 27883, 28027, 28297, 28513, 28607, 28643, 28753, 28807, 29027, 29063, 29243, 29303, 29333, 29387, 29423, 29537, 29717, 29983, 40039, 40111, 40459, 40531, 40591, 40801, 40849, 41039, 41131, 41351, 41389, 41491, 41539, 41609, 41651, 41759, 41761, 41801, 41849, 41879, 41941, 41999, 42179, 42209, 42451, 42461, 42499, 42569, 42571, 42649, 42839, 42859, 42901, 43049, 43151, 43159, 43391, 43669, 44029, 44089, 44129, 44179, 44189, 44249, 44491, 44579, 44621, 44641, 44711, 44789, 45119, 45121, 45289, 45481, 45751, 45779, 45989, 46181, 46199, 46309, 46351, 46489, 46499, 46549, 46559, 46591, 46751, 47111, 47189, 47221, 47309, 47629, 47809, 47981, 48131, 48179, 48449, 48541, 48751, 48799, 48869, 48871, 49009, 49019, 49081, 49391, 60029, 60041, 60089, 60091, 60209, 60271, 60289, 60331, 60659, 60719, 60779, 60821, 60869, 60901, 61141, 61291, 61339, 61651, 61819, 61949, 62099, 62119, 62539, 62581, 62701, 62761, 63131, 63241, 63311, 63331, 63499, 63521, 63599, 63601, 63709, 63761, 63781, 64279, 64661, 64679, 64849, 64901, 64951, 65129, 65269, 65309, 65519, 65539, 65599, 65651, 65719, 65729, 65761, 65881, 66109, 66221, 66569, 66601, 67049, 67061, 67121, 67261, 67349, 67391, 67411, 67421, 67619, 67801, 68171, 68209, 68239, 68389, 68399, 68611, 68729, 68749, 68899, 69019, 69109, 69119, 69439, 69539, 69691, 80153, 80177, 80207, 80233, 80273, 80287, 80363, 80527, 80557, 80567, 80683, 80747, 80803, 81203, 81233, 81283, 81293, 81353, 81667, 81737, 81817, 81847, 81973, 82007, 82037, 82067, 82193, 82267, 82373, 82507, 82613, 82757, 82813, 82963, 83003, 83047, 83137, 83407, 83987, 84143, 84223, 84313, 84377, 84443, 84827, 84967, 85237, 85333, 85447, 85843, 86113, 86263, 86287, 86353, 86357, 86743, 86923, 87133, 87187, 87253, 87433, 87743, 87793, 87887, 88007, 88117, 88223, 88327, 88423, 88513, 88667, 88883, 88937, 89017, 89123, 89153, 89213, 89237, 89293, 89363, 89513, 89767, 89897, 89917, 89923 . There are 412 $\mathbb{L}_2$ numbers smaller then 10^5 .

There are 68 numbers smaller then 10^5 and $\in \mathbb{L}_1 \cap \mathbb{L}_2$: 499, 823, 21157, 21727, 22027, 22147, 22367, 22717, 23557, 24547, 26357, 28027, 28607, 28807, 29027, 41039, 41389, 42179, 42649, 42859, 44129, 44249, 44579, 45119, 46499, 46559, 47809, 48869, 60041, 60271, 60331, 60901, 61651, 63241, 63331, 64661, 64951, 65651, 65881, 66221, 66601, 67061, 67261, 67391, 67411, 69691, 80683, 80803, 81233, 81293, 81973, 82193, 82373, 82813, 82963, 83003, 84223, 84443, 85333, 86113, 86353, 86923, 88223, 89123, 89153, 89213, 89293, 89923 .

Remark 11. At the same time the numbers $q \in \mathbb{L}_1 \cap \mathbb{L}_2$ are the solutions of the following Diophantine systems

$$\begin{cases} p & \in \mathbb{P}\ , \\ p + R(p) & \in \mathbb{P}\ , \\ p^2 + R(p)^2 & \in \mathbb{P}\ . \end{cases} \tag{1}$$

Remark 12. We note with VP$\mathbb{L}_\omega$ the *very probably*$\mathbb{L}_\omega$ numbers, i.e. the primes $p \in \mathbb{P}$ for which $IsPrime(p^\omega + R(p)^\omega) = 1$ and $p^\omega + R(p)^\omega > 10^{16}$.

Using Chai Wah Wu's sequence of numbers $\mathbb{L}_1$ smaller 10^9, A061783, we determined a list of the last 60 numbers that are part of $\mathbb{L}_1 \cap \text{VP}\mathbb{L}_2$:

899704193, 899704453, 899708603, 899709143, 899719433, 899722283, 899726603, 899730683, 899733973, 899753273, 899756623, 899757343, 899768003, 899775293, 899776723, 899778353, 899779813, 899785123, 899786323, 899793673, 899806373, 899806783, 899810843, 899811833, 899819623, 899819663, 899821633, 899824613, 899826113, 899829263, 899842313, 899846863, 899850923, 899854873, 899859823, 899864923, 899871463, 899871853, 899884003, 899885263, 899889073, 899894263, 899910703, 899920873, 899934263, 899951683, 899955283, 899956313, 899963353, 899907143, 899965663, 899974573, 899978483, 899979853, 899980343, 899987903, 899988883, 899988983, 899994353, 899999173 .

Let the function $L_2 : \mathbb{P} \to \mathbb{N}$, $L_2(x) = x^2 + R(x)^2$, where $R(x)$ is reverse the function . Then, Definition 1, $p \in \mathbb{L}_2$ it is equivalent to the statement $L_2(p) \in \mathbb{P}$. Therefor

– if $p \in \mathbb{L}_2$ then, according to Definition 1 $p \in \mathbb{P}$ and $p^2 + R(p)^2 \in \mathbb{P}$, i.e. $L_2(p) \in \mathbb{P}$;

– if $L_2(p) \in \mathbb{P}$ it follows that $p \in \mathbb{P}$ and $p^2 + R(p)^2 \in \mathbb{P}$, i.e. $p \in \mathbb{L}_2$.

We consider the function $L_2^2(x) = L_2(L_2(x))$. If $L_2^2(x) \in \mathbb{P}$, then it follows that $L_2(x) \in \mathbb{L}_2 \subset \mathbb{P}$ and $x \in \mathbb{L}_2$. We denote $\mathbb{L}_2^2 = \{p \mid L_2^2(p) \in \mathbb{P}\}$.

Table 3: Table $\text{VP}\mathbb{L}_2^2$ numbers for $p < 10^5$

$p \in \mathbb{P}$	$L_2(p) \in \mathbb{P}$	$L_2^2(p) \in \text{VP}\mathbb{P}$
491	278717	593023374473
85333	8394477053	82771519543408188653
86923	8642496953	87630748700716215233
88513	8832352913	88202721028208568113
89123	8979620333	91724278252297991693

Therefore up to 10^3, respectively 10^5 we have 1, respectively 5 $\text{VP}\mathbb{L}_2^2$ numbers .

Open problems:

1. If $p < 10^n$, then $Card(\mathbb{L}_2) =?$.
2. Is $\mathbb{L}_2$ finite?
3. Is $\mathbb{L}_2^2$ a finite set?
4. If $p < 10^n$, then $Card(\mathbb{L}_1 \cap \mathbb{L}_2) =?$.
5. Are there a finite number of elements in $\mathbb{L}_1 \cap \mathbb{L}_2$?

6 Luhn primes of order 3

There are no $\mathbb{L}_3$ numbers . Even more so, there are no $\mathbb{L}_\omega$ numbers with $\omega \in \mathbb{N}$, $\omega \neq 2^n$, where $n \in \mathbb{N}$.

Theorem 13. *$\mathbb{L}_\omega = \emptyset$, if $\omega \in \mathbb{N}$ and $\omega \neq 2^n$, where $n \in \mathbb{N}$.*

Proof. We have one identity

$$a^{2m+1} + b^{2m+1} = (a+b) \cdot \sum_{j=0}^{2m} (-1)^j a^{2m-j} b^j \ .$$

1. If $\omega = 2m+1$, then for $\forall p \in \mathbb{P}$ and $\forall m \in \mathbb{N}$ then

   $$p^\omega + R(p)^\omega = p^{2m+1} + R(p)^{2m+1} \notin \mathbb{P} \ ,$$

   then it follows that $\mathbb{L}_\omega = \emptyset$.

2. If ω is an even number, then:

   (a) $\omega = 2^{n-k}(2m+1)$, cu $1 \leq k \leq n-1$, then

   $$\begin{aligned} a^\omega + b^\omega &= (a^{2^{n-k}})^{2m+1} + (b^{2^{n-k}})^{2m+1} \\ &= (a^{2^{n-k}} + b^{2^{n-k}}) \cdot \sum_{j=0}^{2m} (-1)^j (a^{2^{n-k}})^{2m-j} (b^{2^{m-k}})^j \ , \end{aligned}$$

   therefore for $\forall p \in \mathbb{P}$ and $\forall m \in \mathbb{N}$ result that $p^\omega + R(p)^\omega \notin \mathbb{P}$, $\Rightarrow \mathbb{L}_\omega = \emptyset$.

   (b) $\omega = 2^n$, then we can not say anything about $\mathbb{L}_\omega$ sets .

□

7 Luhn primes of order 4

Theorem 14. *The primes $q = p^4 + R(p)^4$, where $p \in \mathbb{P}$, must have 7 as their last digit .*

Proof. With the exception of 2 and 5 all prime p must have 1, 3, 7 or 9 as their last digit . As a result p^4 have 1 on the last position since $1^4 = 1$, $3^4 = 81$, $7^4 = 2401$ and $9^4 = 6561$.

The numbers $R(p)$ must be even in order for $q = p^4 + R(p)^4$ to be odd . An even number raised to the power 4 has 6 on the past position because $2^4 = 16$, $4^4 = 256$, $6^4 = 1296$ and $8^4 = 4096$.

Thus the prime $q = p^4 + R(p)^4$ must have 7 as the last digit because $1 + 6 = 7$. □

Using a program we determined the 539 $\mathbb{L}_4$ numbers out 9592 primes smaller then 10^5 . These are: 23, 43, 47, 211, 233, 239, 263, 419, 431, 487, 491, 601, 683, 821, 857, 2039, 2063, 2089, 2113, 2143, 2203, 2243, 2351, 2357, 2377, 2417, 2539, 2617, 2689, 2699, 2707, 2749, 2819, 2861, 2917, 2963, 4051, 4057, 4127, 4129, 4409, 4441, 4481, 4603, 4679, 4733, 4751, 4951, 4969, 4973, 6053, 6257, 6269, 6271, 6301, 6311, 6353, 6449, 6547, 6551, 6673, 6679, 6691, 6803, 6869, 6871, 6947, 6967, 8081, 8123, 8297, 8429, 8461, 8521, 8543, 8627, 8731, 8741, 8747, 8849, 8923, 8951, 8969, 20129, 20149, 20177, 20183, 20359, 20369, 20593, 20599, 20639, 20717, 20743, 20759, 20903, 20921, 21017, 21019, 21169, 21211, 21341, 21379, 21419, 21503, 21611, 21613, 21661, 21727, 21803, 21821, 21841, 21881, 21893, 21929, 21937, 22031, 22073, 22133, 22171, 22277, 22303, 22343, 22349, 22441, 22549, 22573, 22741, 22817, 22853, 22877, 22921, 23029, 23071, 23227, 23327, 23357, 23399, 23431, 23531, 23767, 23827, 23917, 23977, 24019, 24023, 24113, 24179, 24197, 24223, 24251, 24421, 24481, 24527, 24593, 24659, 24683, 24793, 25171, 25261, 25303, 25307, 25321, 25343, 25541, 25643, 25673, 25819, 25873, 25969, 26083, 26153, 26171, 26267, 26297, 26561, 26833, 26839, 26953, 26993, 27103, 27277, 27337, 27427, 27551, 27617, 27749, 27751, 27791, 27823, 27901, 27919, 27953, 28019, 28087, 28211, 28289, 28297, 28409, 28547, 28631, 28663, 28723, 28793, 28813, 28817, 28843, 28909, 28927, 28949, 28979, 29063, 29173, 29251, 29383, 29663, 29833, 29881, 29989, 40063, 40169, 40189, 40357, 40459, 40471, 40483, 40591, 40627, 40637, 40763, 40813, 40847, 40939, 40961, 41519, 41521, 41593, 41611, 41737, 41941, 41999, 42023, 42071, 42073, 42101, 42193, 42407, 42409, 42461, 42491, 42787, 42793, 42859, 42937, 42961, 43037, 43399, 43411, 43517, 43591, 43597, 43721, 43781, 43867, 44101, 44129, 44171, 44267, 44279, 44371, 44449, 44507, 44777, 44939, 45121, 45127, 45179, 45403, 45439, 45497, 45523, 45533, 45553, 45691, 45751, 45821, 45863, 45989, 46147, 46153, 46273, 46301, 46307, 46451, 46499, 46807, 46817, 46993, 47093, 47221, 47303, 47351, 47417, 47431, 47513, 47701, 47741, 47777, 47791, 47917, 47963, 48017, 48091, 48157, 48247, 48491, 48619, 48761, 48821, 48847, 48883, 48947, 48991, 49033, 49043, 49057, 49169, 49177, 49193, 49333, 49417, 49433, 49523, 49547, 49633, 49711, 49739, 49783, 49853, 60029, 60077, 60107, 60223, 60373, 60443, 60493, 60649, 60773, 60887, 60889, 60953, 61211, 61333, 61487, 61553, 61643, 61819, 62003, 62189, 62219, 62297, 62311, 62327, 62383, 62483, 62597, 62659, 62723, 62903, 62921, 63031, 63211, 63397, 63419, 63527, 63533, 63617, 63689, 63809, 63997, 64123, 64283, 64577, 64633, 64709, 64817, 64871, 64919, 64937, 65003, 65033, 65111, 65179, 65239, 65437, 65557, 65617, 65677, 65731, 65831, 66041, 66553, 66617, 66749, 66919, 66943, 67189, 67273, 67399, 67499, 67567, 67577, 67631, 67853, 67933, 67939, 67993, 68023, 68053, 68059, 68087, 68111, 68489, 68531, 68659, 68699, 68711, 68713, 68743, 68771, 68821, 68881, 68909, 69109, 69149, 69247, 69389, 69403, 69481, 69691, 69697, 69821, 69827, 69997, 80251, 80447, 80611, 80627, 80629, 80671, 80749, 80783, 80833, 80909, 80923, 81223, 81353, 81463, 81509, 81533, 81619, 81727, 81737, 81901, 82037, 82139, 82163, 82193, 82267, 82387, 82393, 82471, 82549, 82567, 82727, 83089, 83383, 83449, 83471, 83869, 84061, 84181, 84349, 84401, 84437, 84827, 84913, 84919, 84991, 85109, 85159, 85259, 85361, 85621, 85643, 85691, 85837, 85889, 86113, 86311, 86353, 86813, 86851, 86929, 86969, 87083, 87119, 87221, 87223, 87427, 87509, 87547, 87587, 87589, 87683, 87793, 87911, 88129, 88211, 88547, 88661, 88843, 88861, 88883, 88969, 89021, 89083, 89123, 89237, 89269, 89273, 89399, 89491, 89513, 89753, 89833, 89839, 89963, 89989.

Remark 15. Applying the $IsPrime(p^4 + R(p)^4)$ test to all primes larger then 20129 we have $p^4 + R(p)^4 > 10^{16}$. If $IsPrime(p^4 + R(p)^4)$ return true, we can say that $p^4 + R(p)^4 \in \mathrm{VP}\mathbb{P}$, i.e. $p \in \mathrm{VP}\mathbb{L}_4$.

We have 539 of numbers $\mathrm{VP}\mathbb{L}_4$ up to 10^5 we have 77 of numbers $\mathbb{L}_4$: 23, 43, 47, 211, 233, 239, 263, 419, 431, 487, 491, 601, 683, 821, 857, 2039, 2063, 2089, 2113, 2143, 2203, 2243, 2351, 2357, 2377, 2417, 2539, 2617, 2689, 2699, 2707, 2749, 2819, 2861, 2917, 2963, 4051, 4057, 4127, 4129, 4409, 4441, 4481, 4603, 4679, 4733, 4751, 4951, 4969, 4973, 6053, 6257, 6271, 6301, 6311, 6353, 6449, 6547, 6551, 6673, 6691, 6803, 6871, 6947, 6967, 8081, 8123, 8297, 8461, 8521, 8543, 8627, 8731, 8741, 8747, 8923, 8951 .

Remark 16. Up to 10^5 have:

- There are 130 of numbers $\in \mathbb{L}_1 \cap \mathrm{VP}\mathbb{L}_4$: 239, 683, 20717, 20759, 21019, 21169, 21211, 21379, 21419, 21611, 21661, 21727, 21841, 21881, 22031, 22171, 22549, 22741, 22877, 23029, 23357, 24019, 24179, 24481, 25171, 25261, 25541, 25819, 25969, 27277, 27617, 27791, 27901, 27919, 28019, 28409, 28817, 28909, 28949, 28979, 29251, 29989, 40169, 40483, 40813, 41519, 42023, 42407, 42409, 42859, 43399, 44129, 44267, 44449, 44939, 45127, 45439, 45497, 45533, 46273, 46499, 46807, 46817, 47417, 47513, 47777, 47917, 48017, 48157, 48847, 48883, 48947, 49043, 49333, 60443, 60773, 61333, 61643, 62311, 62597, 63397, 64123, 64577, 64937, 65437, 65617, 65677, 65831, 66041, 66617, 67273, 67631, 68023, 68087, 68743, 68821, 68881, 69481, 69691, 69697, 69997, 80611, 80833, 81463, 81509, 82139, 82193, 82393, 83383, 83449, 84061, 84181, 84349, 84401, 85109, 85159, 85361, 86113, 86353, 86813, 86929, 88129, 88843, 88969, 89021, 89123, 89269, 89753, 89833, 89963, and 239, 683$\in \mathbb{L}_1 \cap \mathbb{L}_4$.

- 52 of numbers: 23, 233, 419, 491, 857, 2617, 4051, 4129, 4409, 4481, 6301, 6551, 6871, 8543, 21727, 21803, 21937, 22133, 23227, 23327, 24527, 28297, 29063, 40459, 40591, 41941, 41999, 42461, 42859, 44129, 45121, 45751, 45989, 46499, 47221, 60029, 61819, 69109, 69691, 81353, 81737, 82037, 82193, 82267, 84827, 86113, 86353, 87793, 88883, 89123, 89237, 89513$\in \mathbb{L}_2 \cap \mathrm{VP}\mathbb{L}_4$, of which 14 are numbers $\mathbb{L}_2 \cap \mathbb{L}_4$: 23, 233, 419, 491, 857, 2617, 4051, 4129, 4409, 4481, 6301, 6551, 6871, 8543 .

- 9 numbers: 21727, 42859, 44129, 46499, 69691, 82193, 86113, 86353, 89123$\in \mathbb{L}_1 \cap \mathrm{VP}\mathbb{L}_2 \cap \mathrm{VP}\mathbb{L}_4$.

Remark 17. Using Chai Wah Wu sequence of $\mathbb{L}_1$ numbers smaller 10^9 A061783, we determined:

- the last 120 numbers that belong $\mathbb{L}_1 \cap \mathrm{VP}\mathbb{L}_4$:
899705549, 899707373, 899708861, 899709029, 899709583, 899712059, 899716361, 899717579, 899720219, 899722963, 899723603, 899726929, 899727481, 899729659, 899731121, 899737271, 899739923, 899742241, 899742713, 899742751, 899746979, 899747209, 899755559, 899756243, 899758309, 899759339, 899761531, 899763509, 899764781, 899765533, 899767879, 899768629, 899772199, 899772283, 899773883,

899775113, 899776931, 899779499, 899782883, 899784341, 899784733, 899794013, 899797531, 899801311, 899801999, 899803129, 899811403, 899812093, 899812591, 899826071, 899827393, 899827429, 899827609, 899831333, 899835283, 899835641, 899846531, 899846809, 899847419, 899850923, 899851651, 899851679, 899853179, 899853289, 899853569, 899854121, 899854481, 899856421, 899860931, 899861939, 899861971, 899871463, 899871743, 899873131, 899874331, 899877863, 899882213, 899882611, 899885003, 899885683, 899886971, 899888783, 899890261, 899891309, 899892809, 899897093, 899898269, 899899001, 899902511, 899902603, 899902673, 899922949, 899904323, 899905313, 899912473, 899917433, 899918581, 899920169, 899921179, 899928149, 899929781, 899931689, 899939041, 899942213, 899946899, 899947561, 899950391, 899956411, 899959891, 899961941, 899970679, 899984513, 899985089, 899987311, 899988533, 899989789, 899994709, 899995601, 899996191, 899999801 .

- 899850923 and 899871463 $\in \mathbb{L}_1 \cap \text{VP}\mathbb{L}_2 \cap \text{VP}\mathbb{L}_4$.

Let the function $L_4 : \mathbb{P} \to \mathbb{N}$, $L_4(x) = x^4 + R(x)^4$, where $R(x)$ is reverse the function . Then, Definition 1, $p \in \mathbb{L}_4$ it is equivalent to the statement $L_4(p) \in \mathbb{P}$. Therefor

– if $p \in \mathbb{L}_4$ then, according to Definition 1 $p \in \mathbb{P}$ and $p^4 + R(p)^4 \in \mathbb{P}$, i.e. $L_4(p) \in \mathbb{P}$;

– if $L_4(p) \in \mathbb{P}$ it follows that $p \in \mathbb{P}$ and $p^4 + R(p)^4 \in \mathbb{P}$, i.e. $p \in \mathbb{L}_4$.

Table 4: Table $\text{VP}\mathbb{L}_4^2$ numbers for $p < 10^5$

$p \in \mathbb{P}$	$L_4(p) \in \text{VP}\mathbb{P}$
20593	2614709902432843217
21661	296300745474164177
22441	296872969044506417
23227	27512961152730280817
27427	28151373544108562897
28927	29070428390909587217
29063	2410298420861916257
69691	23739342448884098417

$L_4^2(p) \in \text{VP}\mathbb{P}$
2621694179381666992481977442883310252917575789112501205306222374476891845457
361914672908065708348898716726763828848431604521665676539514972523740737
268542116178982326048979213726573127035634315778661489796503798674682817
27161640882133887601449530936365856960694475936522001374002174813314584251126977
41234045311077277686365746725918624370283550114222462692903223281232284510226257
26527054411839253601617336913872467596407776950657274984221533406684612586001217
3242243763630054698046532850581407163377216403484171683501339925588629649297
26436687869623230736108238056365042955148159574074820423234372340404780444335297

We consider the function $L_4^2(x) = L_4(L_4(x))$. If $L_4^2(x) \in \mathbb{P}$, then it follows that $L_4(x) \in \mathbb{L}_4 \subset \mathbb{P}$ and $x \in \mathbb{L}_4$. We denote $\mathbb{L}_4^2 = \{p \mid L_4^2(p) \in \mathbb{P}\}$. Therefore up to 10^5 we have 8 $\text{VP}\mathbb{L}_4^2$ numbers .

Open questions:

1. Is $\mathbb{L}_4$ finite?
2. Is $\mathbb{L}_4^2$ a finite set?
3. If $p < 10^n$, then $Card(\mathbb{L}_4) =$?
4. If $p < 10^n$, then $Card(\mathbb{L}_1 \cap \mathbb{L}_4) =$?
5. If $p < 10^n$, then $Card(\mathbb{L}_2 \cap \mathbb{L}_4) =$?
6. If $p < 10^n$, then $Card(\mathbb{L}_1 \cap \mathbb{L}_2 \cap \mathbb{L}_4) =$?

8 Luhn primes of order 8

Theorem 18. *The number $q = p^8 + R(p)^8$, where $p \in \mathbb{P}$ with first digit 2, 4, 6 or 8, then the last digit is 7 .*

Proof. All prime numbers except 2 and 5 end with 1, 3, 7 or 9 as their last digit . As a consequence p^2 must end with either 1 or 9 because $1^8 = 1$, $3^8 = 6561$, $7^8 = 5764801$ and $9^8 = 43046721$.

If $p \in \mathbb{P}$ with first digit 2, 4, 6 or 8, then $R(p)^8$ has the last digit 6, because $2^8 = 256$, $4^8 = 65536$, $6^8 = 1679616$ and $8^6 = 16777216$.

Therefore the number $q = p^8 + R(p)^8$ has the last digit 7 because $1 + 6 = 7$. □

For $p < 10^5$ we have:

- 204 numbers $\text{VP}\mathbb{L}_8$: 29, 67, 251, 277, 467, 677, 823, 857, 2087, 2111, 2141, 2213, 2383, 2399, 2441, 2531, 2777, 2939, 4027, 4201, 4241, 4561, 4603, 4691, 4787, 4919, 4931, 4951, 4987, 6089, 6287, 6427, 6563, 6577, 6791, 8363, 8501, 8537, 8719, 8753, 8837, 20089, 20129, 20219, 20407, 20549, 20663, 20731, 20771, 20897, 20947, 21187, 21193, 21401, 21757, 21859, 22171, 22787, 23041, 23321, 23833, 23873, 23981, 24071, 24121, 24533, 24677, 24889, 25673, 25819, 26459, 26573, 26591, 26681, 26729, 27487, 27617, 27793, 27967, 28069, 28403, 28493, 28541, 28753, 28771, 28867, 28909, 29129, 29201, 29231, 40099, 40343, 40459, 40867, 41333, 41357, 41381, 41593, 41887, 41903, 41957, 42019, 42227, 42463, 43003, 43049, 43451, 44159, 44623, 44729, 44893, 45053, 45191, 45707, 46027, 46649, 47087, 47387, 47701, 48049, 48313, 48809, 49171, 49463, 60139, 60169, 60259, 60353, 60887, 61703, 62687, 63031, 63131, 63277, 63317, 63589, 63697, 64661, 64817, 64871, 64969, 65089, 65713, 65929, 65981, 66169, 66491, 66601, 67103, 67141, 67843, 68053, 68477, 68543, 69203, 69371, 69383, 69499, 69857, 80329, 80363, 80369, 80557, 80567, 80621, 80687, 80761, 81373, 81401, 81457, 81973, 82009, 82219, 82561, 82571, 82913, 83617, 83641, 83857, 83891, 84053, 84811, 85639, 85661, 85819, 85847, 85909, 86137, 86399, 86693, 86771, 86783, 87491, 87523, 87539, 87553, 88609, 88657, 89041, 89123, 89231, 89449, 89821, 89897;

- 45 numbers: 277, 467, 677, 823, 20897, 21859, 22171, 24071, 24121, 24889, 25819, 26459, 26591, 26681, 27617, 28909, 29129, 40867, 41903, 42227, 42463, 43003, 44159, 44623, 44893, 45053, 45707, 48313, 61703, 63697, 64661, 66601, 68477, 69371, 69857, 80369, 80621, 81373, 81973, 82561, 84053, 85819, 86399, 86783, 89123 $\in \mathbb{L}_1 \cap \mathrm{VP}\mathbb{L}_8$;
- 823, 857 $\in \mathbb{L}_2 \cap \mathrm{VP}\mathbb{L}_8$ and 21757, 22787, 24533, 28753, 40459, 43049, 63131, 64661, 66601, 80363, 80557, 80567, 81973, 89123, 89897 $\in \mathrm{VP}\mathbb{L}_2 \cap \mathrm{VP}\mathbb{L}_8$;
- 857, 4603, 4951 $\in \mathbb{L}_4 \cap \mathrm{VP}\mathbb{L}_8$ and 20129, 22171, 25673, 25819, 27617, 28909, 40459, 41593, 47701, 60887, 63031, 64817, 64871, 68053, 89123 $\in \mathrm{VP}\mathbb{L}_4 \cap \mathrm{VP}\mathbb{L}_8$;
- the number 823 $\in \mathbb{L}_1 \cap \mathbb{L}_2 \cap \mathrm{VP}\mathbb{L}_8$ and 64661, 66601, 81973, 89123 $\in \mathbb{L}_1 \cap \mathrm{VP}\mathbb{L}_2 \cap \mathrm{VP}\mathbb{L}_8$;
- the number 857 $\in \mathbb{L}_2 \cap \mathbb{L}_4 \cap \mathrm{VP}\mathbb{L}_8$ and 40459, 89123 $\in \mathbb{L}_2 \cap \mathrm{VP}\mathbb{L}_4 \cap \mathrm{VP}\mathbb{L}_8$;
- the number 89123 $\in \mathbb{L}_1 \cap \mathbb{L}_2 \cap \mathrm{VP}\mathbb{L}_4 \cap \mathrm{VP}\mathbb{L}_8$;
- not exist numbers $\mathrm{VP}\mathbb{L}_8^2$, up to 10^5 .

9 Luhn primes of order 16

Theorem 19. *The number $q = p^{16} + R(p)^{16}$, where $p \in \mathbb{P}$ with first digit 2, 4, 6 or 8, has last digit 7.*

Proof. All prime numbers except 2 and 5 end with 1, 3, 7 or 9 as their last digit . As a consequence p^{16} must end with either 1 because $1^{16} = 1$, $3^{16} = 43046721$, $7^{16} = 33232930569601$ and $9^{16} = 1853020188851841$.

If $p \in \mathbb{P}$ with first digit 2, 4, 6 or 8, then $R(p)^{16}$ has last digit 6 because $2^{16} = 65536$, $4^{16} = 4294967296$, $6^{16} = 2821109907456$ and $8^{16} = 281474976710656$.

Therefore the number $q = p^{16} + R(p)^{16}$ has last digit 7 because $1 + 6 = 7$. □

For any $p < 10^5$ we have:

- 174 numbers $\mathrm{VP}\mathbb{L}_{16}$: 229, 601, 607, 641, 2221, 2377, 2731, 4127, 4357, 4547, 4597, 4657, 4723, 4787, 4931, 4957, 6269, 6577, 6599, 6793, 6971, 8221, 8243, 8719, 8819, 8837, 8929, 20219, 20393, 20441, 20929, 21163, 21673, 22111, 22481, 22531, 23677, 24179, 24547, 25321, 25411, 25657, 25747, 25793, 25819, 26449, 26497, 26501, 26681, 27253, 27299, 27329, 27337, 27481, 27791, 28307, 28537, 28579, 28949, 29077, 29153, 29399, 29483, 29633, 29671, 29833, 40099, 40129, 40487, 40493, 40879, 41141, 41903, 41927, 42023, 42283, 42689, 42701, 42793, 43049, 43577, 43801, 43991, 44131, 44789, 45127, 45131, 45233, 45767, 46049, 46181, 46337, 46451, 46471, 46619, 46831, 47293, 47381, 47521, 47659, 47933, 48079, 48413, 48479, 48487, 48991, 49523, 49531, 60089, 60289, 60637, 60793, 61379, 61381, 61717, 61781, 61961, 62347, 62417, 62639, 62659, 63149, 63901, 64067, 64091, 65579, 65719, 65951, 66173, 66529, 66877, 67477, 67493, 67783, 67927, 68071, 68371, 68443, 68699, 68891, 68963, 68993, 69473, 69899, 80713, 80933, 81163, 81233, 81569, 81943, 82163, 82339, 82421, 82469, 82567, 82651, 83983, 84163, 84701, 84713, 84919, 84947, 84961, 85081, 85517, 85733, 86287, 86857, 87683, 87881, 88007, 88747, 89057, 89563;

- 38 numbers $\mathbb{L}_1 \cap \mathrm{VP}\mathbb{L}_{16}$: 229, 641, 20441, 22481, 24179, 24547, 25819, 26681, 27329, 27481, 27791, 28307, 28949, 29671, 40129, 40487, 40879, 41903, 42023, 45127, 45233, 46337, 46619, 47659, 48479, 60637, 61381, 61961, 62417, 64091, 68443, 81163, 81233, 82421, 82469, 83983, 84713, 87881;

- 12 numbers $\mathbb{L}_2 \cap \mathrm{VP}\mathbb{L}_{16}$: 8243, 21163, 24547, 43049, 44789, 46181, 60089, 60289, 65719, 81233, 86287, 88007;

- 601, 2377, 4127 $\in \mathbb{L}_4 \cap \mathrm{VP}\mathbb{L}_{16}$ and 6269, 24179, 25321, 25819, 27337, 27791, 28949, 29833, 42023, 42793, 45127, 46451, 48991, 49523, 62659, 68699, 82163, 82567, 84919, 87683 $\in \mathrm{VP}\mathbb{L}_4 \cap \mathrm{VP}\mathbb{L}_{16}$;

- 11 $\mathrm{VP}\mathbb{L}_8 \cap \mathrm{VP}\mathbb{L}_{16}$ numbers : 4787, 4931, 6577, 8719, 8837, 20219, 25819, 26681, 40099, 41903, 43049;

- 24547, 81233 $\in \mathbb{L}_1 \cap \mathbb{L}_2 \cap \mathrm{VP}\mathbb{L}_{16}$;

- the number 25819 $\in \mathbb{L}_1 \cap \mathrm{VP}\mathbb{L}_4 \cap \mathrm{VP}\mathbb{L}_8 \cap \mathrm{VP}\mathbb{L}_{16}$.

Remark 20. The elements of:

- $\mathbb{L}_1 \cap \mathbb{L}_4$, $\mathbb{L}_1 \cap \mathbb{L}_8$, $\mathbb{L}_1 \cap \mathbb{L}_{16}$, $\mathbb{L}_2 \cap \mathbb{L}_4$, $\mathbb{L}_2 \cap \mathbb{L}_8$, $\mathbb{L}_2 \cap \mathbb{L}_{16}$, $\mathbb{L}_4 \cap \mathbb{L}_8$, $\mathbb{L}_4 \cap \mathbb{L}_{16}$, $\mathbb{L}_8 \cap \mathbb{L}_{16}$, are the solutions of Diophantine systems of the type (1);

- $\mathbb{L}_1 \cap \mathbb{L}_2 \cap \mathbb{L}_4$, $\mathbb{L}_1 \cap \mathbb{L}_2 \cap \mathbb{L}_8$, $\mathbb{L}_1 \cap \mathbb{L}_2 \cap \mathbb{L}_{16}$, $\mathbb{L}_1 \cap \mathbb{L}_4 \cap \mathbb{L}_8$, $\mathbb{L}_1 \cap \mathbb{L}_4 \cap \mathbb{L}_{16}$, $\mathbb{L}_1 \cap \mathbb{L}_8 \cap \mathbb{L}_{16}$, $\mathbb{L}_2 \cap \mathbb{L}_4 \cap \mathbb{L}_8$, $\mathbb{L}_2 \cap \mathbb{L}_4 \cap \mathbb{L}_{16}$, $\mathbb{L}_2 \cap \mathbb{L}_8 \cap \mathbb{L}_{16}$, $\mathbb{L}_4 \cap \mathbb{L}_8 \cap \mathbb{L}_{16}$, are the solutions of Diophantine systems of the type
$$\begin{cases} p & \in \mathbb{P} \ , \\ p^{\omega_1} + R(p)^{\omega_1} & \in \mathbb{P} \ , \\ p^{\omega_2} + R(p)^{\omega_2} & \in \mathbb{P} \ , \\ p^{\omega_3} + R(p)^{\omega_3} & \in \mathbb{P} \ , \end{cases}$$
where $\omega_1 < \omega_2 < \omega_3$ cu $\omega_j = 2^k$, where $j = 1, 2, 3$ and $k = 0, 1, 2, 3, 4$;

- $\mathbb{L}_1 \cap \mathbb{L}_2 \cap \mathbb{L}_4 \cap \mathbb{L}_8$, $\mathbb{L}_1 \cap \mathbb{L}_2 \cap \mathbb{L}_4 \cap \mathbb{L}_{16}$, $\mathbb{L}_1 \cap \mathbb{L}_2 \cap \mathbb{L}_8 \cap \mathbb{L}_{16}$, $\mathbb{L}_1 \cap \mathbb{L}_4 \cap \mathbb{L}_8 \cap \mathbb{L}_{16}$, $\mathbb{L}_2 \cap \mathbb{L}_4 \cap \mathbb{L}_8 \cap \mathbb{L}_{16}$ are the solutions of Diophantine systems of the type

$$\begin{cases} p & \in \mathbb{P} \ , \\ p^{\omega_1} + R(p)^{\omega_1} & \in \mathbb{P} \ , \\ p^{\omega_2} + R(p)^{\omega_2} & \in \mathbb{P} \ , \\ p^{\omega_3} + R(p)^{\omega_3} & \in \mathbb{P} \ , \\ p^{\omega_4} + R(p)^{\omega_4} & \in \mathbb{P} \ . \end{cases}$$

  where $\omega_1 < \omega_2 < \omega_3 < \omega_4$ cu $\omega_j = 2^k$, where $j = 1, 2, 3, 4$ si $k = 0, 1, 2, 3, 4$;

- $\mathbb{L}_1 \cap \mathbb{L}_2 \cap \mathbb{L}_4 \cap \mathbb{L}_8 \cap \mathbb{L}_{16}$ are the solutions of Diophantine systems of the type

$$\begin{cases} p & \in \mathbb{P} \ , \\ p^{\omega_1} + R(p)^{\omega_1} & \in \mathbb{P} \ , \\ p^{\omega_2} + R(p)^{\omega_2} & \in \mathbb{P} \ , \\ p^{\omega_3} + R(p)^{\omega_3} & \in \mathbb{P} \ , \\ p^{\omega_4} + R(p)^{\omega_4} & \in \mathbb{P} \ , \\ p^{\omega_5} + R(p)^{\omega_5} & \in \mathbb{P} \ , \end{cases}$$

  where $\omega_1 < \omega_2 < \omega_3 < \omega_4 < \omega_5$ with $\omega_j = 2^k$, where $j = 1, 2, 3, 4, 5$ and $k = 0, 1, 2, 3, 4$.

10 Conclusions

1. We remarked two use cases for $\mathbb{L}_\omega$ numbers, namely:

   (a) Storage of very large primes . Lets assume that $q = p^\omega + R(p)^\omega$ is a large prime . One could store only p and ω and easily reconstruct the large prime .

   (b) The Mersenne numbers $2^n - 1$, the primorial primes $p\# - 1$, the factorial primes $p! - 1, \ldots$ all have a high probability of being large primes, [6] . The $p^\omega + R(p)^\omega$ numbers could also have a high probability of being large primes that end in 7 if $\omega \geq 4$. For example, $281 \in \text{VP}\mathbb{L}_{128}$, i.e.

$$281^{128} + 182^{128} =$$
$$27189983738558379221208088301369231473963644707531280 \ddots$$
$$86161386383724312009857081033255940247862621807230448 \ddots$$
$$61456067290950460888745152869560645261455759481770780 \ddots$$
$$50957569578258913453237529320406329435598073524912577 \ddots$$
$$95581854201826171060795131134190727032606127131205175 \ddots$$
$$80834589109342162169229926118659454265012332124 17 \in \text{VP}\mathbb{P};$$

   This number has 314 digits.

2. $\mathbb{L}_0 \equiv \mathbb{P}$;
3. $\mathbb{L}_1 \neq \emptyset$;
4. $271 \in \mathbb{L}_1^2$;
5. $\mathbb{L}_2 \neq \emptyset$ and $\mathbb{L}_1 \cap \mathbb{L}_2 \neq \emptyset$;
6. $491 \in \mathrm{VP}\mathbb{L}_2^2$;
7. If $\omega \in \mathbb{N}$ si $\omega \neq 2^n$, with $n \in \mathbb{N}$, then $\mathbb{L}_\omega = \emptyset$;
8. $\mathbb{L}_4 \neq \emptyset$ and
    (a) $\mathbb{L}_1 \cap \mathbb{L}_4 \neq \emptyset$,
    (b) $\mathbb{L}_2 \cap \mathbb{L}_4 \neq \emptyset$,
    (c) $\mathbb{L}_1 \cap \mathrm{VP}\mathbb{L}_2 \cap \mathrm{VP}\mathbb{L}_4 \neq \emptyset$;
9. $20593 \in \mathrm{VP}\mathbb{L}_4^2$;
10. $\mathrm{VP}\mathbb{L}_8 \neq \emptyset$ and
    (a) $\mathbb{L}_1 \cap \mathrm{VP}\mathbb{L}_8 \neq \emptyset$,
    (b) $\mathbb{L}_2 \cap \mathrm{VP}\mathbb{L}_8 \neq \emptyset$,
    (c) $\mathrm{VP}\mathbb{L}_4 \cap \mathrm{VP}\mathbb{L}_8 \neq \emptyset$,
    (d) $\mathbb{L}_1 \cap \mathbb{L}_2 \cap \mathrm{VP}\mathbb{L}_8 \neq \emptyset$,
    (e) $\mathbb{L}_1 \cap \mathbb{L}_4 \cap \mathrm{VP}\mathbb{L}_8 \neq \emptyset$,
    (f) $\mathbb{L}_2 \cap \mathbb{L}_4 \cap \mathrm{VP}\mathbb{L}_8 \neq \emptyset$,
    (g) $\mathbb{L}_1 \cap \mathbb{L}_2 \cap \mathrm{VP}\mathbb{L}_4 \cap \mathrm{VP}\mathbb{L}_8 \neq \emptyset$;
11. $\mathrm{VP}\mathbb{L}_{16} \neq \emptyset$,
    (a) $\mathbb{L}_1 \cap \mathrm{VP}\mathbb{L}_{16} \neq \emptyset$,
    (b) $\mathbb{L}_2 \cap \mathrm{VP}\mathbb{L}_{16} \neq \emptyset$,
    (c) $\mathbb{L}_4 \cap \mathrm{VP}\mathbb{L}_{16} \neq \emptyset$,
    (d) $\mathrm{VP}\mathbb{L}_8 \cap \mathrm{VP}\mathbb{L}_{16} \neq \emptyset$,
    (e) $\mathbb{L}_1 \cap \mathbb{L}_2 \cap \mathrm{VP}\mathbb{L}_{16} \neq \emptyset$,
    (f) $\mathbb{L}_4 \cap \mathrm{VP}\mathbb{L}_8 \cap \mathrm{VP}\mathbb{L}_{16} \neq \emptyset$,
    (g) $\mathbb{L}_1 \cap \mathrm{VP}\mathbb{L}_4 \cap \mathrm{VP}\mathbb{L}_8 \cap \mathrm{VP}\mathbb{L}_{16} \neq \emptyset$.

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