Neutrosophic Triplet Group

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Abstract: In this paper, for the first time the authors introduced the notions of neutrosophic triplet group which is completely different from the classical group. In neutrosophic triplet group, we apply the fundamental law of neutrosophy that for an idea A, we have neut(A) and anti(A) and we capture the picture of neutrosophy in algebraic structures.

Key words: groups, homomorphism, neutrosophic triplet group, neutro-homomorphism.

Introduction

Neutrosophy is a new branch of philosophy which studies the origin and features of neutralities in the nature. Florentin Smarandache in 1995, firstly introduced the concept of neutrosophic logic where each proposition in neutrosophic logic is approximated to have the percentage of truth in a subset T, the percentage of indeterminacy in a subset I, and the percentage of falsity in a subset F so that this neutrosophic logic is called an extension of fuzzy logic. In fact neutrosophic set is the generalization of classical sets, conventional fuzzy set [1], intuitionistic fuzzy set [2] and interval valued fuzzy set [3]. This mathematical tool is used to handle problems like imprecise, indeterminacy and inconsistent data etc. By utilizing neutrosophic theory, Vasantha Kandasamy and Florentin Smarandache dig out neutrosophic algebraic structures in [11]. Some of them are neutrosophic fields, neutrosophic vector spaces, neutrosophic groups, neutrosophic bigroups, neutrosophic N-groups, neutrosophic semigroups, neutrosophic bisemigroups, neutrosophic N-semigroup, neutrosophic loops, neutrosophic biloops, neutrosophic N-loop, neutrosophic groupoids, and neutrosophic bigroupoids and so on.

Groups are so much important in algebraic structures and they play the role of back bone in almost all algebraic structures. Groups are count in older algebraic structures. It is so much rich algebraic structure than any other structure. In many algebraic structures, groups are present concrete foundation for it. For example, in rings, fields, vector spaces, etc. Groups are also important in many areas like physics, checmistry, combinatorics, etc. The most important aspect of a group is group action. There are a lot of types of groups as permuation groups, matrix groups, transformation groups, lie-groups, etc.

In this paper, For the first time, we give the idea of a neutrosophic triplet element. This newly born element depends on the operation. Moreover, we introduce neutrosophic triplet group which is different from the classical group in all aspects. Furhtermore, we give some basic and fundamental results with illustrative examples. We also introduce a new type of homomorphism called neutron-homomorphism which is infact a generalization of the classical homomorphism under some conditions.

Neutrosophic Triplet

Definition 1. Let N be a set together with an operation $*$ and $a \in N$. Then a is called neutrosophic triplet if there exist an element $neut(a) \in N$ such that

$$a * neut(a) = neut(a) * \mathrm{a} = \mathrm{a}\ ,$$

where $neut(a)$ is different from unity element. Also there exist $anti(a) \in N$ such that

$$a * anti(a) = anti(a) * a = neut(a)\ .$$

If there are more $anti(a)$'s for a given a, one takes that $anti(a)=b$ that $anti(a)$ in its turn forms a neutrosophic triplet, i.e, there exists $neut(b)$ and $anti(b)$.

We denote the neutrosophic triplet a by $(a, neut(a), anti(a))$. By $neut(a)$, we means *neutral* of a.

Example 1. Consider Z_6 under multiplication modulo 6, where

$$Z_6 = \{0,1,2,3,4,5\}.$$

Then 2 is a neutrosophic triplet because $neut(2)=4$, as $2\times 4=8=2$. Also $anti(2)=2$ because $2\times 2=4$. Thus 2 is a neutrosophic triplet which is denoted by $(2,4,2)$. Similarly 4 is a neutrosophic triplet because $neut(4)=anti(4)=4$. So 4 is denoted by $(4,4,4)$. 3 is not a neutrosophic triplet as $neut(3)=5$ but $anti(3)$ does not exist in Z_6 and last but not the least 0 is a trivial neutrosophic triplet as $neut(0)=anti(0)=0$. This is denoted by $(0,0,0)$.

Neutrosophic Triplet Group

Definition 2. Let $(N,*)$ be a set together with an operation $*$. Then N is called a neutrosophic triplet group, if the following conditions are satisfied.

1) $a*b\in N$ for all $a,b\in N$.

2) $(a*b)*c=a*(b*c)$ for all $a,b,c\in N$.

3) For every $a\in N$, there exists $neut(a)\in \mathrm{N}$ different from the unity element such that

$a*neut(a)=neut(a)*a=a$.

4) For every $a\in N$, there exist $anti(a)$ in N such that $a*anti(a)=anti(a)*a=neut(a)$.

Example 2. Consider $(Z_{14},\times)$. Then $N=\{0,2,4,6,7,8,10,12\}$ be the set of neutrosophic tirplet with multiplication modulo 14 in Z_{14}. Then clearly N is a neutrosophic triplet group under multiplication modulo 14.

Theorem 1. Every idempotent element is a neutrosophic triplet.

Proof. Let a be an idempotent element. Then by definition $a^2 = a$. Since $a^2 = a$, which clearly implies that $neut(a) = a$ and $anti(a) = a$. Hence a is a neutrosophic triplet.

Theorem 2. There are no neutrosophic triplets in Z_n with respect to multiplication if n is a prime.

Proof. It is obvious.

Remark 1. Let $(N,*)$ be a neutrosophic triplet group under $*$ and let $a \in N$. Then $neut(a)$ is not unique in N and also $neut(a)$ depends on the element a and the operation $*$.

To prove the above remark, lets take a look to the following example.

Example 3. Let $N = \{0, 4, 8,$ be a neutrosophic triplet group under multiplication modulo 12 in $(Z_{12}, \times)$. Then $neut(4) = 4$, $neut(8) = 4$ and $neut(9) = 9$. This shows that $neut(a)$ is not unique.

Remark 2. Let $(N,*)$ be a neutrosophic triplet group with respect to $*$ and let $a \in N$. Then $anti(a)$ is not unique in N and also $anti(a)$ depends on the element a and the operation $*$.

To prove the above remark, lets take a look to the following example.

Example 4. Let N be a neutrosophic triplet group in above example. Then $anti(4) = 4$, $anti(8) = 8$ and $anti(9) = 9$.

Proposition 1. Let $(N,*)$ be a neutrosophic triplet group with respect to $*$ and let $a,b,c \in N$. Then

1) $a*b = a*c$ if and only if $neut(a)*b = neut(a)*c$.

2) $b*a = c*a$ if and only if $b*neut(a) = c*neut(a)$.

Proof 1. Suppose that $a*b = a*c$. Since N is a neutrosophic triplet group, so

$anti(a) \in N$. Multiply $anti(a)$ to the left side with $a * b = a * c$.

$$anti(a) * a * b = anti(a) * a * c$$

$$[anti(a) * a] * b = [anti(a) * a] * c$$

$$neut(a) * b = neut(a) * c$$

Conversely suppose that $neut(a) * b = neut(a) * c$.

Multiply a to the left side, we get:

$$a * neut(a) * b = a * neut(\mathrm{a}) * c$$

$$[a * neut(a)] * b = [a * neut(a)] * c$$

$$a * b = a * c$$

2) This proof is similar to (1).

Proposition 2. Let $(N,*)$ be a neutrosophic triplet group with respect to $*$ and let $a,b,c \in N$.

1) If $anti(a) * b = anti(a) * c$, then $neut(a) * b = neut(a) * c$.

2) If $b * anti(a) = c * anti(a)$, then $b * neut(a) = c * neut(a)$.

Proof 1. Suppose that $anti(a) * b = anti(a) * c$. Since N is a neutrosophic triplet group with respect to $*$, so $a \in N$. Multiply a to the left side with $anti(a) * b = anti(a) * c$, we get:

$$a * anti(a) * b = a * anti(a) * c$$

$$[a * anti(a)] * b = [a * anti(a)] * c$$

$$neut(a) * b = neut(a) * c.$$

2) The proof is same as (1).

Definition 3. Let $(N,*)$ be a neutrosophic triplet group with respect to $*$. Then N is called commutative neutrosophic triplet group if $a*b=b*a$ for all $a,b\in N$.

Theorem 3. Let $(N,*)$ be a commutative neutrosophic triplet group with respect to $*$ and

a, $b\in N$. Then

$$neut(a)*neut(b)=neut(a*b)$$

Proof. Consider left hand side, $neut(a)*neut(b)$.

Now multiply to the left with a and to the right with b, we get:

$$a*neut(a)*neut(b)*b=\left[a*neut(a)\right]*\left[neut(\mathrm{b})*\mathrm{b}\right]$$

$$=a*b.$$

Now consider right hand side, we have $neut(a*b)$.

Again multiply to the left with a and to the right with b, we get:

$$a*neut(a*b)*b=\left[a*b\right]*\left[neut(a*b)\right]$$, as $*$ is associative.

$$=a*b.$$

This completes the proof.

Theorem 4. Let $(N,*)$ be a commutative neutrosophic triplet group with respect to $*$ and

$a,b\in N$. Then

$$anti(a)*anti(b)=anti(a*b).$$

Proof. Consider left hand side, $anti(a)*anti(b)$.

Multiply to the left with a and to the right with b, we get:

$$a*anti(a)*\mathrm{anti(b)}*\mathrm{b}=\left[a*anti(a)\right]*\left[anti(\mathrm{b})*b\right]$$

$$=neut(a)*neut(\mathrm{b})$$

$$=neut(a*b)$$, by above theorem.

Now consider right hand side, which is $anti(a*b)$.

Multiply to the left with a and to the right with b, we get:

$$a*anti(a*b)*b=\left[a*b\right]*\left[anti(a*b)\right]$$, since $*$ is associative.

$$=neut(a*b)$$.

This shows that $anti(a)*anti(b)=anti(a*b)$ is true for all $a,b\in N$.

Theorem 5. Let $(N,*)$ be a commutative neutrosophic triplet group under $*$ and $a,b\in N$.

Then

1) $neut(a)*neut(b)=neut(b)*neut(a)$.

2) $anti(a)*anti(b)=anti(b)*anti(a)$.

Proof 1. Consider right hand side $neut(\mathrm{b})*neut(a)$. Since by theorem 3. We have,

$$neut(b)*neut(a)=neut(b*a)$$

$$=neut(a*b)$$, as N is commutative.

$$=neut(a)*neut(b)$$, again by theorem 3.

Hence $neut(a)*neut(b)=neut(b)*neut(a)$.

2). On the similar lines, one can easily obtained the proof of (2).

Definition 4. Let $(N,*)$ be a neutrosophic triplet group under $*$ and let H be a subset of N. Then H is called a neutrosophic triplet subgroup of N if H itself is a neutrosophic triplet group with respect to $*$.

Example 5. Consider $(Z_{12},\times)$. Then N is a neutrosophic triplet group under multiplication modulo 12 in Z_{12}, where

$$N=\{0,4,8,9\}$$

Let H_1 and H_2 be two subsets of N, where

$$H_1=\{4,8\} \text{ and } H_2=\{0,9\}.$$

Then clearly H_1 and H_2 are neutrosophic triplet subgroups of N under $\times$modulo 12.

Proposition 3. Let $(N,*)$ be a neutrosophic triplet group and H be a subset of N. Then H is a neutrosophic triplet subgroup of N if and only if the following conditions are hold.

1) $a*b \in H$ for all $a,b \in H$.

2) $neut(a) \in H$ for all $a \in H$.

3) $anti(a) \in H$ for all $a \in H$.

Proof. The proof is straightforward.

Definition 5. Let N be a neutrosophic triplet group and let $a \in N$. A smallest positive integer $n \geq 1$ such that $a^n = neut(a)$ is called neutrosophic triplet order. It is denoted by $nto(a)$.

Example 6. Let N be a neutrosophic triplet group under multiplication modulo 10 in $(Z_{10},\times)$, where

$$N = \{0,2,4,6,8\}.$$

Then

$$nto(2) = 4, nto(4) = 2,$$
$$nto(2) = 2, nto(8) = 4.$$

Theorem 6. Let $(N,*)$ be a neutrosophic triplet group with respect to $*$ and let $a \in N$.

Then

1) $neut(a)*neut(a) = neut(a)$.

In general $(neut(a))^n = neut(a)$, where n is a non-zero positive integer.

2) $neut(a)*anti(a) = anti(a)*neut(a) = anti(a)$.

Proof 1. Consider $neut(a)*neut(a) = neut(a)$.

Multiply a to the left side, we get;

$$a*neut(a)*neut(a)=a*neut(a)$$

$$\left[a*neut(a)\right]*neut(a)=\left[a*neut(a)\right]$$

$$a*neut(a)=a$$
$$a=a$$

On the same lines, we can see that $(neut(a))^n = neut(a)$ for a non-zero positive integer n.

2). Consider $neut(a)*anti(a)=anti(a)$.

Multiply to the left with a, we get;

$$a*neut(a)*anti(a)=a*anti(a)$$

$$\left[a*neut(a)\right]*anti(a)=neut(a)$$

$$a*anti(a)=neut(a)$$

$$neut(a)=neut(a)$$

Similarly $anti(a)*neut(a)=anti(a)$.

Definition 6. Let N be a neutrosophic triplet group and $a \in N$. Then N is called neutro-cyclic triplet group if $N=\langle a\rangle$. We say that a is the neutrosophic triplet generator of N.

Example 7. Let $N=\{2,4,6,8\}$ be a neutrosophic triplet group with respect to multiplication modulo 10 in $(Z_{10},\times)$. Then clearly N is a neutro-cyclic triplet group as $N=\langle 2\rangle$. Therefore 2 is the neutrosophic tiplet generator of N.

Theorem 7. Let N be a neutro-cyclic triplet group and let a be the neutrosophic triplet generator of N. Then

1) $\langle neut(a)\rangle$ generates neutro-cyclic triplet subgroup of N.

2) $\langle anti(a) \rangle$ also generates neutro-cyclic triplet subgroup of N .

Proof. Straightforward.

Neutro-Homomorphism

Definition 7. Let $(N_1, *_1)$ and $(N_2, *_2)$ be the neutrosophic triplet groups. Let

$$f : \mathrm{N}_1 \to \mathrm{N}_2$$

be a mapping. Then f is called neutro-homomorphism if for all $a, b \in N_1$, we have

1) $f(a *_1 b) = f(a) *_2 f(b)$,

2) $f(neut(a)) = neut(f(a))$, and

3) $f(anti(a)) = anti(f(a))$.

Example 8. Let N_1 be a neutrosophic triplet group with respect multiplication modulo 6 in $(Z_6, \times)$, where

$$N_1 = \{0, 2, 4\}$$

and let N_2 be another neutrosophic triplet group with respect to multiplication modulo 10 in $(\mathrm{Z}_{10}, \times)$, where

$$N_2 = \{0, 2, 4, 6, 8\}.$$

Let

$$f : N_1 \to N_2$$

be a mapping defined as

$$f(0) = 0,\ f(2) = 4,\ f(4) = 6 .$$

Then clearly f is a neutro-homomorphism because condition $(1), (2)$, and (3) are satisfied easily.

Proposition 4. Every neutro-homomorphism is a classical homomorphism by neglecting the unity element in classical homomorphism.

Proof. First we neglect the unity element that classical homomorphism maps unity element to the corresponding unity element. Now suppose that f is a neutro-homomorphism from a neutrosophic triplet group N_1 to a neutrosophic triplet group N_2. Then by condition (1), it follows that f is a classical homomorphism.

Definition 8. A neutro-homomorphism is called neutro-isomorphism if it is one-one and onto.

Conclusion

The main theme of this paper is to find out the neutrosophic approach in the algebraic structures in real sense. This was very helpful which gave a birth to a virgin neutrosophic algebirac structure called neutrosophic triplet group. This neutrosophic triplet group has several extra-ordianry properties as compared to the classical group. Neutrosophic triplet group a lot of extension for the future work.

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