α-D MCDM-TOPSIS Multi-Criteria Decision Making method for n-wise criteria comparisons and inconsistent problems

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The purpose of this paper is to present an extension and alternative of the hybrid approach using Saaty's Analytical Hierarchy Process (AHP) and Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) method (AHP-TOPSIS), that based on the AHP and its use of pairwise comparisons, to a new method called α-D MCDM-TOPSIS(α-Discounting Method for Multi-Criteria Decision Making-TOPSIS). The proposed method works not only for preferences that are pairwise comparisons of criteria as AHP does, but for preferences of any n-wise (with $n \geq 2$) comparisons of criteria. Finally the α-D MCDM-TOPSIS methodology is verified by some examples to demonstrate how it might be applied in different types of matrices and is how it allwos for consistency, inconsistent, weak inconsistent, and strong inconsistent problems.

Categories and Subject Descriptors: []

General Terms:

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1. INTRODUCTION

In recent years, Multi-Criteria Decision Making (MCDM) methods have been widely used to solve real-world problems of complicated technological and social-economic processes [Madjid Tavana 2011]. However the complexity of the modern world, the recent technological advances and the amount of data increases in a decision problem have made MCDM more challenging than ever, so does the importance of the solution quality.

The hybrid approach using Analytical Hierarchy Process (AHP) and Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) method, is one of the simplest, complete, natural and most frequently used multicriteria evaluation methods to improve the reliability of the decision making

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process, [Jingfei Yu 2013], [A. Jayanta 2014], [Wu 2007], [Madjid Tavana 2011], [Mohit Tyagia 2014], [Ghosh 2011] and [Mohit Maheshwarkar 2013].

AHP-TOPSIS is a practical tool for selection and ranking of a number of alternatives, its applications are numerous, several authors have used AHP-TOPSIS to effectively solve complex space exploration problems, [Madjid Tavana 2011] developed a group AHP-TOPSIS framework for human spaceflight mission planning at NASA, [Jingfei Yu 2013] proposed a study on the status evaluation of urban road intersections traffic congestion base on AHP-TOPSIS modal, [A. Jayanta 2014] established a TOPSIS-AHP based approach for selection of reverse logistics service provider: A case study of mobile phone industry, [Mohit Tyagia 2014] constructed a hybrid approach using AHP-TOPSIS for analyzing e-SCM performance, [Wu 2007] proposed a Topsis-AHP simulation model and its application to supply chain management, [Ghosh 2011] developed an AHP and TOPSIS Method to evaluate faculty performance in engineering education, [Mohit Maheshwarkar 2013] combined AHP-TOPSIS Based Approach for the Evaluation of Knowledge Sharing Capabilities of Supply Chain Partners.

In the hybrid AHP-TOPSIS approach, firstly, priority weights for criteria are calculated using AHP technique, and then prioritize the alternatives using TOPSIS approach.

The derivation of weights is a central step in eliciting the decision-maker's preferences, but the hybrid AHP-TOPSIS method meets more difficult: first AHP does not work in inconsistent problems, second it's cannot allowed for the n-wise comparisions criteria cases.

The problem can be abstracted as how to derive weights for a set of activities according to their impact on the situation and the objective of decisions to be made.

Hence, this study will extend AHP-TOPSIS to a MCDM to fit real work. A complete and efficient procedure for decision making will then be provided. The developed model has been analyzed to select a best alternative using α-D MCDM and technique for order preference by similarity to ideal solution (TOPSIS) as a hybrid approach.

The paper is organized as follows. In the next section, the literature survey for consistency is given. Section 3 and 4 will focus on AHP-TOPSIS and the proposed α-D MCDM-TOPSIS model respectively, in a step by- step fashion. Afterwards, the proposed method is tested on the consistent, weak inconsistent and strong inconsistent examples. In the final section, conclusions are drawn and remarks made as regards future study.

2. COMPARISON OF CHARACTERISTICS BETWEEN AHP AND α-D MCDM : CONSISTENCY

2.1 A brief overview of AHP

The Analytic Hierarchy Process (AHP) is a multi-criteria decision-making approach and was introduced by Saaty [Saaty 1994]. it is has proven to be a popular technique for determining weights in multi attribute problems questioned. The importance of AHP and the use of pairwise comparisons in decision making are best illustrated in the more than 1000 references [Evangelos Triantaphyllou 1995].

The AHP and its use of pairwise comparisons has inspired the creation of many other decision-making methods. Besides its wide acceptance, it also created some considerable criticism, both for theoretical and for practical reasons.

2.2 Description of α-D MCDM

α-Discounting Method for Multi-Criteria Decision Making (α-D MCDM), was introduced by Smarandache in [Smarandache 2010], is an alternative and extension of Saaty's Analytical Hierarchy Process (AHP), it works not only for preferences that are pairwise comparisons of criteria as AHP does, but for preferences of any n-wise (with $n \geq 2$) comparisons of criteria that can be expressed as linear homogeneous equations.

The general idea of α-D MCDM is to assign non-null positive parameters α_1, α_2, $\cdots$, α_p to the coefficients in the right-hand side of each preference that diminish or increase them in order to transform the above linear homogeneous system of equations which has only the null-solution, into a system having a particular non-null solution.

After finding the general solution of this system, the principles used to assign particular values to all parameters α's is the second important part of α-D. In this case we herein propose the Fairness Principle, i.e. each coefficient should be discounted with the same percentage (we think this is fair: not making any favouritism or unfairness to any coefficient).

2.2.1 *α-D MCDM method.* The general idea of the α-D MCDM is to discount the coefficients of an inconsistent problem to some percentages in order to transform it into a consistent problem.

Let us assume that $C = \{C_1, C_2, \cdots, C_n\}$, with $n \geq 2$, and $P = \{P_1, P_2, \cdots, P_m\}$, with $m \geq 1$. are a set of Criteria and the set of Preferences, respectively.

Each preference P_i is a linear homogeneous equation of the above criteria $C_1, C_2, \cdots, C_n$:

$$P_i = f(C_1, C_2, \cdots, C_n)$$

Construct a basic belief assignment (*bba*):

$$m : C \to [0, 1]$$

such that $m(C_i) = w_i$, with $0 \leq w_i \leq 1$, and

$$\sum_{i=1}^{n} m(C_i) = \sum_{i=1}^{n} w_i = 1$$

We therefore have an $m \times n$ linear homogeneous system and its associated matrix.

We need to find all variables w_i in accordance with the set of preferences P.

The α-D MCDM method procedure cited above is designed to rank preferences P_i based on C_i criteria, as an alternative method of the AHP method, that is a complete method and used to calculates the weights of criteria C_i and to rank the preferences P_i.

In addition, when the AHP is used with TOPSIS, or other MCDM method, we just benefit from the part of weight calculation criteria and we used TOPSIS to rank preferences or other MCDM methods.

The same for α-D MCDM is just used to calculate the weight of criteria, that will be used later by TOPSIS to rank preferences.

In this paper, we will adapt α-D MCDM for just calculate weight of criteria C_i and not to rank P_i preferences. In this case, when we will calculate the weights of criteria C_i, instead of

$$P_i = f(C_1, C_2, \cdots, C_n)$$

We should have

$$C_i = f(\{C\} \setminus C_i)$$

Then criteria C_i is a linear equation of C_j such as.

$$C_i = \sum_{j=1 j \neq i}^{n} x_{ij} C_j$$

So the comparisons criteria matrix has the number of criteria by rows and columns (rows number n = number of criteria and colones number m also = number of criteria). In the result, we have square matrix ($n = m$), consequently we can calculate the determinant of this matrix. At this point, we have an $n \times n$ linear homogeneous system and its associated matrix.

$$\begin{cases} x_{1,1}w_1 + x_{1,2}w_2 + \cdots + x_{1,n}w_n = 0 \\ \vdots \\ x_{n,1}w_1 + x_{m,2}w_2 + \cdots + x_{n,n}w_n = 0 \end{cases}$$

$$X = \begin{pmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{n,1} & \cdots & x_{n,n} \end{pmatrix}$$

2.3 Classification of Linear Decision-Making Problems

—We say that a linear decision-making problem is consistent if, by any substitution of a variable w_i from an equation into another equation, we get a result in agreement with all equations.

—We say that a linear decision-making problem is weakly inconsistent if by at least one substitution of a variable w_i from an equation into another equation we get a result in disagreement with at least another equation in the following ways:

$$(WD1) \left\{ \begin{array}{l} w_i = \quad k1.w_j,\ k1 > 1 \\ w_i = k2.w_j,\ k2 > 1,\ k2 \neq k1 \end{array} \right\}$$

or

$$(WD2) \left\{ \begin{array}{l} w_i = \quad k1.w_j, 0 < k1 < 1 \\ w_i = k2.w_j, 0 < k2 < 1, k2 \neq k1 \end{array} \right\}$$

or

$$(WD3) \left\{ w_i = k1.w_j, k1 \neq 1 \right\}$$

—We say that a linear decision-making problem is strongly inconsistent if, by at least one substitution of a variable w_i from an equation into another equation, we get a result in disagreement with at least another equation in the following way:

$$(SD4) \left\{ \begin{array}{l} w_i = k1.w_j \\ w_i = k2.w_j \end{array} \right\} \text{ with } 0 < k1 < 1 < k2 \text{ or } 0 < k2 < 1 < k1$$

2.4 Consistency

AHP provides a decision maker with a way of examining the consistency of entries in a pairwise comparison matrix, the problem of accepting/rejecting matrices has been greatly discussed [J. I. Pelaez 2003], [Jose Antonto Alonso 2006], [Vera Jandova 2013], especially the relation between the consistency and the scale used to represent the decision maker's judgments. AHP is too restrictive when the size of the matrix increases, when order n of judgment matrix is large, the satisfying consistency meets more difficult [J. I. Pelaez 2003], [Jose Antonto Alonso 2006].

This problem may become a very difficult when the decision maker is not perfectly consistent, moreover it's seem impossible (where AHP does not work), one when not paiwise comparisons but all kind of comparisons between criteria, such as $n-$wise, because in the AHP there is set a strict consistency condition in order to keep the rationality of preference intensities between compared elements.

In addition, the inconsistency exists in all judgments [J. I. Pelaez 2003], comparing more three alternatives, it is possible that inconsistency exists when there are more than 25 percent of the $3-by-3$ reciprocal matrices with a consistency ratio less than or equal to ten percent. Consequently as the matrix size increases, this percentage of inconsistency decreases dramatically [Jose Antonto Alonso 2006], [Vera Jandova 2013], [J. I. Pelaez 2003].

Furthermore, the Saaty's method measure the inconsistency of the pairwise comparison matrix and set a consistency threshold ($CR(X) > 0.1$) which should not be exceeded, but this requirement for the Saaty's matrix is not achievable in the real situations .

In order to overcome this deficiency, that is why instead of the AHP we suggest an α-D MCDM which is very natural and more suitable for the linguistic descriptions of the Saaty's scale and as a result of it, it is easier to reach this requirement in the real situations.

Moreover, the attractiveness of α-D MCDM is due to its potential use in cases or not only for preferences that are pairwise comparisons of criteria as AHP does, but for preferences of any n-wise (with $n \geq 2$) comparisons of criteria. And one of the most practical issues in the α-D MCDM methodology that it allows for inconsistent, weak inconsistent, and strong inconsistent problems.

2.4.1 *Degrees of consistency and inconsistency in α-D MCDM.*

For α-D MCDM (and Fairness-Principle for coefficients α) in consistent and weak consistent decision-making problems, we have the followings:

—If $0 < \alpha < 1$, then α is the degree of consistency of the decision-making problem, and $\beta = 1 - \alpha$ is the degree of inconsistency of the decision-making problem.

—If $\alpha > 1$, then $\frac{1}{\alpha}$ is the degree of consistency of the decision-making problem, and $\beta = 1 - \frac{1}{\alpha}$ is the degree of inconsistency of the decision-making problem.

2.4.2 *Degrees of consistency in AHP*

—In ideal case : the criteria comparison matrix (X) is fully consistent, the $rank(X) = 1$ and $\lambda = n$ ($n =$ number of criteria).

—In the non perfectly consistent case:

  —If $CR(X) \leq 0.1$, the pairwise comparison matrix is considered to be consistent enough.

  —If $CR(X) > 0.1$, the comparison matrix should be improved.

The consistence ratio (CR) is calculated as the ratio of consistency index and random consistency index (RI).

$$CR(X) = \frac{CI(X)}{RI(n)}$$

The consistency index (CI) is given by :

$$CI = \frac{\lambda_{max} - n}{n - 1}$$

where λ_{max} is the largest eigenvalue of the $n \times n$ pairwise comparison matrix.

The value of RI is based on a simulation of a large number of randomly generated weights and depends on the number criteria being compared.

Table I. Comparison of characteristics between AHP and α-D MCDM

Characteristics	AHP	α-D MCDM
Weight elicitation	Pairwise comparison	n-wise comparison
No. of attributes accommodated	7 ± 2	large inputs
Consistent problems	Provided	Not provided (same result as AHP)
Weakly inconsistent problems	does not works	justifiable results
Strongly inconsistent problems	does not works	justifiable results

3. AHP-TOPSIS METHOD

In the real word decisions problems we have a multiples preferences and diverse criteria. The problem can abstracted as how to :

—Derive weights w_i of criteria C_i.

—Rank preferences (alternatives) A_i.

Let us assume there are n criteria and theirs pair-wise relative importance x_{ij}.

TOPSIS assumes that we have n alternatives (preferences) $A_i (i = 1, 2, ..., m)$ and n attributes/criteria $C_j (j = 1, 2, ..., n)$ and we have the score a_{ij} of preference i with respect to criterion j.

The MCDM approach based on AHP-TOPSIS is explained in the following steps

Step 3.1. Construct decision matrix denoted by $A = (a_{ij})_{m \times n}$

Table II. Decision matrix

	C_1	C_2	$\cdots$	C_n
	w_1	w_2	$\cdots$	w_n
A_1	a_{11}	a_{12}	$\cdots$	a_{1n}
A_2	a_{21}	a_{22}	$\cdots$	a_{2n}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
A_m	a_{m1}	a_{m2}	$\cdots$	a_{mn}

Step 3.2. Using **AHP** to determine the importance weight (w_j) of the criteria such that:

$$\sum_{j=1}^{n} w_j = 1, \quad j = 1, 2, \cdots, n$$

Step 3.2.1. Make a pair-wise comparison matrix of criteria by using a scale 1 to 9 given by Saaty, the pairwise comparison of criterion i with criterion j gives a square matrix $X_{n \times n} = (x_{ij})$ where x_{ij} represents the relative importance of criterion i over the criterion j. In the matrix, $x_{ij} = 1$ when $i = j$ and $x_{ij} = 1/x_{ji}$.

Let X represent an $n \times n$ pair-wise comparison matrix,

Step 3.2.2. Find the relative normalized weight (w_j) of each criterion by normalizing the geometric mean of rows in the comparison matrix.

Let w_i denotes the importance degree for the i^{th} criteria :

$$w_j = \frac{\prod_{j=1}^{n} (x_{ij})^{1/n}}{\sum \prod_{j=1}^{n} (x_{ij})^{1/n}}$$

Step 3.2.3. Calculate matrix $X3$ and $X4$ such that $X3 = X1 * X2$ and $X4 = X3/X2$ where

$$X2 = [w_1, w_2, \cdots, w_j]^T$$

Step 3.2.4. Calculate the maximum eigen value which is the average of matrix $X4$.

According to the Perron-Frobenius theorem, principal eigenvalue λ_{max} always exists for the Saaty's matrix and it holds $\lambda_{max} \geq n$; for fully consistent matrix $\lambda_{max} = n$.

Consistency check is then performed to ensure that the evaluation of the pair-wise comparison matrix is reasonable and acceptable.
Step 3.2.5. Determine the consistency ratio (CR).
The consistency index (CI) is calculated as following

$$CI = \frac{\lambda_{max} - n}{n - 1}$$

Then, the consistence ratio (CR) is calculated as the ratio of consistency index and random consistency index (RI).

$$CR(X) = \frac{CI(X)}{RI(n)}$$

Random consistency Index (RI) depends on the matrix size and is estimated as the average consistency index of randomly generated Saaty's matrices of the dimension n, using the Saaty scale.
According to Saaty and research has shown that if value of consistency ratio is below the threshold of 0.10, the pairwise comparison matrix is considered to be consistent enough and the evaluation of importance of degrees of attributes is considered to be reasonable.

Step 3.3. The normalized decision matrix is obtained, which is given here with r_{ij}
At this point, AHP is used to determine the weights(step 3.2), we will using TOPSIS to rank preferences.

$$r_{ij} = a_{ij} / \left(\sum_{i=1}^{m} a_{ij}^2 \right)^{0.5} ; \; j = 1, 2, \cdots, n; \; i = 1, 2 \cdots, m$$

Step 3.4. Obtain the weighted normalized decision matrix v_{ij}: multiply each column of the normalized decision matrix by its associated weight.

$$v_{ij} = w_j r_{ij}; \; j = 1, 2, \cdots, n; \; i = 1, 2 \cdots, m$$

Step 3.5. Determine the ideal and negative ideal solutions

$$A^+ = (v_1^+, v_2^+, \cdots, v_n^+) = \{(max_i \{v_{ij} \mid j \in B\}), (min_i \{v_{ij} \mid j \in C\})\}$$

$$A^- = (v_1^-, v_2^-, \cdots, v_n^-) = \{(min_i \{v_{ij} \mid j \in B\}), (max_i \{v_{ij} \mid j \in C\})\}$$

where B and C are associated with the benefit and cost attribute sets, respectively
Step 3.6. Calculate the separation measures for each alternative
The separation from the ideal alternative is

$$S_i^+ = \left\{ \sum_{j=1}^{n} (v_{ij} - v_j^+)^2 \right\}^{0.5} ; \; i = 1, 2 \cdots, m$$

Similarly, the separation from the negative ideal alternative is

$$S_i^- = \left\{ \sum_{j=1}^{n} (v_{ij} - v_j^-)^2 \right\}^{0.5} ; \; i = 1, 2 \cdots, m$$

Step 3.7. The relative closeness to the ideal solution of each alternative is calculated as

$$T_i = \frac{S_i^-}{(S_i^+ + S_i^-)}; \;\; i = 1, 2 \cdots, m$$

Step 3.8. A set of alternatives can now be preference ranked according to the descending order of the value of T_i

4. α-D MCDM-TOPSIS METHOD

Probem description is same of AHP-TOPSIS method (section 3.), but in this case we have n-wise comparisons matrix of criteria.

Let us assume that $C = \{C_1, C_2, \cdots, C_n\}$, with $n \geq 2$, and $\{A_1, A_2, \cdots, A_m\}$, with $m \geq 1$, are a set of Criteria and the set of Preferences, respectively.

Let us assume each criteria C_i is a linear homogeneous equation of the other criteria $C_1, C_2, \cdots, C_n$:

$$C_i = f(\{C\} \setminus C_i)$$

The α-D MCDM-TOPSIS method consists of the following steps:

Step 4.1. Construct decision matrix denoted by $A = (a_{ij})_{m \times n}$

Table III. Decision matrix

	C_1	C_2	$\cdots$	C_n
	w_1	w_2	$\cdots$	w_n
A_1	a_{11}	a_{12}	$\cdots$	a_{1n}
A_2	a_{21}	a_{22}	$\cdots$	a_{2n}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
A_m	a_{m1}	a_{m2}	$\cdots$	a_{mn}

Step 4.2. Using α-**D MCDM** to determine the importance weight (w_i) of the criteria

Step 4.2.1. Construct an $n \times n$ linear homogeneous system of equations whose associated matrix denoted by $X = (x_{ij})$, $1 \leq i \leq n$ and $1 \leq j \leq n$ where each criteria C_i be expressed as a linear equation of C_j such as.

$$C_i = \sum_{j=1 j \neq i}^{n} x_{ij} C_j$$

Step 4.2.2. Construct a basic belief assignment (bba):

$$m : C \to [0, 1]$$

such that $m(C_i) = w_i$, with $0 \leq w_i \leq 1$, and

$$\sum_{i=1}^{n} m(C_i) = \sum_{i=1}^{n} w_i = 1$$

We therefore have an $n \times n$ linear homogeneous system and its associated matrix

$$\begin{cases} x_{1,1} w_1 + x_{1,2} w_2 + \cdots + x_{1,n} w_n = 0 \\ \vdots \\ x_{n,1} w_1 + x_{m,2} w_2 + \cdots + x_{n,n} w_n = 0 \end{cases}$$

$$X = \begin{pmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{n,1} & \cdots & x_{n,n} \end{pmatrix}$$

Step 4.2.3. Solve this system, looking for a strictly positive solution (i.e. all unknowns $wi > 0$).
Step 4.2.4. Compute the determinant of X.

—If $det(X) = 0$, the decision problem is consistent, since the system of equations is dependent.

—If $det(X) \neq 0$, the decision problem is inconsistent, since the homogeneous linear system has only the null-solution.

  —If the inconsistency is weak, then parameterize the right-hand side coefficients, and denote the system matrix $X(\alpha)$. Compute $det(X(\alpha)) = 0$ in order to get the parametric equation.

  —If the Fairness Principle is used, set all parameters equal, and solve for $\alpha > 0$. Replace in $X(\alpha)$ and solve the resulting dependent homogeneous linear system. Similarly as in a), replace each secondary variable by 1, and normalize the particular solution in order to get the priority vector.

  —If the inconsistency is strong, the Fairness Principle may not work properly. Another approachable principle might be designed. Or, get more information and revise the strong inconsistencies of the decision-making problem.

Step 4.2.5. Solving this homogeneous linear system (in differents cases abov) we get its general solution that we set as a solution vector

$$S = [s_1, s_2, \cdots, s_n]$$

Step 4.2.6. Normalizing (dividing each vector component by the sum of all vector components) we get the priority vector

$$W = [w_1, w_2, \cdots, w_n]$$

where

$$w_j = \frac{s_j}{\sum_{k=1}^{n} s_k}; \quad i = 1, 2 \cdots, n$$

Step 4.3. The normalized decision matrix is obtained, which is given here with r_{ij}
At this point, α-D MCDM is used to determine the weights(step 4.2), we will be using TOPSIS to rank preferences.

$$r_{ij} = a_{ij} / \left(\sum_{i=1}^{m} a_{ij}^2 \right)^{0.5}; \quad j = 1, 2, \cdots, n; \quad i = 1, 2 \cdots, m$$

Step 4.4. Obtain the weighted normalized decision matrix v_{ij}: multiply each column of the normalized decision matrix by its associated weight.

$$v_{ij} = w_j r_{ij}; \quad j = 1, 2, \cdots, n; \quad i = 1, 2 \cdots, m$$

Step 4.5. Determine the ideal and negative ideal solutions

$$A^+ = (v_1^+, v_2^+, \cdots, v_n^+) = \{(max_i \{v_{ij} \mid j \in B\}), (min_i \{v_{ij} \mid j \in C\})\}$$

$$A^- = (v_1^-, v_2^-, \cdots, v_n^-) = \{(min_i \{v_{ij} \mid j \in B\}), (max_i \{v_{ij} \mid j \in C\})\}$$

where B and C are associated with the benefit and cost attribute sets, respectively

Step 4.6. Calculate the separation measures for each alternative
The separation from the ideal alternative is

$$S_i^+ = \left\{ \sum_{j=1}^{n} (v_{ij} - v_j^+)^2 \right\}^{0.5} \quad ; \ i = 1, 2 \cdots, m$$

Similarly, the separation from the negative ideal alternative is

$$S_i^- = \left\{ \sum_{j=1}^{n} (v_{ij} - v_j^-)^2 \right\}^{0.5} \quad ; \ i = 1, 2 \cdots, m$$

Step 4.7. The relative closeness to the ideal solution of each alternative is calculated as

$$T_i = \frac{S_i^-}{(S_i^+ + S_i^-)}; \ \ i = 1, 2 \cdots, m$$

Step 4.8. A set of alternatives can now be preference ranked according to the descending order of the value of T_i

5. NUMERICAL ILLUSTRATION

We examined a numerical example in which a synthetic evaluation desire to rank four alternatives A_1, A_2, A_3 and A_4 with respect to three benefit attribute C_1, C_2 and C_3.

Table IV. Decision matrix

	C_1	C_2	C_3
	w_1	w_2	w_3
A_1	7	9	9
A_2	8	7	8
A_3	9	6	8
A_4	6	7	8

In the exemples below we used α-D MCDM and AHP (if it work) to calculate the weights of the criteria w_1, w_2 and w_3. We used TOPSIS to rank the four alternatives, the decision matrix below is used for the three following exemples

5.1 Consistent Example 1

We use the α-D MCDM

Let the Set of Criteria be $\{C_1, C_2, C_3\}$ such that:

—C_1 is 4 times as important as C_2

—C_2 is 3 times as important as C_3

—C_3 is one twelfth as important as C_1

Let $m(C_1) = x$, $m(C_2) = y$, $m(C_3) = z$

The linear homogeneous system associated to this decision-making problem is

$$\begin{cases} x = 4y \\ y = 3z \\ z = \frac{x}{12} \end{cases}$$

whose associated matrix is:

$$X1 = \begin{pmatrix} 1 & 4 & 0 \\ 0 & 1 & 3 \\ \frac{1}{12} & 0 & 1 \end{pmatrix}$$

whence $det(X1) = 0$, so the DM problem is consistent with its general solution

$$S = \left[\ 12z \ \ 3z \ \ z\ \right]$$

Replacing $z = 1$ into the solution vector, the solution vector becomes

$$S = \left[\ 12 \ \ 3 \ \ 1\ \right]$$

and then normalizing (dividing by $12 + 3 + 1 = 16$ each vector component) we get the priority vector:

$$W = \left[\ \frac{12}{16} \ \ \frac{3}{16} \ \ \frac{1}{16}\ \right]$$

Using AHP, we get the same result
The pair-wise comparison matrix of criteria is:

$$X1 = \begin{pmatrix} 1 & 4 & 12 \\ \frac{1}{4} & 1 & 3 \\ \frac{1}{12} & \frac{1}{3} & 1 \end{pmatrix}$$

whose maximum eigenvalue is $\lambda_{max} = 3$ and its corresponding normalized eigenvector (Perron-Frobenius vector) is

$$W = \left[\ \frac{12}{16} \ \ \frac{3}{16} \ \ \frac{1}{16}\ \right]$$

We use TOPSIS to rank the four alternatives

Table V. Calculate (a_{ij}^2) for each column

a_{ij}^2	C_1	C_2	C_3
	12/16	3/16	1/16
A_1	49	81	81
A_2	64	49	64
A_3	81	36	64
A_4	36	49	64
$\sum_{i=1}^{n} a_{ij}^2$	230	215	273

Table VI. Divide each column by $(\sum_{i=1}^{n} a_{ij}^2)^{1/2}$ to get r_{ij}

r_{ij}	C_1	C_2	C_3
	12/16	3/16	1/16
A_1	0.4616	0.6138	0.5447
A_2	0.5275	0.4774	0.4842
A_3	0.5934	0.4092	0.4842
A_4	0.3956	0.4774	0.4842
$\sum_{i=1}^{n} a_{ij}^2$	230	215	273

Table VII. Multiply each column by w_j to get v_{ij}

v_{ij}	C_1	C_2	C_3
	12/16	3/16	1/16
A_1	0.3462	0.1151	0.0340
A_2	0.3956	0.0895	0.0303
A_3	0.4451	0.0767	0.0303
A_4	0.2967	0.0895	0.0303
v_{max}	0.4451	0.1151	0.0340
v_{min}	0.2967	0.0767	0.0303

Table VIII presents the separation measure of each alternative from the positive ideal solution (PIS) and from the negative ideal solution (NIS) and the overall ranking of the alternatives (A_1, A_2, A_3, A_4)in wish the weighted values are calculated by AHP or α-D MCDM (for consistent problems, AHP and α-D MCDM/Fairness-Principle give the same result) .

Table VIII. The distance values and the final rankings for dicision matrix (Table II) (using AHP-TOPSIS, α-D MCDM-TOPSIS

Alternative	S_i^+	S_i^-	T_i	Rank
A_1	0.0989	0.0627	0.3880	3
A_2	0.0558	0.0997	0.6412	2
A_3	0.0385	0.1484	0.7938	1
A_4	0.1506	0.0128	0.0783	4

5.2 Weak Inconsistent Examples where AHP does not Work

Consider another example investigated by [Smarandache 2010] for which AHP does not work,

Then we use the α-D MCDM to clculate the weights values and TOPSIS to rank the four alternatives (see Table IV)

The Set of Criteria be:

—C_1 is as important as 2 times C_2 plus 3 times C_3

—C_2 is half as important as C_1

—C_3 is one third as important as C_1

Let $m(C_1) = x$, $m(C_2) = y$, $m(C_3) = z$

The linear homogeneous system associated to this decision-making problem is

$$\begin{cases} x = 2y + 3z \\ y = \frac{x}{2} \\ z = \frac{x}{3} \end{cases}$$

whose associated matrix is:

$$X1 = \begin{pmatrix} 1 & -2 & -3 \\ \frac{-1}{2} & 1 & 0 \\ \frac{-1}{3} & 0 & 1 \end{pmatrix}$$

If we solve this homogeneous linear system of equations as it is, we get $x = y = z = 0$, but the null solution is not acceptable since the sum $x + y + z$ has to be 1

Let us parameterise each right-hand side coefficient and get the general solution of the above system:

$$x = 2\alpha_1 y + 3\alpha_2 z \quad (1)$$
$$y = \frac{\alpha_3 x}{2} \quad (2)$$
$$z = \frac{\alpha_4 x}{3} \quad (3)$$

where

$$\alpha_1, \alpha_2, \alpha_3, \alpha_4 > 0$$

Replacing (eq. 2) and (eq. 3) in (eq. 1) we get,

$$x = 2\alpha_1(\frac{\alpha_4 x}{3}) + 3\alpha_2(\frac{\alpha_4 x}{3})$$

so

$$1.x = (\alpha_1\alpha_3 + \alpha_2\alpha_4).x$$

whence

$$\alpha_1\alpha_3 + \alpha_2\alpha_4 = 1$$

The general solution of the system is:

$$S = \begin{cases} y = \frac{\alpha_3 x}{2} \\ z = \frac{\alpha_4 x}{3} \end{cases}$$

whence the priority vector:

$$S = \begin{bmatrix} x & \frac{\alpha_3 x}{2} & \frac{\alpha_4 x}{3} \end{bmatrix}$$

$$S = \begin{bmatrix} 1 & \frac{\alpha_3}{2} & \frac{\alpha_4}{3} \end{bmatrix}$$

Fairness Principle: discount all coefficients with the same percentage, so, replace $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha$ in (4) we get $\alpha^2 + \alpha^2 = 1$ whence $\alpha = \frac{\sqrt{2}}{2}$
Priority vector becomes

$$S = \begin{bmatrix} 1 & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{6} \end{bmatrix}$$

and normalizing it:

$$W = \begin{bmatrix} 0.62923 & 0.22246 & 0.14831 \end{bmatrix}$$

Topsis is used to rank the four alternative: calculate and application of TOPSIS method is in the same manner as in the previous example (the overall ranking order of the four alternatives (Ai) presented in following Table XII)

5.3 Jean Dezert's Strong Inconsistent Example

[Smarandache 2010] introduced a Jean Dezert's Strong Inconsistent example

Table IX. Calculate (a_{ij}^2) for each column

a_{ij}^2	C_1	C_2	C_3
	0.62923	0.22246	0.14831
A_1	49	81	81
A_2	64	49	64
A_3	81	36	64
A_4	36	49	64
$\sum_{i=1}^n a_{ij}^2$	230	215	273

Table X. Divide each column by $(\sum_{i=1}^n a_{ij}^2)^{1/2}$ to get r_{ij}

r_{ij}	C_1	C_2	C_3
	0.62923	0.22246	0.14831
A_1	0.4616	0.6138	0.5447
A_2	0.5275	0.4774	0.4842
A_3	0.5934	0.4092	0.4842
A_4	0.3956	0.4774	0.4842
$\sum_{i=1}^n a_{ij}^2$	230	215	273

Table XI. Multiply each column by w_j to get v_{ij}

v_{ij}	C_1	C_2	C_3
	0.62923	0.22246	0.14831
A_1	0.2904	0.1365	0.0808
A_2	0.3319	0.1062	0.0718
A_3	0.3734	0.0910	0.0718
A_4	0.2489	0.1062	0.0718
v_{max}	0.3734	0.1365	0.0808
v_{min}	0.2489	0.0910	0.0718

Table XII. The distance values and the final rankings for dicision matrix (Table II) using α-D MCDM-TOPSIS

Alternative	S_i^+	S_i^-	T_i	Rank
A_1	0.0830	0.0622	0.4286	3
A_2	0.0522	0.0844	0.6178	2
A_3	0.0464	0.1245	0.7285	1
A_4	0.1284	0.0152	0.1057	4

$$X = \begin{pmatrix} 1 & 9 & \frac{1}{9} \\ \frac{1}{9} & 1 & 9 \\ 9 & \frac{1}{9} & 1 \end{pmatrix}$$

$$\begin{cases} x{=}9y, x > y \\ x{=}\frac{1}{9}z, x < z \\ y{=}9z, y > z \end{cases}$$

From $x > y$ and $y > z$ (first and third above inequalities) we get $x > z$, but the second inequality tells us the opposite: $x < z$; that is why we have a strong contradiction/inconsistency. Or, if we combine all three we have $x > y > z > x$ strong contradiction again.

Parameterize:

$$\begin{cases} x = 9\alpha_1 y \\ x = \frac{1}{9}\alpha_2 z \\ y = 9\alpha_3 z \end{cases}$$

where

$$\alpha_1, \alpha_2, \alpha_3 > 0$$

we get

$$y = \frac{1}{9\alpha_1}x, z = \frac{1}{9\alpha_2}x$$

and we get:

$$y = 9\alpha_3\left(\frac{9}{\alpha_2 x}\right) = \frac{81\alpha_3}{\alpha_2}x$$

So

$$\frac{1}{9\alpha_1}x = \frac{81\alpha_3}{\alpha_2}x \text{ or } \alpha_2 = 729\alpha_1\alpha_3$$

The general solution of the system is:

$$S = \left[\, x \;\; \tfrac{1}{9\alpha_1}x \;\; \tfrac{9}{\alpha_2}x \,\right]$$

The general priority vector is:

$$S = \left[\, 1 \;\; \tfrac{1}{9\alpha_1} \;\; \tfrac{9}{\alpha_2} \,\right]$$

Consider the fairness principle, then $\alpha_1 = \alpha_2 = \alpha_3 = \alpha > 1$ are replaced into the parametric equation: $\alpha = 729\alpha^2$, whence $\alpha = 0$ (not good) and $\alpha = \frac{1}{9^3}$

The particular priority vector becomes

$$S = \left[\, 1 \;\; 81 \;\; 6561 \,\right]$$

$$W = \left[\, \tfrac{1}{6643} \;\; \tfrac{81}{6643} \;\; \tfrac{6561}{6643} \,\right]$$

We use TOPSIS to rank the four alternatives

Table XIII. Calculate (a_{ij}^2) for each column

a_{ij}^2	C_1	C_2	C_3
	0.0002	0.0122	0.9877
A_1	49	81	81
A_2	64	49	64
A_3	81	36	64
A_4	36	49	64
$\sum_{i=1}^{n} a_{ij}^2$	230	215	273

Table XIV. Divide each column by $(\sum_{i=1}^{n} a_{ij}^2)^{1/2}$ to get r_{ij}

r_{ij}	C_1	C_2	C_3
	0.0002	0.0122	0.9877
A_1	0.503	0.699	0.623
A_2	0.574	0.543	0.553
A_3	0.646	0.466	0.553
A_4	0.431	0.543	0.553
$\sum_{i=1}^{n} a_{ij}^2$	230	215	273

Table XV. Multiply each column by w_j to get v_{ij}

v_{ij}	C_1	C_2	C_3
	0.0002	0.0122	0.9877
A_1	0.0001	0.0075	0.5380
A_2	0.0001	0.0058	0.4782
A_3	0.0001	0.0050	0.4782
A_4	0.0001	0.0058	0.4782
v_{max}	0.0001	0.0075	0.5380
v_{min}	0.0001	0.0050	0.4782

Table XVI. The distance values and the final rankings for dicision matrix (Table II) (using AHP-TOPSIS, α-D MCDM-TOPSIS

Alternative	S_i^+	S_i^-	T_i	Rank
A_1	0.0000	0.0598	0.999668	1
A_1	0.0598	0.0008	0.013719	2
A_1	0.0598	0.0000	0.000497	4
A_1	0.0598	0.0008	0.013715	3

6. CONCLUSION

For consistent decision-making problems with pairwise comparisons, α-D MCDM -TOPSIS give the same result as AHP-TOPSIS, But for weak inconsistent and strong inconsistent decision-making problems, α-D MCDM -TOPSIS give a justifiable results, in that AHP-TOPSIS does not works.

The proposed hybrid approach using α-Discounting Method for Multi-Criteria Decision Making (α-D MCDM) and Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) method can uses to solve real-life problems in wish criteria are not only pairwise but n-wise comparisons and the problems are not perfectly consistent.

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