Emergence of Space in Quantum Shape Kinematics

Furkan Semih Dündar
Physics Department, Boğaziçi University
34342, İstanbul, Turkey[*]

Barış Tamer Tonguç
Physics Department, Sakarya University
54050, Sakarya, Turkey[†]
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A model universe is analyzed with N protons and electrons where there are electromagnetic and spin interactions in the Hamiltonian is investigated in the context of quantum shape kinematics. A similar classical model where there is only gravitational attraction between masses was investigated by Barbour, Koslowski and Mercati before [2]. We have found that quantum shape space exists for $N \geq 4$ particles has $2N-7$ functional degrees of freedom in the case of spin-1/2 particles. Emergence of space is associated with non-vanishing expectation value $\langle L^2 \rangle$. Finally we have shown that space emerges for almost all states that satisfy the constraint equations, because the states that satisfy $\langle L^2 \rangle$ constitute a measure-zero set.

INTRODUCTION

Shape dynamics is a fully relational theory of gravitation. In the case of N-body problem, it states that only the relative distances and angles between them are dynamical [1]. In this scenario universe cannot have a non-vanishing angular momentum, otherwise it would define an absolute space in which the universe is rotating [2]. Similarly non-vanishing total energy implies an external absolute time according to which the universe evolves, therefore we require total energy to vanish [2]. Another constraint comes from working in the center of mass frame, the total momentum should be zero. Some works [3] also put another constraint on the system, e.g. the vanishing of the dilational momentum: $\sum_a \mathbf{r}^a \cdot \mathbf{p}^a = 0$. This is required for scale invariance [3], however we will not impose it. Overall, we have three constraints:

- $\mathrm{H} = \sum_a E^a = 0$
- $\mathbf{P} = \sum_a \mathbf{p}^a = 0$
- $\mathbf{L} = \sum_a \mathbf{r}^a \times \mathbf{p}^a = 0$

There is a need to calculate Poisson brackets of constraints with each other in order to classify them. This distinction will be important when the theory is quantized. The list is as follows:

$$\{P_i, P_j\} = 0, \tag{1}$$
$$\{L_i, L_j\} = \varepsilon^{ijk} L_k, \tag{2}$$
$$\{L_i, P_j\} = \varepsilon^{ijk} P_k, \tag{3}$$
$$\{H, P_i\} = 0, \tag{4}$$
$$\{H, L_i\} = 0. \tag{5}$$

The results of these commutators are either zero or another constraint. Hence they vanish weakly. Therefore all the constraints are first class in the terminology of Dirac [4]. We did the Dirac analysis in order to show that our system is consistent and have seen that it is well defined.

QUANTIZATION OF THE MODEL

The model is quantized by promoting positions and momenta to quantum operators. The Poisson bracket $\{\cdot,\cdot\}$ is mapped to $i\hbar[\cdot,\cdot]$. Between position and momenta is the following expression:

$$[\hat{r}_i^a, \hat{p}_j^b] = i\hbar \delta_b^a \delta_j^i. \tag{6}$$

Momenta are represented by operators, $\hat{p}_i^a = -i\hbar\partial/\partial r_i^a$. It is time to consider what happens to constraints in this case. As the readers can verify easily, the constraint algebra survives the quantization. In particular there is no anomaly. In the presence of dilational momentum constraint ref. [3] reports the existence of scale anomaly. It is then argued in [3] that this anomaly may give rise to a gravitational arrow of time. However, the arrow of time is outside our scope in this study.

At the quantum level the constraints become operators acting on the quantum state of the system. For example the Hamiltonian constraint becomes:

$$H|\psi\rangle = 0, \tag{7}$$

where H includes kinetic terms, potential terms and angular momentum couplings. We see that we have obtained a time independent Schrödinger equation. Wavefunctions do not evolve in time and are static. This is similar to what happens with the Wheeler-DeWitt equation.

The momentum and angular momentum constraints become:

$$\mathbf{P}|\psi\rangle = -i\hbar \sum_a \nabla_a |\psi\rangle = 0, \tag{8}$$

$$\mathbf{J}|\psi\rangle = -i\hbar \sum_a \mathbf{r}^a \times \nabla_a |\psi\rangle + \sum_a \mathbf{S}_a |\psi\rangle = 0, \tag{9}$$

where the spin operator for each particle $\mathbf{S}_a$ is added. We interpret equations (7) (8) and (9) as operator equations that determines the allowed kinematic states of the system.

EMERGENCE OF SPACE

The Hamiltonian constraint, momentum constraint and angular momentum constraint adds up to total of 7 constraints in three dimensional space. For a single particle the momentum constraint implies that $|\psi\rangle$ is a constant spinor. Because it has not orbital angular momentum $\mathbf{L}|\psi\rangle = 0$, however $\mathbf{S}|\psi\rangle$ cannot vanish. Hence the theory disallows the existence of a single particle in the universe. In fact the theory disallows the existence of two and three particles as well, because there are at most six functional degrees of freedom. This is contrast to classical shape dynamics, because there the shape space is defined for $N \geq 3$ [1]. The source of this mismatch is because in classical theory every particle is assigned a three dimensional vector where in the quantum theory every particle is assigned a *two* dimensional spinor.

For more than three particles quantum shape kinematic states are allowed since there are now at least eight functional degrees freedom. Quantum shape space is $2N-7$ dimensional, where as the classical shape space is $3N-7$ dimensional. Of course the dimension of the quantum shape space changes with species other than spin-1/2 particles, the result we give is for spin-1/2 particles.

Let $|\psi\rangle$ be an element of quantum shape space. Due to angular momentum constraint $\mathbf{L}|\psi\rangle = -\mathbf{S}|\psi\rangle$ holds, which implies $\langle L^2\rangle = \langle S^2\rangle$ where $\mathbf{L} = \sum_a \mathbf{L}_a, \mathbf{S} = \sum_a \mathbf{S}_a$. Hence for states $|\psi\rangle$ such that $\langle S^2\rangle \neq 0$, there is a non-vanishing classical+orbital angular momentum. Therefore classical absolute space emerges in which the universe has a non-vanishing classical angular momentum. We now focus our attention to existence of states that allows the emergence of space.

THE STATES FOR WHICH SPACE EMERGES

We need to show that $\langle S^2\rangle \neq 0$ for some states. The method embraced is the matrix form of this operator. We expand S^2 as follows:

$$S^2 = \sum_{i,j} \mathbf{S}_i \odot \mathbf{S}_j = \sum_i S_i^2 + \sum_{i\neq j} \mathbf{S}_i \odot \mathbf{S}_j. \tag{10}$$

Observe that this matrix is diagonal. It is known that $S_i^2 = \hbar^2/4$. So, it is a factor of identit matrix of dimension $2N$. On the other hand, the second term in (10) places $2(N-1)\hbar^2/4(\sigma_x + \sigma_y + \sigma_z)$ in N blocks of the matrix. We find S^2 as follows:

$$S^2 = \begin{pmatrix} \sigma & & \\ & \ddots & \\ & & \sigma \end{pmatrix}, \tag{11}$$

where

$$\sigma = \frac{\hbar^2}{4}\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} + \frac{\hbar^2}{2}(N-1)\begin{pmatrix} 1 & 1-i \\ 1+i & -1 \end{pmatrix}. \tag{12}$$

The spectrum S^2 is important. In order to find the eigenvalues of S^2, it just suffices to calculate the corresponding values for σ. The eigenvalues of S^2 will be N many copies of them. The spectrum of σ is easily found to be:

$$\frac{\hbar^2}{4}\left(3 \pm 2\sqrt{3}(N-1)\right). \tag{13}$$

It is easily seen that S^2 has no zero eigenvalue. However it has N positive and N negative eigenvalues. Therefore for some states $\langle S^2\rangle = 0$ may hold. However states that satisfy this relation constitute a measure-zero set. Therefore one may conclude that for almost all states space emerges.

CONCLUSION

In this study, quantum shape kinematics is studied for N electrons and protons in an otherwise empty universe for a general Hamiltonian that may include all types of conceivable interactions. For $N = 1, 2, 3$ no solution is found, hence there is no quantum shape space for these numbers of particles. Since the quantum shape space has $2N-7$ free functional degrees of freedom for spin-1/2 particles, solutions exist for $N \geq 4$.

The dimensionality of the quantum shape space crucially depends on the type of existing species. For scalars, quantum shape space has $N-7$ functional degrees of freedom. For a mixture of N spin-1/2 particles and M scalar particles, quantum shape space has $2N+M-7$ free functions.

The emergence of space is associated with non-vanishing of classical angular momentum of the universe.

Because if so, there exists a *container*, i.e. absolute space, in which universe has a classical angular momentum. We have shown that space almost always emerges because $\langle L^2 \rangle \neq 0$ for almost all states. States for which this expectation value vanishes constitute a measure-zero set.

Currently, the solutions are static and there is no time. The main result in this paper concerns the emergence of space, not time. Interesting phenomena can be expected to occur for quantum shape *dynamics* solutions when this theory is developed. Because once the evolution of states becomes available, the expectation value $\langle L^2 \rangle$ may vanish at certain times.

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* furkan.dundar@boun.edu.tr
† btonguc@sakarya.edu.tr

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